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# Nonlinear and Modern Mathematical Physics

NMMP-2022, Tallahassee, Florida, USA (Virtual), June 17–19

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Editors

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# Preface

The 6th International Virtual Workshop on Nonlinear and Modern Mathematical Physics (NMMP2022) took place virtually from June 17 to 19, 2022, hosted by Florida Agricultural and Mechanical University. This workshop is part of a series of conferences organized periodically, starting with the inaugural workshop held in China from July 15 to 21, 2009. Subsequent events took place in Tampa at the University of South Florida from March 9 to 11, 2013, at the African Institute for Mathematical Sciences in Cape Town, South Africa from April 9 to 11, 2015, in Kuala Lumpur, Malaysia, from May 4 to 8, 2017, and the 5th edition, which was successfully conducted in Honolulu, Hawaii, from May 20 to 24, 2019.

The 6th edition of the NMMP workshop served as a dynamic forum, bringing together scholars and researchers from various institutions worldwide. Florida A&M University led the organization, with support from the University of South Florida, Florida State University, Embry-Riddle Aeronautical University, Savannah State University, Prairie View A&M University, and Beijing Jiaotong University. The focus of the workshop was on recent advances and prevailing trends in nonlinear science, with a specific emphasis on nonlinear partial differential equations and their applications. Featuring 42 distinguished speakers, the three day event attracted over 300 participants globally, fostering collaboration and knowledge exchange in the field.

This book, a compilation of papers from both speakers and participants of NMMP2022, aims to showcase new ideas and discoveries in the field of partial differential equations (PDEs), integrable systems, and related areas in mathematical physics. In the dynamic landscape of mathematical physics, the exploration of nonlinear phenomena takes center stage, and this compendium, titled "Nonlinear and Modern Mathematical Physics," endeavors to encapsulate the forefront of research and discourse in this field. As customary, each contribution in the book has undergone standard double-blind refereeing.

Nonlinearity, with its intriguing and often unpredictable nature, has emerged as a central theme in contemporary mathematical physics. From the theoretical realms of chaos theory to the practical applications in fluid dynamics, the study of nonlinear phenomena has opened up new avenues of exploration and understanding. One remarkable example of this is the discovery of solitons, which has had a profound impact on mathematical physics, reshaping our understanding of nonlinear dynamics and leaving a lasting imprint on various scientific disciplines. The introduction of solitons has not only revolutionized our conceptual framework but has also brought forth powerful mathematical methods. Techniques such as the inverse scattering transform and Hirota's method have been developed, offering sophisticated tools to solve a wide range of nonlinear equations across diverse fields. These methods have not only expanded our analytical capabilities but have also facilitated deeper insights into the behavior of nonlinear systems.

This compilation of works boldly explores the forefront of advancements in nonlinear theories, offering a comprehensive examination of the richness and diversity inherent in this dynamic field. The contributors, by delving into the intricacies of nonlinear dynamics, illuminate the multifaceted nature of nonlinear phenomena. Their collective efforts shed light on the profound implications and versatile applications of nonlinear theories across various scientific domains. This volume serves as a testament to the far-reaching impact and ongoing exploration within the captivating realm of nonlinear mathematical physics.

As editors, our aim is to curate a collection that not only reflects the current state of nonlinear mathematical physics but also serves as an intellectual catalyst for future explorations. The breadth and depth of topics covered herein cater to both seasoned researchers navigating the cutting edge and aspiring scholars embarking on their journey into this captivating realm. May this compilation serve as both a testament to the vibrant state of nonlinear mathematical physics and an inspiration for those who embark on the quest to unravel the mysteries that lie beyond the linear veil.

Tallahassee, USA — Solomon Manukure
Tampa, USA — Wen-Xiu Ma

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The editors also express their gratitude for the financial support provided by *Mathematics*, an open-access journal by MDPI, for the plenary speakers during the conference.

Much appreciation also goes to the reviewers for their invaluable contributions to the peer review process. Their expertise and thorough evaluations have played a crucial role in ensuring the quality and rigor of the content presented in this volume. The editors extend their sincere appreciation for the time and effort invested by these esteemed reviewers.

# Contents

# A Hamiltonian Set-Up for 4-Layer Density Stratified Euler Fluids

**R. Camassa, G. Falqui, G. Ortenzi, M. Pedroni, and T. T. Vu Ho**

**Abstract** By means of the Hamiltonian approach to two-dimensional wave motions in heterogeneous fluids proposed by Benjamin [1] we derive a natural Hamiltonian structure for ideal fluids, density stratified in four homogenous layers, constrained in a channel of fixed total height and infinite lateral length. We derive the Hamiltonian and the equations of motion in the dispersionless long-wave limit, restricting ourselves to the so-called Boussinesq approximation. The existence of special symmetric solutions, which generalise to the four-layer case the ones obtained in [11] for the three-layer case, is examined.

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## 1 Introduction

Density stratification in incompressible fluids is an important aspect of fluid dynamics, and plays an important role in variety of phenomena occurring in both the ocean and the atmosphere. In particular, displacement of fluid parcels from their neutral buoyancy position within a density stratified flow can result in internal wave motion. Effective one-dimensional models (in particular, their quasi-linear limit) were introduced to study these phenomena, and were the subject of a number of investigations (see, e.g., [6–10, 13, 14, 16] and references therein). Although most of the theoretical and numerical results that can be found in the literature are focussed on the 2-layer case, multiply-layered fluid configurations appear as effective models of physical phenomena, e.g., in the atmosphere or in mountain lakes. The extension to the $n > 2$ layers case can also be seen as a refined approximation to the real-world continuous stratification of incompressible fluids.

The focus of the present paper is on the dynamics of an ideal (incompressible, inviscid) stably stratified fluid consisting of 4 layers of constant density $\rho_1 < \rho_2 < \rho_3 < \rho_4$, confined in a channel of fixed height $h$ (see Fig. 1 for a schematic of our setup), and, in particular, on its Hamiltonian setting. This will be obtained by a suitable reduction of the Hamiltonian structure introduced by Benjamin [1] in the study of general density stratifications for Euler fluids in 2 dimensions.

We shall follow the approach set forth in our recent paper [4], where the 3-layer case was considered by extending to the multiple layer case a technique introduced in [3]. In particular, after having discussed in details the construction of the Hamiltonian operator for an effective 1D model, we shall consider the so-called Boussinesq limit of the system, and explicitly determine its Hamiltonian structure and Hamiltonian functional, as well as point out the existence of special symmetric solutions.

Our mathematical model is based on some simplifying hypotheses. At first, we assume that an inviscid model suffices to capture the essential features of the dynamics since the scales associated with internal waves are large, and consequently the Reynolds number is typically high ($>10^5$). Although in the ocean and the atmosphere (as well as in laboratory experiments) the density stratification arises as a consequence of diffusing quantities such as temperature and salinity, we can neglect diffusion and mixing since the time scales associated with diffusion processes are far larger than the time scale of internal wave propagation. Finally, we use the rigid lid assumption for the upper surface since the scales associated with internal wavemotion are greatly exceeding the scales of the surface waves (see, e.g., [17] for further details on these assumptions).

The Hamiltonian 4-layer model herewith discussed is a natural extension of the 2 and 3-layer model. Indeed, when two adjacent densities are equal (and as a consequence the relative interface becomes meaningless) we fully recover the dynamics of the 3 layer model (see, e.g., [4, 11]). Similarly, the 3-layer model reduces to the ordinary 2-layer model when two mass densities coincide.

The layout of the paper is the following. In Sect. 2 we briefly review the Hamiltonian representation for 2-dimensional incompressible Euler fluids of [1]. Section 3 is devoted to a detailed presentation of our Hamiltonian reduction scheme, which endows the dynamics of the set of 4-layer stratified fluids with a natural Hamiltonian structure. In Sect. 4 we compute the reduced Hamiltonian and the ensuing equations of motion, confining ourselves to the case of the so-called Boussinesq approximation. In Sect. 5 a class of special evolutions, selected by a symmetry of the Hamiltonian, is found and briefly examined.

## 2 The 2D Benjamin Model for Heterogeneous Fluids in a Channel

Benjamin [1] proposed and discussed a set-up for the Hamiltonian formulation of an incompressible stratified Euler system in 2 spatial dimensions, which we hereafter summarize for the reader's convenience.

The Euler equations for a perfect inviscid and incompressible but heterogeneous fluid in 2D, subject to gravity $-g\mathbf{k}$, are usually written for the the density $\rho(x, z)$ and the velocity field $\mathbf{u} = (u, w)$ as

$$\frac{D\rho}{Dt} = 0, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \rho\frac{D\mathbf{u}}{Dt} + \nabla p + \rho g\mathbf{k} = 0 \tag{1}$$

together with appropriate boundary conditions, where, as usual, $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative.

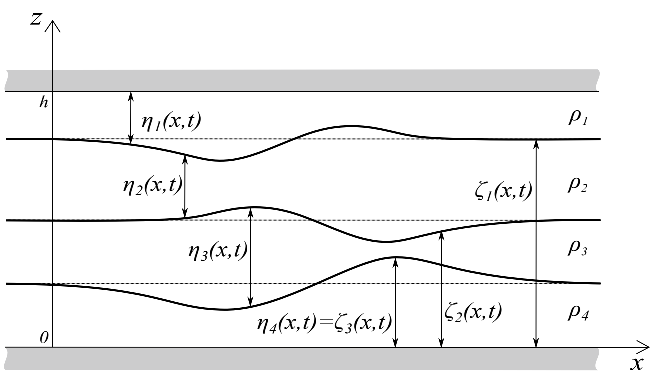


**Fig. 1** Four-layer rigid lid setup and relevant notation: $\zeta_i$ are the surface heights and $\eta_i$ are the layer thicknesses

Benjamin's contribution was to consider, as basic variables for the evolution of such a fluid, the density $\rho$ together with the "weighted vorticity" $\Sigma$ defined by

$$\Sigma = \nabla \times (\rho \mathbf{u}) = (\rho w)_x - (\rho u)_z . \tag{2}$$

The equations of motion for these two fields, ensuing from the Euler equations for incompressible fluids, are

$$\begin{aligned} &\rho_t + u\rho_x + w\rho_z = 0 \\ &\Sigma_t + u\Sigma_x + w\Sigma_z + \rho_x\big(gz - \tfrac{1}{2}(u^2+w^2)\big)_z + \tfrac{1}{2}\rho_z\big(u^2+w^2\big)_x = 0\,. \end{aligned} \tag{3}$$

They can be written in the form

$$\rho_t = -\left[\rho, \frac{\delta H}{\delta \Sigma}\right], \qquad \Sigma_t = -\left[\rho, \frac{\delta H}{\delta \rho}\right] - \left[\Sigma, \frac{\delta H}{\delta \Sigma}\right], \tag{4}$$

where, by definition, the bracket is $[A, B] \equiv A_x B_z - A_z B_x$, and the functional

$$H = \int_{\mathcal{D}} \rho \left(\frac{1}{2}|\mathbf{u}|^2 + gz\right) \mathrm{d}x\, \mathrm{d}z \tag{5}$$

is simply given by the sum of the kinetic and potential energy, $\mathcal{D}$ being the fluid domain. The most relevant feature of this coordinate choice is that $(\rho, \Sigma)$ are physical, directly measurable, variables. Their use, though confined to the 2D case with the above definitions, allows one to avoid the introduction of Clebsch variables (and the corresponding subtleties associated with gauge invariance and limitations of the Clebsch potentials) which are often used in the Hamiltonian formulation of both 2D and the general $3D$ case (see, e.g., [18]).

As shown by Benjamin, Eq. (4) are a Hamiltonian system with respect to a Lie-theoretic Hamiltonian structure, that is, they can be written as

$$\rho_t = \{\rho, H\}, \qquad \Sigma_t = \{\Sigma, H\},$$

for the Poisson bracket defined by the Hamiltonian operator

$$J_B = -\begin{pmatrix} 0 & \rho_x\partial_z - \rho_z\partial_x \\ \rho_x\partial_z - \rho_z\partial_x & \Sigma_x\partial_z - \Sigma_z\partial_x \end{pmatrix}. \tag{6}$$

## 3 The Hamiltonian Reduction Process

As mentioned in the Introduction, we shall consider special stratified fluid configurations, consisting of a fluid with $n = 4$ layers of constant density $\rho_1 < \rho_2 < \rho_3 < \rho_4$ and respective thicknesses $\eta_1, \eta_2, \eta_3, \eta_4$, confined in a channel of fixed height $h$. We define, as in Fig. 1, the locations of the interfaces at $z = \zeta_k$, $k = 1, 2, 3$, related to the thickness $\eta_j$ by

$$\zeta_3 = \eta_4, \quad \zeta_2 = \eta_4 + \eta_3, \quad \zeta_1 = \eta_4 + \eta_3 + \eta_2. \tag{7}$$

The velocity components in each layer are denoted by $(u_i(x, z), w_i(x, z))$, $i = 1, \dots, 4$. By means of the Heaviside $\theta$ and Dirac $\delta$ generalized functions, a four-layer fluid configuration can be described within Benjamin's setting as follows. First, the $2D$ density and velocity variables can be written as

$$\begin{aligned}
\rho(x, z) &= \rho_4 + (\rho_3 - \rho_4)\theta(z - \zeta_3) + (\rho_2 - \rho_3)\theta(z - \zeta_2) + (\rho_1 - \rho_2)\theta(z - \zeta_1) \\
u(x, z) &= u_4 + (u_3 - u_4)\theta(z - \zeta_3) + (u_2 - u_3)\theta(z - \zeta_2) + (u_1 - u_2)\theta(z - \zeta_1) \\
w(x, z) &= w_4 + (w_3 - w_4)\theta(z - \zeta_3) + (w_2 - w_3)\theta(z - \zeta_2) + (w_1 - w_2)\theta(z - \zeta_1)\,.
\end{aligned} \tag{8}$$

Thus, the density-weighted vorticity $\Sigma = (\rho w)_x - (\rho u)_z$ can be computed as

$$\begin{aligned}
\Sigma =& \sum_{j=1}^{3} \left(\rho_{j+1}\Omega_{j+1} - \rho_j\Omega_j\right)\theta(z - \zeta_j) + \rho_4\Omega_4 \\
&+ \sum_{j=1}^{3} \left((\rho_{j+1}u_{j+1} - \rho_j u_j) + (\rho_{j+1}w_{j+1} - \rho_j w_j)\zeta_{j\,x}\right)\delta(z - \zeta_j)\,,
\end{aligned} \tag{9}$$

where $\Omega_i = w_{i\,x} - u_{i\,z}$ for $i = 1, \dots, 4$ are the bulk vorticities.

Next, we assume the bulk motion in each layer to be irrotational, so that $\Omega_i = 0$ for all $i = 1, \dots, 4$. Thus the density weighted vorticity is explicitly given by

$$\begin{aligned}
\Sigma =& \left((\rho_4 u_4 - \rho_3 u_3) + (\rho_4 w_4 - \rho_3 w_3)\zeta_{3x}\right)\delta(z - \zeta_3) \\
&+ \left((\rho_3 u_3 - \rho_2 u_2) + (\rho_3 w_3 - \rho_2 w_2)\zeta_{2x}\right)\delta(z - \zeta_2) \\
&+ ((\rho_2 u_2 - \rho_1 u_1) + (\rho_2 w_2 - \rho_1 w_1)\zeta_{1x})\,\delta(z - \zeta_1)\,.
\end{aligned} \tag{10}$$

A further assumption we make right from the outset is that of the long-wave asymptotics, with small parameter $\epsilon = h/L \ll 1$, $L$ being a typical horizontal scale of the motion such as wavelength. This assumption implies (see, e.g., [8] for further details) that at the leading order as $\epsilon \to 0$ we have

$$u_i \sim \overline{u}_i\,, \qquad w_i \sim 0\,,$$

i.e., we can neglect the vertical velocities $w_i$ and trade the horizontal velocities $u_i$ with their layer-averaged counterparts,

$$\overline{u}_i = \frac{1}{\eta_i}\int_{\zeta_i}^{\zeta_{i-1}} u(x, z)\mathrm{d}z, \text{ where } \zeta_0 \equiv h,\ \zeta_4 \equiv 0\,. \tag{11}$$

Hence, from (10) and recalling the first of (8), we obtain

$$\begin{aligned} \rho(x,z) &= \rho(x,z) = \rho_4 + \sum_{i=1}^{3} (\rho_i - \rho_{i+1})\,\theta(z-\zeta_i) \\ \Sigma(x,z) &= \sum_{i=1}^{3} \sigma_i\,\delta(z-\zeta_i)\,, \end{aligned} \tag{12}$$

where, hereafter,

$$\sigma_i \equiv \rho_{i+1}\overline{u}_{i+1} - \rho_i\overline{u}_i \tag{13}$$

is the horizontal averaged momentum shear. We remark that field configurations of the form (12) can be regarded as defining a submanifold, which will be denoted by $\mathcal{I}$, of Benjamin's Poisson manifold $\mathcal{M}$ described in Sect. 2.

The $x$ and $z$-derivative of the Benjamin's variables given by Eq. (12) are generalized functions supported at the interfaces $\{z=\zeta_1\} \cup \{z=\zeta_2\} \cup \{z=\zeta_3\}$, and are computed as

$$\begin{aligned} \rho_x &= -\,\Sigma_{i=1}^3 (\rho_i - \rho_{i+1})\delta(z-\zeta_i)\zeta_{ix} \\ \rho_z &= \Sigma_{i=1}^3 (\rho_i - \rho_{i+1})\delta(z-\zeta_i)\,, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \Sigma_x &= -\,\Sigma_{i=1}^3 \sigma_i \zeta_{ix} \delta'(z-\zeta_i) + \Sigma_{i=1}^3 \sigma_{ix}\delta(z-\zeta_i) \\ \Sigma_z &= \Sigma_{i=1}^3 \sigma_i \delta'(z-\zeta_i)\,. \end{aligned} \tag{15}$$

To invert the map (12) we choose to integrate along the vertical direction $z$. To this end, we define the two intermediate isopycnals

$$\overline{\zeta}_{12} = \frac{\zeta_1 + \zeta_2}{2}, \qquad \overline{\zeta}_{23} = \frac{\zeta_2 + \zeta_3}{2}\,. \tag{16}$$

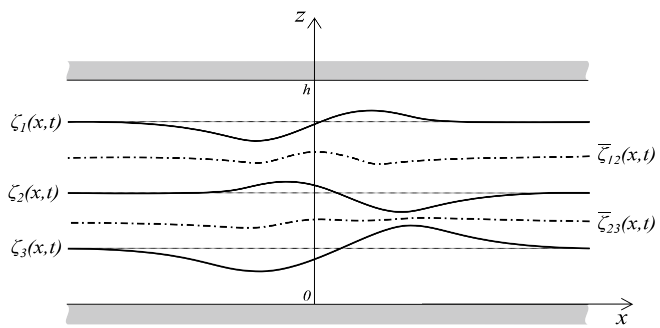


**Fig. 2** Choice of the isopycnals: $\zeta_i$ are the surface heights and $\overline{\zeta}_{12}$, $\overline{\zeta}_{23}$ the intermediate isopycnals

Remarking that $\overline{\zeta}_{12}$ lies in the $\rho_2$-layer and $\overline{\zeta}_{23}$ in the $\rho_3$-layer (see Fig. 2), by means of this choice we can introduce on $\mathcal{I}$ the "projection" $\pi$ by

$$\begin{aligned}&\pi\ (\rho(x,z), \Sigma(x,z)) \equiv (\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3)\\ &= \begin{pmatrix} \int_0^h (\rho(x,z)-\rho_4)\,\mathrm{d}z, \int_0^{\overline{\zeta}_{12}} (\rho(x,z)-\rho_4)\,\mathrm{d}z, \int_0^{\overline{\zeta}_{23}} (\rho(x,z)-\rho_4)\,\mathrm{d}z,\\ \int_0^h \Sigma(x,z)\mathrm{d}z, \int_0^{\overline{\zeta}_{12}} \Sigma(x,z)\mathrm{d}z, \int_0^{\overline{\zeta}_{23}} \Sigma(x,z)\mathrm{d}z \end{pmatrix}\end{aligned} \tag{17}$$

which maps Benjamin's manifold of $2D$ fluid configurations to the space of effective $1D$ fields $\mathcal{S}$, parameterized by the six quantities $(\zeta_k, \sigma_k)$. A straightforward computation shows that the relations

$$\begin{aligned}\xi_1 &= (h-\zeta_1)(\rho_1-\rho_2) + (h-\zeta_2)(\rho_2-\rho_3) + (h-\zeta_3)(\rho_3-\rho_4)\\ \xi_2 &= \frac{\rho_2-\rho_3}{2}(\zeta_1-\zeta_2) + \frac{\rho_3-\rho_4}{2}(\zeta_1+\zeta_2-2\zeta_3)\\ \xi_3 &= \frac{1}{2}(\rho_3-\rho_4)(\zeta_2-\zeta_3)\\ \tau_1 &= \sigma_1+\sigma_2+\sigma_3, \quad \tau_2 = \sigma_1+\sigma_2, \quad \tau_3 = \sigma_3\end{aligned} \tag{18}$$

hold.

To obtain a Hamiltonian structure on the manifold $\mathcal{S}$ by reducing Benjamin's parent structure (6), we have to perform, as per the Hamiltonian reduction scheme of [15], the following steps:

1. Starting from a 1-form on the manifold $\mathcal{S}$, represented by the 6-tuple

$$(\alpha_S^1, \alpha_S^2, \alpha_S^3, \alpha_S^4, \alpha_S^5, \alpha_S^6)\,,$$

   we construct its lift to $\mathcal{I}$, that is, a 1-form $\boldsymbol{\beta}_M = (\beta_\rho, \beta_\Sigma)$ satisfying the relation

$$\int_{-\infty}^{+\infty}\int_0^h (\beta_\rho\dot{\rho} + \beta_\Sigma\dot{\Sigma})\,\mathrm{d}x\,\mathrm{d}z = \int_{-\infty}^{+\infty}\sum_{k=1}^{6}\alpha_S^k\cdot\left(\pi_*(\dot{\rho},\dot{\Sigma})\right)^k\,\mathrm{d}x\ , \tag{19}$$

   where $\pi_*$ is the tangent map to (17) and $(\dot{\rho}, \dot{\Sigma})$ are generic infinitesimal variations of $(\rho, \Sigma)$ in the tangent space to $\mathcal{I}$.
2. We apply Benjamin's operator (6) to the lifted one form $\boldsymbol{\beta}_M$ to get the vector field

$$\begin{pmatrix}\dot{\rho}\\ \dot{\Sigma}\end{pmatrix} = \begin{pmatrix} Y_M^{(1)}\\ Y_M^{(2)}\end{pmatrix} = J_B\cdot\begin{pmatrix}\beta_\rho\\ \beta_\Sigma\end{pmatrix}. \tag{20}$$

3. We project the vector $(Y_M^{(1)}, Y_M^{(2)})$ under $\pi_*$ and obtain a vector field on $\mathcal{S}$. The latter depends linearly on $\{\alpha_S^{(i)}\}_{i=1,\ldots,6}$, and defines the reduced Poisson operator $P$ on $\mathcal{S}$.

As in the three layer case of [4] this construction essentially works as in the two-layer case considered in [3], provided one subtle point is taken into account. Thanks to the relations (12) and the definition of $\pi$, we have that, for tangent vectors $(\dot\rho, \dot\Sigma)$,

$$\pi_* \begin{pmatrix} \dot\rho \\ \dot\Sigma \end{pmatrix} = \begin{pmatrix} \int_0^h \dot\rho \, dz \\ \int_0^{\overline{\zeta}_{23}} \dot\rho \, dz + \dot{\overline{\zeta}}_{23} \left(\rho(x, \overline{\zeta}_{23}) - \rho_4\right) \\ \int_0^{\overline{\zeta}_{12}} \dot\rho \, dz + \dot{\overline{\zeta}}_{12} \left(\rho(x, \overline{\zeta}_{12}) - \rho_4\right) \\ \int_0^h \dot\Sigma \, dz \\ \int_0^{\overline{\zeta}_{23}} \dot\Sigma \, dz + \dot{\overline{\zeta}}_{23}\, \Sigma(x, \overline{\zeta}_{23}) \\ \int_0^{\overline{\zeta}_{12}} \dot\Sigma \, dz + \dot{\overline{\zeta}}_{12}\, \Sigma(x, \overline{\zeta}_{12}) \end{pmatrix} . \tag{21}$$

Note that in this formula we have an explicit dependence on the variations $\dot{\overline{\zeta}}_{12}$ and $\dot{\overline{\zeta}}_{23}$. To express these quantities in terms of $\dot\rho$, which is needed to perform the abovementioned steps of the Poisson reduction, we can use the analogue of relations (14), that is

$$\dot\rho = \Sigma_{i=1}^3 (\rho_{i+1} - \rho_i)\dot\zeta_i \delta(z - \zeta_i) . \tag{22}$$

Integrating this with respect to $z$ on the relevant intervals $[0, h]$, $[0, \overline{\zeta}_{12}]$ and $[0, \overline{\zeta}_{23}]$ yields

$$\begin{aligned} \int_0^h \dot\rho \, dz &= (\rho_4 - \rho_3)\dot\zeta_3 + (\rho_3 - \rho_2)\dot\zeta_2 + (\rho_2 - \rho_1)\dot\zeta_1 , \\ \int_0^{\overline{\zeta}_{12}} \dot\rho \, dz &= (\rho_4 - \rho_3)\dot\zeta_3 + (\rho_3 - \rho_2)\dot\zeta_2 , \\ \int_0^{\overline{\zeta}_{23}} \dot\rho \, dz &= (\rho_4 - \rho_3)\dot\zeta_3 . \end{aligned} \tag{23}$$

Solving the linear system (23) with respect to the $\dot\zeta_k$'s, we can obtain $\dot{\overline{\zeta}}_{12}$ and $\dot{\overline{\zeta}}_{23}$ in terms of integrals of $\dot\rho$ along $z$, and thus trade Eq. (21) for

$$\pi_* \begin{pmatrix} \dot\rho \\ \dot\Sigma \end{pmatrix} = \begin{pmatrix} \int_0^h \dot\rho \, dz \\ c_1 \int_0^h \dot\rho \, dz + (1 + c_3 - c_2) \int_0^{\overline{\zeta}_{12}} \dot\rho dz - c_3 \int_0^{\overline{\zeta}_{23}} \dot\rho dz \\ c_2 \int_0^{\overline{\zeta}_{12}} \dot\rho dz + \left(\frac{1}{2} - c_2\right) \int_0^{\overline{\zeta}_{23}} \dot\rho dz \\ \int_0^h \dot\Sigma \, dz \\ \int_0^{\overline{\zeta}_{12}} \dot\Sigma \, dz \\ \int_0^{\overline{\zeta}_{23}} \dot\Sigma \, dz \end{pmatrix} , \tag{24}$$

where, for the sake of compactness, we use the notation

$$c_1 = \frac{1}{2}\frac{\rho_2 - \rho_4}{\rho_2 - \rho_1}, \qquad c_2 = \frac{1}{2}\frac{\rho_3 - \rho_4}{\rho_3 - \rho_2}, \qquad c_3 = \frac{1}{2}\frac{\rho_2 - \rho_4}{\rho_3 - \rho_2} . \tag{25}$$

We now have at our disposal all the elements to perform the Poisson reduction process.

**Step 1:** The construction of the lifted 1-form $(\beta_\rho, \beta_\Sigma)$ satisfying (19), i.e.,

$$\int_{-\infty}^{+\infty}\int_0^h (\dot{\rho}\,\beta_\rho + \dot{\Sigma}\,\beta_\Sigma)\,\mathrm{d}x\,\mathrm{d}z = \int_{-\infty}^{+\infty} \Sigma_{k=1}^{6}\alpha_S^k \pi_\star(\dot{\rho},\dot{\Sigma})^k \mathrm{d}x\,, \tag{26}$$

yields

$$\begin{aligned}\beta_\rho &= \alpha_S^1 + \left(c_1 + (1+c_3-c_1)\theta(\overline{\zeta}_{12}-z) - c_3\theta(\overline{\zeta}_{23}-z)\right)\alpha_S^2 + \\ &\quad \left(c_2\theta(\overline{\zeta}_{12}-z) + (\frac{1}{2}-c_2)\theta(\overline{\zeta}_{23}-z)\right)\alpha_S^3 \\ \beta_\Sigma &= \alpha_S^4 + \theta(\overline{\zeta}_{12}-z)\,\alpha_S^5 + \theta(\overline{\zeta}_{23}-z)\alpha_S^6\,.\end{aligned} \tag{27}$$

In this equation, Heaviside $\theta$'s appear and enable the computation of integrals from the bottom to the chosen isopycnals $\overline{\zeta}_{12}$ and $\overline{\zeta}_{12}$ along the full channel $[0, h]$.
**Step 2:** The computation of the vector fields $(Y_M^1, Y_M^2)$ from the relation

$$\begin{pmatrix} Y_M^{(1)} \\ Y_M^{(2)} \end{pmatrix} = J_B \cdot \begin{pmatrix} \beta_\rho \\ \beta_\Sigma \end{pmatrix} = \begin{pmatrix} 0 & \rho_x\partial_z - \rho_z\partial_x \\ \rho_x\partial_z - \rho_z\partial_x & \Sigma_x\partial_z - \Sigma_z\partial_x \end{pmatrix} \cdot \begin{pmatrix} \beta_\rho \\ \beta_\Sigma \end{pmatrix} \tag{28}$$

is greatly simplified by the specific dependence of the lifted 1-form $(\beta_\rho, \beta_\Sigma)$ of (27) on $z$ and and on the crucial fact that the inequalities

$$\zeta_3 < \frac{\zeta_3+\zeta_2}{2} = \overline{\zeta}_{23} < \zeta_2 < \overline{\zeta}_{12} = \frac{\zeta_1+\zeta_2}{2} < \zeta_1$$

hold in the strict sense, so that the terms $\rho_x\partial_z$ and $\Sigma_x\partial_z$ when acting on $(\beta_\rho, \beta_\Sigma)$ generate products of Dirac $\delta$'s supported at different points, which vanish *qua* generalized functions. Moreover,

$$\begin{aligned}\Sigma_z\cdot\partial_x(\beta_\Sigma) &= (\Sigma_{i=1}^3\sigma_i\delta'(z-\zeta_i))\left(\alpha_S^4 + \theta(\overline{\zeta}_{12}-z)\,\alpha_S^5 + \theta(\overline{\zeta}_{23}-z)\alpha_S^6\right)_x \\ &= (\Sigma_{i=1}^3\sigma_i\delta'(z-\zeta_i))\left(\alpha_{S,x}^4 + \theta(\overline{\zeta}_{12}-z)\,\alpha_{S,x}^5 + \theta(\overline{\zeta}_{23}-z)\alpha_{S,x}^6\right) \\ &\quad + (\Sigma_{i=1}^3\sigma_i\delta'(z-\zeta_i))\left(\delta(\overline{\zeta}_{12}-z)\overline{\zeta}_{12,x}\alpha_S^5 + \delta(\overline{\zeta}_{23}-z)\overline{\zeta}_{23,x}\alpha_S^6\right) \\ &= (\Sigma_{i=1}^3\sigma_i\delta'(z-\zeta_i))\left(\alpha_{S,x}^4 + \theta(\overline{\zeta}_{12}-z)\,\alpha_{S,x}^5 + \theta(\overline{\zeta}_{23}-z)\alpha_{S,x}^6\right)\,,\end{aligned} \tag{29}$$

still due to the above observation about the supports of the Dirac $\delta$'s. Denoting by $\Delta^{(2)}$ this term, we can write (28) as

$$Y_M^{(1)} = -\rho_z(\beta_\Sigma)_x\,, \quad Y_M^{(2)} = -\rho_z(\beta_\rho)_x - \Delta^{(2)}\,. \tag{30}$$

We obtain

$$
\begin{aligned}
Y_M^{(1)} &= \left(\sum_{k=1}^{3}(\rho_k-\rho_{k+1})\delta(z-\zeta_k)\right)\alpha^4_{S,x}\\
&+\left(\sum_{k=2}^{3}(\rho_k-\rho_{k+1})\delta(z-\zeta_k)\right)\alpha^5_{S,x}+(\rho_3-\rho_4)\delta(z-\zeta_3)\alpha^6_{S,x}\,,
\end{aligned}
\tag{31}
$$

as well as the more complicated formula for $Y_M^{(2)}$,

$$
Y_M^{(2)} = (\sum_{i=1}^{3}(\rho_i-\rho_{i+1})\delta(z-\zeta_i))\,(\alpha^1_{S,x}+K_2\alpha^2_{S,x}+K_3\alpha^3_{S,x})-\Delta^{(2)}\,,
\tag{32}
$$

where

$$
\begin{aligned}
K_2 &= c_1+(1+c_3-c_1)\theta(\overline{\zeta}_{12}-z)-c_3\theta(\overline{\zeta}_{23}-z)\\
K_3 &= c_2\theta(\overline{\zeta}_{12}-z)+\left(\frac{1}{2}-c_2\right)\theta(\overline{\zeta}_{23}-z)\,.
\end{aligned}
\tag{33}
$$

**Step 3:** The computation of the push-forward under the map $\pi_*$ of the vector field $(Y_M^{(1)}, Y_M^{(2)})$, to obtain the six-component vector field $(\dot{\xi}_k, \dot{\tau}_k)$ on $\mathcal{S}$ is a direct but tedious task. Thanks to the explicit expressions (25) and (33), substituting in (24) and noticing that, due to the presence of the $z$-derivatives of the Dirac $\delta$, $\Delta^{(2)}$ is in the kernel of $\pi_*$, yields

$$
\begin{aligned}
\dot{\xi}_1 &= \alpha^4_{S,x}(\rho_1-\rho_4)+\alpha^5_{S,x}(\rho_2-\rho_4)+\alpha^6_{S,x}(\rho_3-\rho_4)\\
\dot{\xi}_2 &= \frac{1}{2}(\rho_2-\rho_4)\alpha^5_{S,x}+(\rho_3-\rho_4)\alpha^6_{S,x}\\
\dot{\xi}_3 &= \frac{1}{2}(\rho_3-\rho_4)\alpha^6_{S,x}\\
\dot{\sigma}_1 &= (\rho_1-\rho_4)\alpha^1_{S,x}\\
\dot{\sigma}_2 &= (\rho_2-\rho_4)\alpha^1_{S,x}+\frac{1}{2}(\rho_2-\rho_4)\alpha^2_{S,x}\\
\dot{\sigma}_3 &= (\rho_3-\rho_4)\alpha^1_{S,x}+(\rho_3-\rho_4)\alpha^2_{S,x}+\frac{1}{2}(\rho_3-\rho_4)\alpha^3_{S,x}\,.
\end{aligned}
\tag{34}
$$

Thus, the Poisson tensor $P$ on the manifold $\mathcal{S}$ in the coordinates $(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3)$ becomes

$$
P=\begin{pmatrix}0 & A\\ A^T & 0\end{pmatrix}\partial_x\,,\ \text{where } A=\begin{pmatrix}\rho_1-\rho_4 & \rho_2-\rho_4 & \rho_3-\rho_4\\ 0 & \dfrac{\rho_2-\rho_4}{2} & \rho_3-\rho_4\\ 0 & 0 & \dfrac{\rho_3-\rho_4}{2}\end{pmatrix}.
$$

Recalling relations (18), a straightforward computation shows that in the coordinates $(\zeta_1, \zeta_2, \zeta_3, \sigma_1, \sigma_2, \sigma_3)$ the reduced Poisson operator acquires the particularly simple form

$$P = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \partial_x \, . \tag{35}$$

**Remark 1** According to the terminology favored by the Russian school, for Hamiltonian quasi-linear systems of PDEs the coordinates $(\xi_l, \tau_l)$ and, *a fortiori*, the coordinates $(\zeta_l, \sigma_l)$, are "flat" coordinates for the system. In view of the particularly simple form of the Poisson tensor (35), the latter set could be called a system of flat *Darboux* coordinates.

**Remark 2** In [4] we conjectured that in the $n$-layered case, with a stratification given by densities $\rho_1 < \rho_2 < \cdots < \rho_n$ and interfaces $\zeta_1 > \zeta_2 > \cdots > \zeta_{n-1}$, a procedure yielding a natural Hamiltonian formulation for the averaged problem was to consider intervals

$$I_1 = [0, h], \; I_2 = \left[0, \frac{\zeta_1 + \zeta_2}{2}\right], I_3 = \left[0, \frac{\zeta_2 + \zeta_3}{2}\right], \; \ldots, \; I_n = \left[0, \frac{\zeta_{n-2} + \zeta_{n-1}}{2}\right]. \tag{36}$$

We explicitly proved it here for $n = 4$, together with the conjecture that the quantities

$$(\zeta_1, \zeta_2, \zeta_3, \sigma_1, \sigma_2, \sigma_3), \tag{37}$$

where $\sigma_k = \rho_{k+1}\overline{u}_{k+1} - \rho_k\overline{u}_k$, are flat Darboux coordinates for the reduced Poisson structure.

## 4 The Reduced Hamiltonian Under the Boussinesq Approximation

The energy of the $2D$ fluid in the channel is just the sum of the kinetic and potential energy,

$$H = \int_{-\infty}^{+\infty} \int_0^h \frac{\rho}{2} \left(u^2 + w^2\right) \mathrm{d}x \, \mathrm{d}z + \int_{-\infty}^{+\infty} \int_0^h g(\rho - \rho_0) z \, \mathrm{d}x \, \mathrm{d}z \, , \tag{38}$$

where $\rho_0$ is the reference density fixed by the far field constant values of the layers' thicknesses. In our case we have $\rho_0 = \sum_{i=1}^4 \rho_i \eta_i^{(\infty)}$, where $\eta_i^{(\infty)}$ are the asymptotic values of the $\eta_i$'s as $|x| \to \infty$.

The potential energy is thus readily reduced, using the first of (8), to

$$U = \int_{-\infty}^{+\infty} \frac{1}{2} \left( g\,(\rho_2 - \rho_1)\,{\zeta_1}^2 + g\,(\rho_3 - \rho_2)\,{\zeta_2}^2 + g\,(\rho_4 - \rho_3)\,\zeta_3^2 \right)\, \mathrm{d}x + U_\Delta\,, \tag{39}$$

where $U_\Delta$ contains constant and linear in the $\zeta_k$'s terms, which ensure the convergence of the integral, but that do not affect the equations of motion in view of the form (35) of the Poisson tensor.

To obtain the reduced kinetic energy density, we use the fact that at order $O(\epsilon^2)$ we can disregard the vertical velocity $w$, and trade the horizontal velocities with their layer-averaged means. Thus the $x$-density is computed as

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\left( \int_0^{\zeta_3} \rho_4 \overline{u}_4^2\, \mathrm{d}z + \int_{\zeta_3}^{\zeta_2} \rho_3 \overline{u}_3^2\, \mathrm{d}z + \int_{\zeta_2}^{\zeta_1} \rho_2 \overline{u}_2^2\, \mathrm{d}z + \int_{\zeta_1}^{h} \rho_1 \overline{u}_1^2\, \mathrm{d}z \right) \\ &= \frac{1}{2}\left( \rho_4 \zeta_3 \overline{u}_4^2 + \rho_3(\zeta_2 - \zeta_3)\overline{u}_3^2 + \rho_2(\zeta_1 - \zeta_2)\overline{u}_2^2 + \rho_1(h - \zeta_1)\overline{u}_1^2 \right)\,. \end{aligned} \tag{40}$$

The so-called Boussinesq approximation consists of the double scaling limit

$$\rho_i \to \bar{\rho},\ \ i = 1, \ldots, 4, \quad g \to \infty \text{ with } g(\rho_{j+1} - \rho_j) \text{ finite},\ \ j = 1, 2, 3, \tag{41}$$

where

$$\bar{\rho} = \frac{1}{4} \sum_{i=1}^{4} \rho_i$$

denotes the average density. This approximation then consists of neglecting density differences in the inertia terms of stratified Euler fluids, while retaining these differences in the buoyancy terms, owing to the relative magnitude of gravity forces with respect to those from inertia. This results in the Boussinesq energy density

$$\begin{aligned} \mathcal{E} &= \frac{\bar{\rho}}{2}\left( \zeta_3 \overline{u}_4^2 + (\zeta_2 - \zeta_3)\overline{u}_3^2 + (\zeta_1 - \zeta_2)\overline{u}_2^2 + (h - \zeta_1)\overline{u}_1^2 \right) \\ &\quad + \frac{1}{2}\left( g\,(\rho_2 - \rho_1)\,\zeta_1^2 + g\,(\rho_3 - \rho_2)\,\zeta_2^2 + g\,(\rho_4 - \rho_3)\,\zeta_3^2 \right)\,. \end{aligned} \tag{42}$$

To express this energy in terms of the Hamiltonian variables $(\zeta_i, \sigma_i)$, $i = 1, 2, 3$, we use the dynamical constraint

$$(h - \zeta_1)\overline{u}_1 + (\zeta_1 - \zeta_2)\overline{u}_2 + (\zeta_2 - \zeta_3)\overline{u}_3 + \zeta_3 \overline{u}_4 = 0\,, \tag{43}$$

as well as the definitions (13) that, in the Boussinesq approximation, are turned into

$$\sigma_k = \bar{\rho}(\overline{u}_{k+1} - \overline{u}_k)\,. \tag{44}$$

We get

$$\begin{aligned}
\bar{u}_1 &= -\frac{\zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3\sigma_3}{h\bar{\rho}},\\
\bar{u}_2 &= -\frac{\zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3\sigma_3 - h\sigma_1}{h\bar{\rho}},\\
\bar{u}_3 &= -\frac{\zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3\sigma_3 - h\sigma_1 - h\sigma_2}{h\bar{\rho}},\\
\bar{u}_4 &= -\frac{\zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3\sigma_3 - h\sigma_1 - h\sigma_2 - h\sigma_3}{h\bar{\rho}}\,.
\end{aligned} \tag{45}$$

Hence, from (42), the Hamiltonian functional acquires its final form in the Boussinesq approximation as

$$\begin{aligned}
H_B = \int_{\mathbb{R}} \Bigg( &\frac{1}{2\,h\bar{\rho}} \left(\sigma_1^2\zeta_1\,(h-\zeta_1) + \sigma_2^2\,(h-\zeta_2)\,\zeta_2 + \sigma_3^2\,(h-\zeta_3)\,\zeta_3 + \right.\\
&2\sigma_1\sigma_2\zeta_2\,(h-\zeta_1) + 2\sigma_1\sigma_3\zeta_3\,(h-\zeta_1) + 2\sigma_2\sigma_3\zeta_3\,(h-\zeta_2)) + \\
&\frac{g}{2}\left((\rho_2-\rho_1)\zeta_1^2 + (\rho_3-\rho_2)\zeta_2^2 + (\rho_4-\rho_3)\zeta_3^2\right)\Bigg)\,\mathrm{d}x\,.
\end{aligned} \tag{46}$$

Thanks to the simple form of the Poisson tensor (35), the ensuing equations of motion can be written as the conservation laws

$$\begin{aligned}
&\zeta_{1t} + \left(\frac{\sigma_1\zeta_1\,(h-\zeta_1)}{h\rho} + \frac{\sigma_3\zeta_3\,(h-\zeta_1)}{h\rho} + \frac{\sigma_2\zeta_2\,(h-\zeta_1)}{h\rho}\right)_x = 0\\
&\zeta_{2t} + \left(\frac{\sigma_2\,(h-\zeta_2)\,\zeta_2}{h\rho} + \frac{\sigma_3\,(h-\zeta_2)\,\zeta_3}{h\rho} + \frac{\sigma_1\zeta_2\,(h-\zeta_1)}{h\rho}\right)_x = 0\\
&\zeta_{3t} + \left(\frac{\sigma_3\,(h-\zeta_3)\,\zeta_3}{h\rho} + \frac{\sigma_2\,(h-\zeta_2)\,\zeta_3}{h\rho} + \frac{\sigma_1\zeta_3\,(h-\zeta_1)}{h\rho}\right)_x = 0\\
&\sigma_{1t} + \left(\frac{(h-2\zeta_1)\,\sigma_1^2}{2h\rho} - \frac{\sigma_1\sigma_2\zeta_2}{h\rho} - \frac{\sigma_1\sigma_3\zeta_3}{h\rho} + g\,(\rho_2-\rho_1)\,\zeta_1\right)_x = 0\\
&\sigma_{2t} + \left(\frac{(h-2\zeta_2)\,\sigma_2^2}{2h\rho} - \frac{\sigma_2\sigma_3\zeta_3}{h\rho} + \frac{\sigma_1\sigma_2\,(h-\zeta_1)}{h\rho} + g\,(\rho_3-\rho_2)\,\zeta_2\right)_x = 0\\
&\sigma_{3t} + \left(\frac{(h-2\zeta_3)\,\sigma_3^2}{2h\rho} + \frac{\sigma_2\sigma_3\,(h-\zeta_2)}{h\rho} + \frac{\sigma_1\sigma_3\,(h-\zeta_1)}{h\rho} + g\,(\rho_4-\rho_3)\,\zeta_3\right)_x = 0\,.
\end{aligned} \tag{47}$$

The Hamiltonian formalism easily shows the existence of the eight conserved quantities

$$\begin{aligned}
&Z_j = \int_{-\infty}^{+\infty} \zeta_j\,\mathrm{d}x, \qquad S_j = \int_{-\infty}^{+\infty} \sigma_j\,\mathrm{d}x, \quad j = 1,2,3\ ,\\
&K = \int_{-\infty}^{+\infty} \sum_{k=1}^{3} \zeta_k\sigma_k\,\mathrm{d}x \quad \text{and } H_B \text{ given by (4.9).}
\end{aligned} \tag{48}$$

**Remark 3** The first six quantities are Casimir functionals for the Darboux Poisson tensor (35), while the seventh one, $K$, is the generator of $x$-translations. Note that, formulas (45) imply that the total linear momenta of the individual layers are conserved quantities. This is consistent with the fact that the dispersionless limit of the $N$-layer equations are conservation laws for the averaged momenta, and no pressure imbalances can arise in the Boussinesq approximation [2].

**Remark 4** The steps leading to the computation of the effective Hamiltonian (46) can be performed also by dropping the assumptions (41) of the Boussinesq approximation. In this case, the kinetic energy acquires a non trivial rational dependence on the density differences $\rho_i - \rho_{i+1}$, and the equations of motion become much more complicated (as already seen in the 2 and 3-layer cases). However, they are still Hamiltonian equations of motion that preserve, together with their Hamiltonian, the quantities $Z_j,\ S_j,\ j = 1, 2, 3$ and the generator of $x$-translations $K$ of Eq. (48). Note that, as shown in [2] and further discussed in [4], once beyond the Boussinesq approximation pressure imbalances can appear. Hence the individual layer momenta are no longer conserved quantities and $K$ does not even coincide with the total horizontal momentum.

## 5 Symmetric Solutions

Symmetric solutions of the three-layer configurations were ingeniously found in [11] by a direct inspection of the equations of motion (written in velocity – thickness coordinates). They exist provided a certain relation is enforced on the density differences of the individual layers, and were interpreted in [4] as the fixed point of a suitable canonical involution of the phase space of the 3-layer model.

Here we shall adopt the latter point of view, and identify an involution of the phase space of the 4-layer model above that leads to the existence of a family of symmetric solutions. First, we focus on the kinetic energy part of the Boussinesq model (46),

$$\mathcal{T}_B = \frac{1}{2\,h\bar{\rho}}\Big(\sigma_1^2\zeta_1\,(h-\zeta_1) + \sigma_2^2\,(h-\zeta_2)\,\zeta_2 + \sigma_3^2\,(h-\zeta_3)\,\zeta_3 + \\ 2\sigma_1\sigma_2\zeta_2\,(h-\zeta_1) + 2\sigma_1\sigma_3\zeta_3\,(h-\zeta_1) + 2\sigma_2\sigma_3\,(h-\zeta_2)\,\zeta_3\Big). \tag{49}$$

This expression is clearly invariant under the involutive map

$$\zeta_1 \to h-\zeta_3,\quad \zeta_2 \to h-\zeta_2,\quad \zeta_3 \to h-\zeta_1,\quad \sigma_1 \to -\sigma_3,\quad \sigma_2 \to -\sigma_2,\quad \sigma_3 \to -\sigma_1\,. \tag{50}$$

If we assume that the densities $\rho_k$ fulfill the relations

$$\rho_4 - \rho_3 = \rho_2 - \rho_1 \equiv \rho_{\scriptscriptstyle\triangle}\,, \tag{51}$$

the Hamiltonian density (46) is invariant as well, up to the addition of linear terms in the $\zeta$'s, that is, up to constant terms and Casimir densities of the Poisson tensor $P$ of (35) which do not affect the equations of motion. A straightforward computation shows that the Poisson tensor (35) is left invariant by the above involution. Hence, the manifold $\mathcal{F}$ of fixed points of the involution (50) is invariant under the Hamiltonian flow (47).

The above statement can be cast in a more geometrical light. Suppose that we are given a Poisson manifold $(M, P)$ with Hamilton equations written generically as

$$z_t = P\, dH\,, \tag{52}$$

and suppose that $z \to \varphi(z)$ is an involution preserving $P$, i.e.,

(i) $\varphi \circ \varphi = \mathrm{Id}$
(ii) $\varphi_*\, P\, \varphi^* = P$, where $\varphi_*$ is the (Fréchet) derivative of $\varphi$, and $\varphi^*$ is its pull-back (from the linear algebra perspective, the adjoint map).

Then

$$\varphi(z)_t = \varphi_* z_t = \varphi_* P\, dH = \varphi_* P \varphi^* \varphi^* dH = P \varphi^* dH = P d\varphi^* H\,. \tag{53}$$

Hence, if $z$ satisfies $\varphi(z) = z$ we have $\varphi(z)_t - z_t = 0$ so that initial data fixed by the involution $\varphi$ remain on the invariant submanifold during the time evolution. In our case, the invariant manifold can be explicitly described as the submanifold of $\mathcal{S}$ characterized by the constraints (see Fig. 3)

$$\zeta_1 + \zeta_3 - h = 0\,, \quad \zeta_2 - \frac{h}{2} = 0\,, \quad \sigma_1 + \sigma_3 = 0\,, \quad \sigma_2 = 0\,, \tag{54}$$

and is parametrized by two of the remaining variables, for instance the two quantities

$$\sigma \equiv \sigma_3\,, \quad \zeta \equiv \zeta_3. \tag{55}$$

The reduced equations of motion on $\mathcal{F}$ in these variables are

$$\begin{cases} \zeta_t - \dfrac{2(\zeta^2\sigma)_x}{h\bar{\rho}} + \dfrac{(\zeta\sigma)_x}{\bar{\rho}} = 0 \\ \sigma_t + \dfrac{1}{2}\dfrac{((h-4\zeta)\sigma^2)_x}{h\bar{\rho}} + 2g\rho_{\scriptscriptstyle\triangle}\zeta\zeta_x = 0 \end{cases}, \tag{56}$$

while the restriction of the Hamiltonian (46) is

$$H_{\mathcal{F}} = \int_{\mathbb{R}} \left( \frac{\zeta\,(h-2\zeta)\,\sigma^2}{h\bar{\rho}} + g\rho_{\scriptscriptstyle\triangle}\zeta^2 \right) \mathrm{d}x\,. \tag{57}$$

One can readily check that Eq. (56) are the Hamiltonian equations of motion.

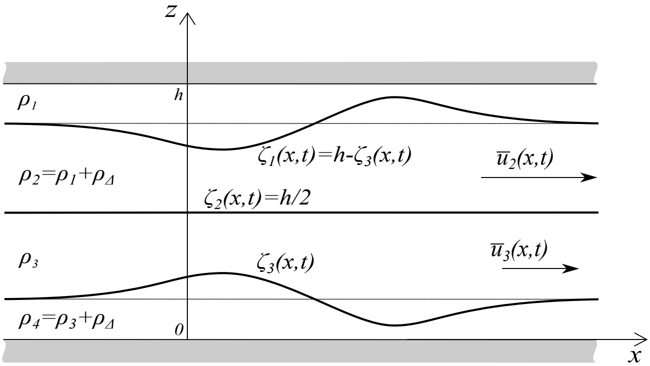


**Fig. 3** Example of a symmetric solution

$$\begin{pmatrix} \zeta_t \\ \sigma_t \end{pmatrix} = P_{\mathcal{F}} \begin{pmatrix} \delta_\zeta H_{\mathcal{F}} \\ \delta_\sigma H_{\mathcal{F}} \end{pmatrix} \text{ with } P_{\mathcal{F}} = \begin{pmatrix} 0 & -\frac{1}{2}\partial_x \\ -\frac{1}{2}\partial_x & 0 \end{pmatrix} . \tag{58}$$

The appearance of the factor $1/2$ in the expression of $P_{\mathcal{F}}$ is readily explained within Dirac's theory of constrained Hamiltonian systems. Indeed, if we consider the constraints (54), we notice that, renaming the constraint densities as

$$\Phi_1 = \zeta_1 + \zeta_3 - h\,, \quad \Phi_2 = \zeta_2 - h/2\,, \quad \Phi_3 = \sigma_1 + \sigma_3\,, \quad \Phi_4 = \sigma_2\,, \tag{59}$$

the sixtuple $(\zeta = \zeta_3, \sigma = \sigma_3, \Phi_1, \ldots, \Phi_4)$ is clearly a set of coordinates. The Poisson tensor in these coordinates is given by the block matrix

$$P = -\partial_x \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} , \tag{60}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{61}$$

In this formalism, Dirac's formula [12] for the $2 \times 2$ reduced tensor $P^D$ with respect to the pair of coordinates $(\zeta, \sigma)$ on the constrained manifold is

$$P^D = \left(A - B^T \cdot C^{-1} \cdot B\right) \partial_x\,, \tag{62}$$

by which we recover the tensor $P_{\mathcal{F}}$ of (58).