Science Networks
Historical Studies
63

Ralf Krömer
Emmylou Haffner
Editors

Duality in 19th and 20th Century Mathematical Thinking

Birkhäuser

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Science Networks. Historical Studies

Volume 63

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Editors

Duality in 19th and 20th Century Mathematical Thinking

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ISSN 1421-6329 ISSN 2296-6080 (electronic)
Science Networks. Historical Studies
ISBN 978-3-031-59796-1 ISBN 978-3-031-59797-8 (eBook)
https://doi.org/10.1007/978-3-031-59797-8

Mathematics Subject Classification (2020): 01A55, 18-XX, 01Axx, 01A60, 22D35, 46B10, 01A99, 55M05

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

From 2016 to 2019, a collective research project entitled *Dualität – Ein Archetypus mathematischen Denkens* was conducted at the Bergische Universität Wuppertal (Germany) with the financial support of the Deutsche Forschungsgemeinschaft (DFG). The present book is one of two resulting from this project.

In March 2019, the group of investigators of this project organized a one-week international workshop at the *Centre International de Rencontres Mathématiques* (Luminy, France), to which a number of international collaborators (including the contributors to this volume and many of the referees) participated. Most of the chapters included in the books emerged from talks given during the workshop.

During the 19th century, duality was mostly restricted to the context of its first occurrence, namely geometry. This context is studied in our first book (Etwein, Voelke, and Volkert, 2019), while the present volume aims at studying duality phenomena in 19th and 20th century mathematics beyond this context. In fact, mathematicians started to consider duality (or dualities) in logic in the middle of the 19th century, and in still other contexts towards the end of the 19th century. On the other hand, the development of duality in geometry continued in the 20th century. The first part of this book deals with this temporal overlap. Parts II and III cover the period 1880-1945, and part IV the period after WWII.

The reader will find 16 historical case studies, covering a large range of mathematical subjects and disciplines, as well as a complex web of—related and unrelated—parallel developments. This situation prompted the editors to adopt a primarily thematic organization, rather than a strictly chronological one. The dualities studied in the present book occur in the following contexts: curve theory; mathematical logic, Boolean algebra, lattice theory; algebraic topology; linear algebra and functional analysis; the theory of abelian groups; category theory and related fields. Some cases also belong to applied mathematics.

The book is not intended as a mere collection of independent essays, but as a comprehensive whole, as shown by its organization as described in the Introduction. Our aim, in this volume, is to investigate whether there is such a thing as duality in mathematics—or are there just several things called by the same name and similar in some respect? Gathering the individual case studies presented in each chapter allows us to provide a more global view and, more importantly, to put forth possible comparisons that can give elements of answer to this question.

The width of this book is such that the reader will encounter, in addition to the mathematical subjects just mentioned, a number of more general historical, historiographical and epistemological issues, e.g. concerning the diffusion of knowledge about duality, the introduction of new notations or terminology, or the status and roles of duality in various mathematical investigations. As such, it is a testimony of what we hope can be seen as the fruitfulness of "duality" as a subject of study for the history and philosophy of mathematics. Let us sum up our findings by stating that "duality" is neither a mathematical theory nor a notion, but can be considered as a basic structure of mathematical thinking.

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Chapter 1

Introduction

Ralf Krömer and Emmylou Haffner

The word "duality" is used in modern mathematics for many different phenomena. Some of these have been recognized long ago in history, albeit under different names, such a the duality of Platonic solids in book XV of the *Elements* and in Kepler's *Harmonices mundi* books II and V, or the polar triangle in spherical geometry. The idea became more prominent and explicit in the context of projective geometry starting around 1810, and it was in this context that the term "duality" was coined (although "reciprocity" or "polarity" were also used). With the emergence of modern mathematics during the 19th and 20th centuries, the number of phenomena labelled "duality" exploded.

Yet, even today, no general definition of the term is available, and there is no systematic and comprehensive study of its occurrences, let alone of its history. There have only been a few individual studies on this topic so far, which usually refer to duality in a particular context and in a limited period of time (Bollinger, 1972; Chemla and Pahaut, 1988; Becker and Gottlieb, 1999). In other cases, duality is discussed as an aspect among others in larger contexts—we think here of works on the history of projective geometry or algebraic topology, for example (Scholz, 1980; Dieudonné, 1989; Scholz, 1989; James, 1999b; Lorenat, 2015b). Only occasionally is the importance of duality addressed in a comprehensive manner (Geymonat, 1990; Atiyah, 2007). The question of whether duality played an essential role in the development of modern mathematics and corresponding changes in the understanding of the foundations of mathematics has so far only been addressed by Nagel (1939)—albeit thematically restricted to geometry, in particular to the development of the abstract-axiomatic understanding, and temporally limited to the period 1820-1920.[1] Other reflections on particular

[1] Nagel held that (along with the problem of the parallels) it was the discovery of duality in projective geometry in the first half of the 19th century which has paved the way for the

R. Krömer, E. Haffner (eds.), *Duality in 19th and 20th Century Mathematical Thinking*,
Science Networks. Historical Studies 63, https://doi.org/10.1007/978-3-031-59797-8_1

dualities from a more philosophical point of view have been published in Grosholz (1985), Schlimm (2008) and Jarry (2020).[2] Krömer and Corfield (2014) compared the epistemological features of classical projective duality and several more recent dualities, in particular category-theoretical ones. To date, there has been no study that includes both historical and systematic aspects and that also allows links to epistemological questions, including the countless dualisms of philosophy.

In the absence of a definition of the term "duality", one possible approach to the situation is to focus on the historical dimension. This is the approach we chose. Our aim was to investigate the different forms in which duality appeared and still appears in mathematics, to study their respective histories and to analyze interactions between the different forms of duality.

This book is one of two resulting from a collective research project entitled *Dualität – Ein Archetypus mathematischen Denkens*, conducted from 2016 to 2019 at the Bergische Universität Wuppertal (Germany) with the financial support of the Deutsche Forschungsgemeinschaft (DFG; grant number KR 4706/1-1). The group of investigators included, along with the editors of this book, F. Etwein, A. Jarry, S. Oltmanns, E. Scholz, K. Volkert and A. Waldvogel. In early 2019, a one-week international workshop at the *Centre International de Rencontres Mathématiques* (Luminy, France) brought together the group of investigators and a number of international collaborators (including the contributors to this volume and many of the referees). Some of the talks given during the workshop constituted early versions of the chapters included in the books.

One part of the group concentrated on duality in the theory of polyhedra and classical (*i.e.*, projective and spherical) geometry. Their results have been published in a first book, see F. Etwein, J.-D. Voelke and K. Volkert, *Dualität als Archetypus mathematischen Denkens. Klassische Geometrie und Polyedertheorie* (mit Beiträgen von J.-P. Friedelmeyer und E. Scholz), Göttingen: Cuvillier, 2019.[3] For the reader's convenience, we give an overview of the contents of that book in section 1.3 below.

The present volume opens with another study of an aspect of duality in geometry (chapter 2) but otherwise gathers studies of duality phenomena in mathematics of the 19th and 20th century outside the above mentioned contexts. All in all, we brought together 16 historical case studies roughly covering the period 1820-1980.

One problem which occurred when trying to integrate these case studies in a historical narrative is that we faced a complicated network of chronologically parallel developments exhibiting various forms of interdependencies and indepen-

logical approach to geometry, such as Hilbert's, in which the basic concepts of geometry (point, line, plane) are no longer objects but variables (since the interchangeability of points and lines suggested abandoning a too narrow conception of their ontology). Etwein, Voelke, and Volkert (2019) showed that the thesis is largely a retrospective projection of modern ideas (*ibid.*, epilog).

[2]See the end of section 13.6.1 for information on the first two papers, and below for the third one.

[3]Cited in the following as Etwein, Voelke, and Volkert (2019).

dencies. One possibility here is to decide on one guiding question and then interpret the particular contributions as case studies of that question by looking at what the contributors say about it. The following is our guiding question:

Is there such a thing as duality in mathematics, or are there just several things called by the same name and similar in some respect?[4]

This is of course a question one cannot answer in each individual case study but rather by comparing them all. In fact, since the present book is not intended to be just a collection of independent essays, we made considerable efforts to give it a unity as a book. The table of contents is very detailed in order to give a comprehensive overview of the book. There is one common bibliography, testifying both the sheer amount of writings relevant to the history of duality in mathematics and, in many cases, the relevance of particular publications for several of the case studies (as indicated by the pages of the book where the publications are respectively cited). Similarly, there is a common index of names, showing in particular that some mathematicians played an influential role in this history. In the chapters, we freely use cross-references to other parts of the book wherever useful, thus stressing the interconnections between the case studies. Finally, in this introduction, we try to carve out the general conclusions derivable from the particular conclusions of the case studies taken together.

"Duality" is neither a mathematical theory nor a notion. The studies included in the two books suggest that duality can be considered as a basic structure of mathematical thinking. Therefore, an investigation like ours can open new perspectives of research in the history and philosophy of mathematics and on the reflection on mathematics in general.

1.1 Historical issues

Our aim in exploring historical issues related to duality was to incorporate our reflection on duality all at once upon the different stages of the development of mathematics in general, and to examine the characteristic methods and basic ideas of each period in relation to the notion of duality.

Our desire to embrace a widespread historical reflection on the matter led us to unpack the guiding question into a number of subordinate questions. Let us mention two of them here (others will follow below): First, what we recognize as duality has many names, and we wanted to know whether the shifts and changes in vocabulary were significant and if so, to what extent. Secondly, although we know *a priori* that duality appears independently in different parts of mathematics, we wanted to understand what is covered by the same name: are there commonalities? or, conversely, can 'duality' designate completely different things?

Let us give an overview of the various **contexts** in which phenomena of duality occurred in the more recent parts of the history of mathematics. Initially, we

[4](MathStack, 2013) asks exactly this question.

did not adopt a systematic method for identifying relevant contexts, but rather started from a preliminary thematic list. This list has been added to over time by further discussions, and by examination of the *Jahrbuch für die Fortschritte der Mathematik* (see below). Finally, we should also mention that many questions remain open, as we will detail below.

The main contexts in which dualities occur and (most of) which are studied in the present book are:

(a) the theory of polyhedra;

(b) spherical and projective geometry;

(c) mathematical logic, Boolean algebra, lattice theory;

(d) algebraic topology (Poincaré, Alexander);

(e) linear algebra and functional analysis;

(f) the theory of abelian groups (Pontrjagin duality)[5]

(g) category theory and related fields, such as homological algebra and Grothendieck's contributions to algebraic geometry;

(h) interactions with other fields, such as physics and statics.

Roughly speaking, we can assign these topics to three periods (where (h) spreads over all of them):

Before 1880: (a), (b), first phase of (c);

1880-1945: second phase of (c), (d), (e), (f);

1945 and later: (g).

Let us note that most of the fields of mathematical research mentioned here were, at the time at which we are studying them, emerging fields with roots in more traditional fields.

As a first approximation, we could say that during the 19th century, duality as a phenomenon in mathematics was mostly restricted to the context where it made its first occurrence, namely geometry. This situation changed towards the end of the century when mathematicians started to consider duality (or dualities) in other contexts. Of course, this statement comes with some considerable simplifications. In particular, such a boundary seems to overlook, on the one hand, that the relevant developments in logic started in the middle of the 19th century and, on the other hand, that the development of duality in geometry naturally

[5]In current literature written in English, Pontrjagin's last name is spelled "Pontryagin", in accordance with the current transliteration of Russian. However, we decided to stick with the spelling he used himself in his papers written in languages using the Latin alphabet, including his own publications in English. This spelling was also used by all his contemporaries citing his work. A similar remark applies to Paul Alexandroff (now spelled "Aleksandrov"), Andrej Kolmogoroff and others.

continued in the 20th century. The first part of this book deals with this temporal overlap, in what we called "The 19th century heritage". Parts II and III cover the period 1880-1945, and part IV the period after WW II.

In investigating duality in all these different contexts, we were particularly interested in uncovering historical **connections** between domains. We wondered whether it is possible to identify "missing links", and to what extent classical dualities influenced other views and conceptions of duality. In that last case, can one speak about "constructed traditions"? The most important example (among many others discussed in the book) certainly is the fact that the duality of classical geometry in the 20th century explicitly served as a model, with the use of the typical dualistic columnar writing and other elements of seeking connection to the classical case.

In fact, Gergonne's notation in columns is but one example among several which highlight the role played by the chosen forms of **notation** or representation (think of symmetries in formulae or coordinates, arrow-diagram notation...).

Terminology played a prominent role in this history. The variability of terms (dual, adjoint, reciprocal, conjugated, correlative,...) is considerable, and a complete standardization still has not been achieved. One can nevertheless ask when, how, and why the term "duality" (proposed by J. D. Gergonne in 1825-26) has prevailed over competing terminologies.

Note that new dualities are sometimes not called this from the outset. Our criterion for including a phenomenon in our investigation is *not* whether it was called duality when first introduced but whether it has been called duality at all, possibly only by later authors. In such cases, we wish to make sense of such attributions.

During the writing of the book, we also investigated the possibility of making more detailed observations on the development of terminology in context with the help of digital tools, for instance a bibliometric evaluation of the *Jahrbuch für die Fortschritte der Mathematik* (1868-1942). The aim would be to find the number of relevant *Jahrbuch* entries

- in absolute numbers;
- in relation to the respective total number of entries (are numbers only increasing absolutely or do they also increase relatively to each other?);
- by subdisciplines (which subdisciplines appear and when? do dualities "disappear" in certain subdisciplines?;
- by comparing the various terms (when does one term prevail over another?).

For example, a query for "dual*" yields approximately 1400 individual publications in the database. Of course, much fine tuning would be necesary to yield interesting and valid observations. Indeed, there is an obvious problem with the relevance of a hit ("polari*" yields polarization of light, or "dual*" yields dualism as opposed to monism, etc.). Besides, there is the problem that an entry might be found just because of an MSC 2000 keyword added later to the database but without historical significance. And most importantly, how should one attribute a publication to a subdiscipline? We tried to use the titles of the *Jahrbuch* sections for this purpose, but of course these titles also develop over time. Ultimately, the work to be done turned out to be considerable, and we decided to pursue this in a separate publication.

The nature of our topic of investigation led us to focus also on **backgrounds and networks.** Researchers have backgrounds (their primary field of work, their Ph.D. thesis advisors or "schools", the community of colleagues in regular contact with them...). How do these backgrounds influence the development of their work? This is of course a very general question relevant for virtually every investigation of the history of mathematics. We stress it here since the history of duality offers many examples to study. Especially in the 20th century, one can observe both a strong conceptual network of mathematical fields and a personal network of correspondence, conferences and seminars. The Bourbaki group plays a particularly important role here.

The problem of backgrounds is, of course, also related to the history of terminology, since, for example, there are terminological traditions in given "schools". More generally, issues of background can, at least partly, be traced methodologically: one can analyze biographical data, follow up explicit references and the terminology used, and so on. But in many cases, one also has to follow less reliable paths, such as personal reminiscences or autobiographical accounts which sometimes come with problems of invented tradition.

Another, related question is that of **knowledge diffusion**: how was the knowledge about duality diffused? Did the peculiarities of each language area have beneficial or obstructive effects in the reception of new theories? Besides published work, which role was played by the above-mentioned informal parts of scientific exchange, such as contacts between the mathematicians involved in the form of correspondence, teacher-pupil-relations, "schools" and so on?

In the second third of the 20th century, the situation with respect to language areas had, of course, changed greatly. Indeed, mathematics as a research discipline had by then obtained a strongly internationalized and globalized structure. Still, peculiar routes have been taken in the development and dissemination of mathematics, such as the one followed by the very influential French Bourbaki group, for instance.

1.2 Epistemological aspects

The questions in the foregoing section clearly were of a historical nature, to be answered using historical data, i.e. sources. But our investigation also touches on epistemological problems.[6] When addressing them, sources still played an impor-

[6]We use "epistemology" as a synonym of "theory of science" here.

tant role, but the methods used for their interpretation included philosophical reflection.

Emphasized as an epistemological issue, our guiding question addresses the **essence** of duality and would amount to the following:

What is "duality"?
What are the common features (if any) of the different manifestations?

From the historian's point of view, such a question comes with a problem of method; the task is more philosophical than historical. The usage of the term "duality" in mathematical discourse is most probably a case of Wittgensteinian "family resemblance", and, thus, no mathematical definition of the term "duality" is possible. The "complementary" (or dual?) problem is methodologically simpler: study the usage of the term "dual", or the development of the terminology for phenomena today called "duality phenomena" (the fact that such a thing is now called "duality" suggests that it became seen as a duality for a reason). From this perspective, the focus on terminology turns out to be more of a method than a question, which is useful in how we address the guiding question.

In the literature, we find several answers on the question of what duality is. Sometimes, it is called a "principle", for instance by Atiyah (2007):

> Duality in mathematics is not a theorem, but a "principle". It has a simple origin, it is very powerful and useful, and has a long history going back hundreds of years. Over time it has been adapted and modified and so we can still use it in novel situations. It appears in many subjects in mathematics (geometry, algebra, analysis) and in physics. Fundamentally, duality gives two different points of view of looking at the same object. There are many things that have two different points of view and in principle they are all dualities. (Atiyah, 2007, p. 69)

Of course, in this quote, the term "principle" itself remains largely undefined. Mainly, Atiyah seems to think of a "principle" as something opposed to a "theorem", something with many facets. It is clear that the term is used here in a larger sense than in the quite narrow one which is used when we speak about a "duality principle"; *i.e.*, a metatheorem stating that a theorem obtained from a valid theorem by some dualization procedure is valid. What Atiyah stresses about duality is that it "gives two different points of view of looking at the same object". Similarly, "Duality" in math really just means having two ways to think about a problem" (MathStack, 2013).[7]

As an alternative to the term "principle", Etwein, Voelke, and Volkert (2019) characterized duality as an "archetype" of mathematical thinking—comparable to "analogy" or "symmetry". In thi book, we avoid the notion of "archetype" since it needs clarification as well; rather, we would like to speak about a *basic structure of mathematical thinking*.

[7] See the beginning of Tinne Hoff Kjeldsen's Chapter 16 for a collection of similar quotes found on the Internet.

Other epistemological questions concern the **functions** of duality as well as its **justifications**: which roles did duality play in the development of mathematics between 1810 and 1980? What are the foundations of different dualities and how are they ensured? Has there been a shift towards "abstract-axiomatic foundation"? Of course, these questions should again be addressed historically, *i.e.*, including possible shifts in meaning: what are the various ways in which duality is understood by the authors who are interested in it?

Several chapters in this book address topics that belong to the philosophy of mathematics in their own right. To name a few: contributions to duality in formal logic, which, especially in its early days, stands completely between mathematics and philosophy; the role of duality in Albert Lautman's philosophy of mathematics (in which both superficial and deeper references to duality appear) as studied by Christophe Eckes; Saunders Mac Lane's reflections on different types of duality as studied by Jean-Pierre Marquis, as well as Marquis' own conclusions on the mathematical-philosophical significance of category-theoretical concepts of duality.

1.3 Duality in geometry: an overview

As a background to the present collection of essays, we start by summarizing the results of Etwein, Voelke, and Volkert (2019) related to geometry. This allows the reader to get a fuller picture of the history of duality. However, our intention is only to give an overview. Readers interested in detailed discussions should refer to that book itself. In most cases, the precise references for primary sources are also to be found there.[8]

1.3.1 The theory of polyhedra

The oldest duality, albeit without being called so in the beginning, can already be found in Euclid's *Elements*. There, we observe an (implicit) use of duality in the inscription and circumscription of regular polygons (Book IV). Already in this case, a characteristic feature of duality becomes visible: the *economy of work* ("*one results in two*"). Indeed, if one has constructed the inscribed regular pentagon (IV, 11), the construction of the circumscribed one (IV, 12) is reduced to drawing the tangents in the corners of the inscribed pentagon. From a modern standpoint, this is a special case of polar reciprocity where poles and polars coincide—but of course, this is an anachronistic interpretation.

A rich terrain for duality is provided by the theory of polyhedra: the duality of Platonic solids is considered in Book XV of the *Elements* (not written by Euclid). Later, an interplay emerged between the definition of dualization and various extensions of the classes of the polyhedra under consideration (*e.g.*, transition from the Platonic to the Archimedean solids or to the regular star bodies,

[8] Some parts of the content of Etwein, Voelke, and Volkert (2019) are also discussed in 1.4.4 and 1.4.6 below; they were placed there for thematic reasons.

inclusion of unilateral polyhedra). Since even in simple cases such as that of the cuboctahedron, the dualization by surface centers fails, dualization had to be formulated more generally. This led to the combinatorial view of polyhedra which is dominant today and was suggested by Eugène Catalan (1865). This approach makes it possible to dualize in a completely abstract manner, without having to take into account the question of the geometrical feasibility of the dual body. In connection with dualization, symmetries gained in importance, since the symmetries of a polyhedron and that of its dual coincide.[9] Ultimately, this led to detailed investigations of vertex, edge and face transitivity, as found under different names in the work of Edmund Hess. The desire to be able to dualize polyhedra thus was an important driving force for new investigations. This development continued in emerging combinatorial topology, as discussed in Chapter 7 in this book.

One of the central results of the theory of polyhedra, *Euler's polyhedron formula*, due to the symmetrical occurrence of the number of vertices and faces, refers, in its very formulation, to a duality between vertices and faces, between points and planes. This relates to the dualization of circumscribed or inscribed spheres, where points of contact correspond to tangential planes and vice versa. Joseph Diez Gergonne established this connection and thus achieved an interesting interweaving of two lines of development in the early history of duality. Similar to the dualization of polyhedra, the history of the polyhedron formula is characterized by the endeavour to extend it to larger classes of polyhedra (*e.g.*, not simply connected ones).[10]

1.3.2 Spherical geometry and trigonometry

Spherical geometry and trigonometry is another line of development in which duality occurred (here, usually called polarity). An important notion in this context is the polar triangle, which, for a long time, was used mainly as an auxiliary figure, e.g. to derive the angular cosine theorem from the lateral cosine theorem. Etwein, Voelke, and Volkert (2019) offer an analysis of the rather complex history of the polar triangle, beginning with the contributions of Arabic mathematics and those of Pitiscus, Viète and Snellius (to name only a few). They also discuss interesting developments in the 19th century that had previously received little attention, such as the emergence of the "elementary" spherical geometry by Christoph Gudermann and others or the work of Sorlin-Gergonne (1824-25) on the formulae of spherical trigonometry which was one of the first places where the two-column notation was used. In addition to the fact that is saves labour, another important achievement of duality is evident here: it *structures theories*, an aspect that Gergonne always emphasized. In Gergonne's work, the various dualities of the theory of polyhedra, projective geometry and spherical geometry are still seen in

[9] See Etwein, Voelke, and Volkert (2019), paragraph 4.2.

[10] See Etwein, Voelke, and Volkert (2019), paragraph 1.1, 2.1 and chapter 4. See also section 7.1.2 of the present book on a generalization of the formula that Poincaré attributes to de Jonquières.

close relation. In later developments, they are increasingly treated as separated things. Duality is replaced by a variety of area-specific dualities, a development that suggests increasing specialization. However, the search for dualities remains an important research project in all relevant disciplines.

1.3.3 Projective geometry

As is well known, in the beginning of the 19th century, projective geometry had a particularly strong influence on considerations of duality—it became, so to speak, the geometry of duality. This was already easily recognizable in the use of the two-column notation, a fairly recurrent feature of writings on duality which we will encounter in other mathematical contexts as well.

From a constructive point of view, which in the beginning was mainly advocated by Jean-Victor Poncelet, the central conception of duality in projective geometry is polar reciprocity (with respect to a conic section in the plane case and with respect to a quadric in the spatial case). Etwein, Voelke, and Volkert (2019) examine the prehistory of this, that is, the rather incidental occurrences of polar reciprocity in Philippe de la Hire's, Gaspard Monge's and J.-J. Livet's works. Ideas related to duality can also be found early on in the works of Charles Brianchon and—albeit with limitations—of Étienne Bobillier, but they remain implicit. The best-known, often misrepresented, example of this is Brianchon's proof of the theorem that bears his name. Etwein, Voelke, and Volkert (2019) also extensively investigate the role of Gergonne and Poncelet, especially their different views and methods (analytical and synthetical, respectively). The book offers, as well, a detailed overview, compiled by Jean-Pierre Friedelmeyer, of the publications related to duality in Gergonne's *Annales de Mathématiques Pures et Appliquées* (ibid., pp. 141-146).

Gergonne's journal in the first third of the 19th century played a key role with regard to the development of duality and the dissemination of these ideas. In a thorough analysis, Etwein, Voelke, and Volkert (2019) show that, unlike what is often claimed, it was not the Pascal/Brianchon pair of theorems that played the central role in the development of duality, but that it was first and foremost the Cramer-Castillon problem that provided decisive motives. It is, to our knowledge, in the context of projective geometry that duality became, for the first time, explicit knowledge and was considered to be important. The sensational debate between pioneers Poncelet and Gergonne, was also decisive in widening the popularity of questions related to duality.

Other important contributions to the research on projective duality came from Jakob Steiner and Karl Georg Christian von Staudt. Through the introduction of so-called absolutes ("*Fundamentalgebilde*") (such as point ranges or pencils of lines and planes), Steiner succeeded in providing a new view of duality as based on the properties of the absolutes themselves, organically developing from them. Von Staudt further developed Steiner's approach by introducing the abstract conceptions of correlations, which—in modern terms—can be understood as incidence-preserving functions mapping points to straight lines and straight lines to points, and of colineations which correspondingly map points to points and straight lines to straight lines. His observation that correlations of the projective plane with self-conjugate elements (*i.e.*, poles that coincide with their polars) can always be represented as polar reciprocities reconciled, in a certain way, the approaches of Gergonne and Poncelet.[11]

Another important line of development was the *analytical approach to projective geometry*, made possible by the introduction of homogeneous coordinates, especially by August Ferdinand Möbius and Julius Plücker. Since, in the plane case, formulae can be read in both point and straight line coordinates, this approach provided a convincing justification of duality. The analytical approach was continued by Ludwig Otto Hesse, who—similarly to Bobillier—applied it to the theory of curves. Here, a central role was played by singularities and the discovery of the laws governing their behaviour under dualization, expressed in the Plücker formulae. Investigations of this kind were deepened by Hesse's student Alfred Clebsch and continued by members of his school, such as Alexander von Brill and Max Noether. In this context, let us also mention Hesse's rather little-known ideas on so-called transfer principles, of which traditional duality is one example. Further aspects of the history of duality in curve theory are examined in Chapter 2 in this book.

Von Staudt's attempt to separate *projective* geometry from *Euclidean* geometry—*i.e.*, to give it an autonomous structure—also raised the question of the relationship of duality to Euclidean geometry. This led, on the one hand, to Hermann Hankel's approach, who constructed a duality in plane Euclidean geometry and examined its properties, and on the other hand, to Wilhelm Fiedler's considerations on the conditions for the possibility of duality, which he found to be, to a large extent, in the existence of an absolute measure of length.[12]

From a modern point of view, a foundation for the duality of projective geometry is, of course, to be achieved via the *axiomatic* approach. This approach entered a new stage of development with Moritz Pasch, followed by the works of several Italian mathematicians, such as Giuseppe Peano, Mario Pieri and Gino Fano, who made important contributions to the axiomatic treatment of projective geometry. Finally, with Hilbert's *Grundlagen der Geometrie* (1899), the axiomatic approach became a widely acclaimed and discussed research program. While projective geometry was still left out by Hilbert in this framework, it was discussed by Oswald Veblen and John Wesley Young in their book *Projective Geometry* (1910). From such an axiomatic point of view, duality exists because the axioms of projective geometry are self-dual, and because it is possible to give a model of dualized geometry. Remarkably, Steiner's fundamental structures, once again, play a role.[13]

[11] See Etwein, Voelke, and Volkert (2019), paragraph 1.3 as well as Chapter 2 and Subchapter 3.1.

[12] See Etwein, Voelke, and Volkert (2019), Subchapter 3.2. Both developments have been largely unknown so far.

[13] See Etwein, Voelke, and Volkert (2019), paragraph 5.4.

Finally, another attempt to give a foundation for the duality of projective geometry is based on the consideration of *lattices*. This approach was developed in the 1930s and ultimately led to an embedding of projective geometry in linear algebra, as presented by Reinhold Baer (1952).[14]

1.4 The contents of this book

Before turning our attention to the occurrences of duality examined in the present volume, a general remark is in order. When investigating developments in the history of mathematics, one can often adopt broader or narrower perspectives—let us speak of macro perspective and micro perspective in such cases. Macro perspective studies investigate a longer development including several stages, often at the expense of skipping details. The micro perspective, on the other hand, focuses on the work of a single author, or even a small set of papers by that author, and analyses this in considerable detail. In these cases, the analysis of context-related issues may be left out more than we would have liked. To avoid any kind of one-sidedness, we took care to combine the macro and the micro perspective. Readers who are interested in a general overview should first read the "macro" chapters, *i.e.*, this introduction and Chapters 4, 7, 11 and 17.

1.4.1 Geometry

In the first chapter of this book (chapter 2, "On "contour apparent", "courbe de contact" and ramification curves: Duality between a principle and a tool"), Michael Friedman follows up the investigation by Etwein, Voelke, and Volkert (2019, 308ff) of what has been called "Plücker's paradox" on the degrees of a curve and its dual. In this context, he points out a recurrent feature of duality. To wit, one thinks about the situation as paradoxical here precisely because one expects the dual of the dual to be the original curve, or put differently, one expects the dualization procedure to be an involution. The fact that this is not the case in a certain respect motivated the search for clarification of the issue.

Further aspects of the history of duality in curve theory are examined, from Bobillier's and Plücker's works to that of George Salmon and later Oscar Zariski. In particular, Friedman highlights various shifts in this development, beginning with a conceptual shift in the case of the polar curve. More generally, "research on ramification and branch curves under the setting of duality was characterized by shifts in context" which "can be noted with the shifts of terminology regarding the ramification curve". By contrasting the use of duality as a principle (Gergonne's approach, see section 2.3) to that as a tool (in the later developments, see section 2.4), Friedman sets a first benchmark for the general reflection undertaken in this volume.

[14] See Etwein, Voelke, and Volkert (2019), paragraph 5.2 and 5.3.

1.4.2 Logic, Boolean algebra, Lattice theory

In the 19th century, duality began to play a role in the emergence of *formal logic*. In this context, what springs to mind is De Morgan's rules. In Chapter 3, Juan Luis Gastaldi examines their history and shows that De Morgan's work is related to both medieval syllogistics on the one hand and Gergonne's work on the other. In particular, to display duality, De Morgan used the two-column layout popularized by Gergonne. Gastaldi's micro perspective study contributes to the history of duality "by providing an analytical insight into the place occupied by duality in the emergence of mathematized systems of logic at the turn of the 19th century." In this respect, Gastaldi's interpretation of De Morgan's work goes beyond the one offered by Ivor Grattan-Guinness. In Gastaldi's words, "the recourse to an embryonic practice of class complementation provided De Morgan with a powerful semantics for an original treatment of logical negation" which, however, is "far from reducible to a simple form of propositional negation" but rather adopts several forms. More generally, Gastaldi's "analysis repeatedly showed that any attempt to project upon the resulting system the image of a propositional calculus as we know it, where the duality of the so-called 'De Morgan's laws' follow from definitions, should be regarded with suspicion."

If De Morgan's work still belongs to the early history of formal logic in the 19th century, the latter reached its heyday with Boole, Jevons, Peirce, Schröder and others. The role of duality in this development is examined from the macro perspective by Dirk Schlimm in Chapter 4. Schlimm shows, in particular, how an initially rather vague "analogy" investigated by Boole becomes a precise duality. Following a broadly chronological line, he highlights the main steps in the development of propositional logic in the 19th century towards the emergence of duality as the symmetry between formulae in Boolean algebra (that is, any valid formula can be turned into another valid formula by interchanging the symbols), beginning with Boole's application of symbolical algebra to logic and including a number of authors up to Russell. He studies the role of both formal representations (or lack thereof) and individual goals in identifying duality and its importance. By doing so, Schlimm shows that "the emergence of the notion of duality in the algebra of logic runs in parallel with a turn towards a more theoretical perspective on logic".

In Chapter 5, Emmylou Haffner examines the genesis of the theory of lattices in Dedekind's works, with special attention to duality. These beginnings are not, as one might expect, situated in a logical context, but rather in a number-theoretical one. In particular, Dedekind's papers from 1897 and 1900 on the concept of *Dualgruppe* (formally equivalent to a lattice) turn out to be the result of twenty years of research, which started with the observation of a "peculiar dualism" between the GCD and LCM (which Dedekind writes as $+$ and $-$) of modules in the Dedekind sense. These ideas were developed in interesting, previously unpublished material,[15] whose analysis provides elements to understand the genesis of

[15] A digital edition of the manuscripts is now available: Haffner, E., *Brouillons de Richard Dedekind: étude génétique*, http://eman-archives.org/Dedekind/, édition génétique textuelle