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K. Ratnavelu *Editors*

Discrete Mathematics and Mathematical Modelling in the Digital Era

ICDM3DE-2023, Gandhigram, India, March 23–25

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Editors

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Editors
P. Balasubramaniam
Department of Mathematics
Gandhigram Rural Institute
Dindigul, Tamil Nadu, India

P. Raveendran
School of Information Technology
Monash University
Selangor, Malaysia

G. Mahadevan
Department of Mathematics
Gandhigram Rural Institute
Dindigul, Tamil Nadu, India

K. Ratnavelu
Institute of CS and Digital Innovation
UCSI University
Kuala Lumpur, Malaysia

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Preface

Discrete Mathematics serves as the foundational framework for numerous concepts and technologies that drive the digital age, impacting aspects ranging from hardware, and algorithm design to the security of digital communication and the analysis of extensive datasets. Its principles are deeply woven into the fabric of contemporary computer science and technology.

Within the digital era, Mathematical Modeling has emerged as an essential tool cutting across diverse disciplines, transforming the way we comprehend, dissect, and resolve intricate problems. In a landscape where copious amounts of data are generated, and computational power is increasingly accessible, Mathematical Modeling plays a pivotal role in translating real-world phenomena into Mathematical expressions for simulation, analysis, and prediction. The integration of Discrete Mathematics and Mathematical Modeling is fundamental to the digital epoch, providing both theoretical underpinnings and practical foundations for a broad spectrum of applications in technology and computer science. These disciplines facilitate the creation of efficient algorithms, secure communication protocols, network designs, and the optimization of digital systems, significantly contributing to the evolution of the digital realm.

The 9th International Conference on Discrete Mathematics and Mathematical Modeling in Digital Era (ICDMMMDE-2023) was organized by the Department of Mathematics, The Gandhigram Rural Institute-Deemed to be University on March 23–25, 2023. This conference is intended to provide a common forum for budding researchers, scientists, and engineers throughout the world to share their ideas and recent developments. Further, the conference plays a vital role as far as the application part is concerned and many real-life problems can be solved with proper investigations of appropriate Mathematical Modeling of realistic situations. The scope of ICDMMMDE-2023 is to bring all the modernistic researchers from various fields to discuss the latest developments in Mathematics.

In total, 192 research articles were presented at ICDMMMDE-2023 which were reviewed by experts from various fields from both India and abroad. Based on the reviewer's comments 16 research articles were accepted for publication in this volume. A total of 77 experts reviewed the presented articles all around the world.

We express our greatest gratitude for spending their valuable time and remarkable suggestions.

We extend our hearty thanks to the national funding agencies for their grand support in the successful completion of ICDMMMDE-2023.

1. University Grants Commission (UGC), Government of India, New Delhi.
2. Council of Scientific and Industrial Research (CSIR), New Delhi, India.

We would like to express our sincere thanks to Dr. Shamim Ahmad, Senior Editor, and all the Editors, Springer India, and his team at Springer for accepting our proposal to publish the papers in the Springer series.

Dindigul, India — P. Balasubramaniam
Selangor, Malaysia — P. Raveendran
Dindigul, India — G. Mahadevan
Kuala Lumpur, Malaysia — K. Ratnavelu
November 2023

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We wish to convey our heartfelt gratitude to the Vice-Chancellor and the Registrar of The Gandhigram Rural Institute-Deemed to be University, Gandhigram, Tamil Nadu, India, for their gracious acceptance and encouragement. Our gratitude extends to the faculty members and research scholars of the Department of Mathematics, as well as the administrative staff of GRI-(DTBU), Gandhigram. We also extend profound thanks to the Honorary Chairs, Co-chairs, Technical Program Chairs, Organizing Chair, and all committee members who collaborated tirelessly to ensure the success of ICDMMMDE-2023.

Special thanks are due to the keynote speakers who graciously accepted our invitations. Specifically, we acknowledge Dr. K. Ratnavelu (UCSI University, Malaysia), Dr. Wan Ainun Mior Othwaman (University of Malaya, Malaysia), Dr. Meng Joo Er (Dalian Maritime University, China), Dr. C. P. Lim (Deakin University, Australia), Dr. S. Sanjeewa Nishantha Perara (University of Colombo, Sri Lanka), Dr. Raveendran (Monash University, Malaysia), Dr. Grienggrai Rajchakit (Maejo University, Thailand), Dr. Fathalla A. Rihan (United Arab Emirates University and Helwan University, UAE), Dr. C. Sivagnanam (University of Technology and Applied Sciences-Sur Oman), Dr. S. Ponnusamy (IITM-Chennai), Dr. K. Balachandran (Bharathiar University, Coimbatore), Dr. B. V. Appa Rao (KL University, Andhra Pradesh), Dr. R. Sakthivel (Bharathiar University, Coimbatore), Dr. Raju K. George (IIST-Trivandrum), Dr. S. Marshal Antoni (Anna University, Coimbatore), Dr. Rajeswari Seshadri (Pondicherry University, Pondicherry), Dr. Suresh Singh (University of Kerala, Kerala), Dr. K. B. Rajesh (Chikkanna Government Arts College, Tirupur), Dr. A. Selvam (VHNSN College, Virudhunagar), Dr. N. Sukavanam (IIT-Roorkee), Dr. T. Asir (Pondicherry University, Pondicherry), Dr. M. Seenivasan (Annamalai University, Chidambaram) for delivering plenary talks and contributing to the grand success of ICDMMMDE-2023.

A total of 77 experts from around the world dedicated their time to review the presented papers on various topics. We express our deepest gratitude for their invaluable contributions to the review process, which significantly contributed to the success of ICDMMMDE-2023. We also thank the funding agencies for their generous support, including the University Grants Commission (UGC), the Government of India, New Delhi, and the Council of Scientific and Industrial Research (CSIR), New Delhi, India.

Finally, we extend our sincere thanks to Dr. Shamim Ahmad, Senior Editor, and all the Editors at Springer India, along with their team, for accepting our proposal to publish the conference papers in the Springer series.

Salient Features

1. The primary objective of the conference proceedings is to establish an inclusive platform fostering advancements in Discrete Mathematics and Mathematical Modeling in the Digital Era.
2. Within this compilation of conference proceedings, a diverse array of disciplines are explored including the development of efficient algorithms, secure communication protocols, network designs, and the optimization of digital systems. These collective efforts significantly shape the evolution of the digital landscape.
3. The integration of Discrete Mathematics and Mathematical Modeling holds pivotal importance in the digital age, offering both theoretical foundations and practical frameworks for a wide range of applications in technology and computer science.
4. The editorial team overseeing these conference proceedings comprises experts from various fields, including neurobiology, computer science, engineering, networks, and physics.
5. A rigorous evaluation of original research articles was conducted with a focus on their potential lasting value, a criterion that underscores the strength of the conference proceedings.
6. The articles that appeared in this proceedings conform to the latest developments in the fields of image processing, controllability, stability, graph theory, topology, queuing theory, and analysis.

Contents

About the Editors

P. Balasubramaniam received a Ph.D. in Mathematics in 1994 in the area of control theory at Bharathiar University, Coimbatore, Tamil Nadu, INDIA, and a D.Sc. in the year 2022 in the same area at Gandhigram Rural Institute-Deemed to be University, Gandhigram, Dindigul, Tamil Nadu, India. His research interest includes stochastic fractional systems, image processing, and stability analysis. He received awards including TANSA-2005, Senior Scientist-2018, and the Mid-Career Award in 2019, and is recognized among the top 2% of scientists by Stanford University in the field of Artificial Intelligence and Image Processing, USA, from 2020 to till date. He is one of the best scientists in the field of Mathematics as well as in the field of Electronics and Electrical Engineering respectively with the national rank of 3 and 7.

P. Raveendran obtained both his bachelor's and master's degrees in electrical engineering from South Dakota State University, USA in 1984 and 1985 respectively. He then worked briefly as a System Designer at Daktronics, South Dakota, before joining as a lecturer at the Department of Electrical Engineering, University of Malaya (UM) in 1986. In 1994, he obtained his Doctor of Engineering for his research in image processing and neural networks from the University of Tokushima, Japan. He became a Professor in 2003 and achieved the highest grade for a Professor in 2014. He retired from UM in 2018 and later was employed as an honorary Professor at the Department of Electrical Engineering till 2024. He was also employed as a Professor at the Institute of Computer Science and Digital Innovation, UCSI University from 2019 to 2021. Currently, he is working as a Professor at the School of Information Technology, Monash University Malaysia. His research and teaching interests have been in the areas of Image and Video Analysis, Analysis and Applications of EEG signals, machine learning algorithms, and Soft Computing.

G. Mahadevan received a Ph.D. in Mathematics in 2005 in the area of Graph Theory at Manonmaniam Sundaranar University, Tirunelveli, India, and having 29 years of Teaching and Research Experience in various Colleges and Universities including Head of the Department of Mathematics at Anna University of Technology, Tirunelveli. His research area is Discrete Mathematics especially Graph Theory, Networks, and Domination theory. He has published more than 170 Research articles in Many International and National Journals and served as Guest Editor for special issues of some of the reputed Journals. He received several National and International awards. He is the Associated Editor of the *International Journal of Applied Graph Theory* and served as a reviewer for many International/National Journals. He has written a book *A Text Book of Calculus* and written three books on Engineering Mathematics. He delivered plenty of invited talks as a resource person at various International/National Conferences organized by many Universities and colleges in India and abroad.

K. Ratnavelu is the founder and Director of the Institute of Computer Science and Digital Innovations at UCSI University, Malaysia. He served as Deputy Dean, Science and Deputy Vice-Chancellor (Development), and Head of UM Strategic Planning Unit, University of Malaya. He was involved in the HIR Program at the University of Malaya. He received a Ph.D. in atomic physics from Flinders University, Australia, with a Flinders Scholarship in 1990. He is a Council Member of the Association of Asia-Pacific Physical Societies (2016–2019, 2020–2023). He received the 19th Fellow of Persatuan-Sains-Matematik Malaysia (2018) and the Malaysian Toray Science Foundation Science and Technology Award (2004). His research interests are theoretical atomic collision processes, social networks, and stability theory.

Part I
Mathematical Modelling

Chapter 1
On the Approximate Controllability of Second-Order Hilfer Fractional Integro-Differential Equations via Measure of Non-compactness

B. Ram Kumar, P. Balasubramaniam, and K. Ratnavelu

Abstract This manuscript investigates the approximate controllability results for a class of second-order Hilfer fractional integro-differential equations (HFIDEs). A new set of appropriate mild solutions has been derived. Further, the existence of mild solutions for the proposed system has been verified using Mönch fixed point theorem (MFPT) and derived sufficient condition utilizing the measure of non-compactness (MNC). The approximate controllability results of the proposed system have been established by presuming that the associated linear system is an approximate control system. Finally, for the understanding, a numerical example is included.

Keywords Approximate controllability · Hilfer fractional derivative · Mönch fixed point theorem

AMS Subject classification 34A08 · 45J05 · 93B05 · 47H08

1.1 Literature Review

Fractional calculus and its applicability have attracted the attention of many scientists and researchers in subjects like engineering, chemistry, physics, economics, and others. It is also considered an excellent tool for describing inherited features of different materials and diffusion phenomena. Fractional differential equations (FDEs) are the critical tool for modeling and analyzing many physical processes; it is a generalized form of integer-order differential equations; see [1, 2].

B. Ram Kumar · P. Balasubramaniam (✉)
Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram 624302, Tamil Nadu, India
e-mail: balugru@gmail.com

K. Ratnavelu
Institute of Computer Science and Digital Innovation (ICSDI), UCSI University, Kuala Lumpur, Malaysia
e-mail: kurunathan@ucsiuniversity.edu.my

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Although numerous fractional derivatives have been published in the literature, the Riemann–Liouville derivative (RLD) and Caputo derivative (CD) are preferred in applications due to their greater flexibility. Hilfer recently proposed a generalized Hilfer fractional derivative (HFD) that interpolates smoothly between RLD and CD [3, 4]. In recent years, researchers have been interested in Hilfer fractional differential equations in both finite- and infinite-dimensional frameworks.

In 1960, Kalman introduced the concept of controllability; gradually, it was generalized to infinite-dimensional spaces. It is a key concept in control theory and studies the possibilities for steering a system to a specified state using an effective control function. Numerous authors investigated the controllability of Hilfer fractional systems in recent years [5–10]. In general, infinite-dimensional systems are not controllable in most of the cases.

For infinite-dimensional spaces, approximate controllability is explicitly investigated to overcome the issue. In the approximate controllability evaluation of a system, the system is steered from the initial position to an arbitrarily small neighborhood of the final position. As a result, a reduced basic concept of approximative controllability has been provided for applications. Many researchers have studied solvability and approximate controllability results for Hilfer fractional evolution equations [11–14].

Studying second-order HFIDEs is crucial due to their ability to accurately model complex phenomena that defy traditional differential equations. These systems find applications in diverse fields, including physics, biology, materials science, and engineering. They are essential for describing anomalous diffusion, viscoelastic behavior in materials, control theory applications, and signal processing, as well as improving our understanding of biological systems, geophysical processes, and financial modeling. Furthermore, this study fosters innovation and scientific advancements, enabling precise modeling and problem-solving in various domains.

Hilfer fractional differential systems are a powerful mathematical tool to comprehend and address real-world process intricacies, ultimately leading to improved technologies and scientific insights. To this extent, the approximate controllability of second-order HFIDEs has still not been examined. Inspired by this aspect, this present work aims to verify the existence and approximate controllability results of the stated second-order problem. The significance of the derived key result is stated as follows:

- This study is the first attempt to analyze the solvability of HFIDEs.
- A new set of mild solutions has been derived for HFIDEs.
- The continuity and bonding properties of the control function are explicitly examined.
- The existence of a mild solution is efficiently proved by utilizing MFPT.
- The Hausdorff MNC property is used to verify the relatively compact conditions.
- An example is provided to validate the theoretical result.

The manuscript is structured as follows. Section 1.2 presents the system description of HFIDEs. The essential terminology, function spaces, definitions, and required early results are given in Sect. 1.3. In Sect. 1.4, we derive the respective characteristics of solution operators and present an appropriate definition of mild solutions to the problem. Through Sect. 1.5, we prove the existence results of the mild solutions of the problem via MFPT and investigate the approximate controllability of the proposed system. Finally, in Sect. 1.6, a numerical example is provided to enhance understanding.

1.2 System Description

This manuscript aims to extend the study of solvability and approximate controllability for a class of HFIDEs given by

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha,\beta} v(\zeta) = \mathcal{A}v(\zeta) + \mathcal{B}y(\zeta) + f(\zeta, v(\zeta)) + \int_0^\zeta a(\zeta,\aleph)g(\aleph, v(\aleph))d\aleph, \ \zeta \in \mathcal{J}_1, \\ \mathcal{I}_{0^+}^{\kappa} v(\zeta)\big|_{\zeta=0} = v_0, \ \frac{d}{d\zeta}\mathcal{I}_{0^+}^{\kappa} v(\zeta)\big|_{\zeta=0} = v_1. \end{cases} \tag{1.1}$$

where $\mathcal{D}_{0^+}^{\alpha,\beta}$ represents the HFD of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$ and $\mathcal{I}_{0^+}^{\kappa}$ represents the conventional fractional integral of order $\kappa := (1-\beta)(2-\alpha)$. Let $\mathfrak{H}$ and $\mathfrak{U}$ be real Hilbert spaces. We suppose that $\mathcal{A}$ is the infinitesimal generator which generates a strongly continuous uniformly bounded cosine family $\{\mathfrak{C}(\zeta)\}_{\zeta\in\mathbb{R}}$ in $\mathfrak{H}$; i.e., $\exists\ \mathfrak{M} \in \mathbb{R}^+ \ni$: $\|\mathfrak{C}(\zeta)\|_{\mathfrak{L}(\mathfrak{H})} \leq \mathfrak{M}$, $\zeta \in \mathbb{R}$. Let $\mathfrak{D}(\mathcal{A}) = \Big\{v \in \mathfrak{H} | \mathfrak{C}(\zeta)v \in \mathcal{C}^2(\mathbb{R}, \mathfrak{H})\Big\} \subset \mathfrak{H}$ be the domain of $\mathcal{A}$ with norm $\|v\|_{\mathfrak{D}(\mathcal{A})} = \|v\| + \|\mathcal{A}v\|$, $\forall v \in \mathfrak{D}(\mathcal{A})$. Clearly, $\mathcal{A}$ is closed and densely-defined in $\mathfrak{H}$.Take $\mathcal{J} := [0, b]$ and $\mathcal{J}_1 := (0\ , b]$. $\mathcal{B} \in \mathfrak{L}(\mathfrak{U}, \mathfrak{H})$ symbolizes a bounded linear operator with $\|\mathcal{B}\|_{\mathfrak{L}(\mathfrak{U},\mathfrak{H})} \leq \mathfrak{M}_{\mathcal{B}}$. The control function $y(\zeta)$ takes values in $\mathfrak{L}^2(\mathcal{J}, \mathfrak{U})$. The functions f and g will be defined with suitable conditions in the sequel. The integral kernel $a \in \mathcal{C}(\Delta, \mathbb{R}^+)$, where $\Delta = \{(\zeta, \aleph) : 0 \leq \aleph \leq \zeta \leq b\}$ with $\tilde{a} = \sup_{\zeta\in\mathcal{J}} \int_0^\zeta \|a(\zeta, \aleph)\|d\aleph$.

1.3 Basic Framework

1.3.1 Weighted Continuous Function Space

The space $\mathfrak{L}(a, b)$ of Lebesgue measurable functions $v(\zeta)$ on $[a, b]$ $(b > a)$ of $\mathbb{R}$ is defined by

$$\mathfrak{L}(a,b) = \Big\{v : \|v(\zeta)\|_{\mathfrak{L}(a,b)} = \int_a^b |v(\zeta)|d\zeta < \infty\Big\}.$$

Let $\mathfrak{AC}[a,b]$ denotes the space of absolute continuous functions $v(\zeta)$ on $[a,b]$. For $n\in\mathbb{N}$, we define the space $\mathfrak{AC}^n[a,b]$ of absolute continuous functions $v(\zeta)$ as follows,

$$\mathfrak{AC}^n[a,b]=\left\{v:[a,b]\to\mathbb{R}\middle|\frac{d^{n-1}v}{d\zeta^{n-1}}\in\mathfrak{AC}[a,b]\right\}.$$

Let $\mathcal{C}(\mathcal{J},\mathfrak{H})$ denotes the space of all continuous functions from $\mathcal{J}$ to $\mathfrak{H}$ satisfying $\sup_{\zeta\in\mathcal{J}}\|v(\zeta)\|<\infty$. Clearly, $\mathcal{C}(\mathcal{J},\mathfrak{H})$ is a Banach space with norm $\|v\|=\sup_{\zeta\in\mathcal{J}}\|v(\zeta)\|$.

To establish a suitable definition for the mild solution of problem (1.1), we construct the weighted space given below:

$$\mathfrak{Y}=\mathcal{C}_{(1-\beta)(2-\alpha)}(\mathcal{J},\mathfrak{H})=\left\{v(\zeta)\in\mathcal{C}(\mathcal{J}_1,\mathfrak{H})\middle|\zeta^{(1-\beta)(2-\alpha)}v(\zeta)\in\mathcal{C}(\mathcal{J},\mathfrak{H})\right\},$$

for $\alpha\in(1,2)$ and $\beta\in[0,1]$ with $\sup_{\zeta\in\mathcal{J}}\left\|\zeta^{(1-\beta)(2-\alpha)}v(\zeta)\right\|<\infty$.

Note that $\mathfrak{Y}$ is a Banach space with norm $\|v\|_{\mathfrak{Y}}=\sup_{\zeta\in\mathcal{J}}\left\|\zeta^{(1-\beta)(2-\alpha)}v(\zeta)\right\|$. For some $r\in\mathbb{R}^+$, define

$$\mathfrak{R}_r=\left\{v\in\mathfrak{Y}:\|v\|_{\mathfrak{Y}}\le r\right\}.$$

Clearly, $\mathfrak{R}_r$ is closed bounded and convex set in $\mathfrak{Y}$.

1.3.2 Generalized Hilfer Fractional Derivative

Now, we describe the generalized Hilfer fractional derivatives (GHFDs) and their particular versions. The GHFD has the property that it smoothly interpolates between the Riemann–Liouville derivative (RLD) and Caputo derivative (CD).

Definition 1.3.1 The fractional integral of order $\alpha>0$ for a function $v:\mathbb{R}^+\to\mathfrak{H}$ with right limit zero is depicted as

$$\mathcal{I}_{0^+}^{\alpha}v(\zeta)=\frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\aleph)^{\alpha-1}z(\aleph)d\aleph,\quad \zeta>0. \tag{1.2}$$

Definition 1.3.2 [4] The GHFD of order α and type β for a function $v:\mathbb{R}^+\to\mathfrak{H}$ with right limit zero is depicted as

$$\mathcal{D}_{0^+}^{\alpha,\beta}v(\zeta)=\mathcal{I}_{0^+}^{\beta(n-\alpha)}\frac{d^n}{d\zeta^n}\mathcal{I}_{0^+}^{(1-\beta)(n-\alpha)}v(\zeta),\ n-1\le\alpha\le n \text{ and } 0\le\beta\le 1\ n\in\mathbb{N}, \tag{1.3}$$

where $\frac{d^n}{d\zeta^n}$ is the classical nth-order derivative.

Definition 1.3.3 The RLD of a function $\upsilon : \mathbb{R}^+ \to \mathfrak{H}$ with right limit zero is depicted as

$$\mathcal{D}^{\alpha}_{0^+}\upsilon(\zeta) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{d\zeta^n}\int_0^{\zeta}\frac{\upsilon(\aleph)d\aleph}{(\zeta-\aleph)^{1+\alpha-n}},\ \zeta > 0,\ \alpha \in (n-1, n),\ n \in \mathbb{N}. \tag{1.4}$$

Definition 1.3.4 The CD of a function $\upsilon : \mathbb{R}^+ \to \mathfrak{H}$ with right limit zero is depicted as

$$\mathcal{D}^{\alpha}_{0^+}\upsilon(\zeta) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\zeta}\frac{d^n\upsilon(\aleph)}{d\aleph^n}\frac{1}{(\zeta-\aleph)^{1+\alpha-n}}d\aleph,\ \zeta > 0,\ \alpha \in (n-1, n),\ n \in \mathbb{N}. \tag{1.5}$$

Remark 1.3.1 • If $\beta = 0$, Definition 1.3.2 corresponds to Definition 1.3.3.
• If $\beta = 1$, Definition 1.3.2 corresponds to Definition 1.3.4.

Note 1 Definitions 1.3.1,1.3.2,1.3.3,1.3.4 are valid whenever the right-hand side exists.

1.3.3 Measure of Non-compactness

Now, we present the basic results of the theory of MNC in Banach spaces. Let us begin with the general definition of MNC [15]. The most familiar example of MNC is the MNC due to Hausdorff.

Definition 1.3.5 Let $\mathfrak{H}$ be a Banach space and $(\mathfrak{E}, \geq)$ be a partially ordered set. A mapping $\nu : \mathfrak{P}(\mathfrak{H}) \to \mathfrak{E}$ is called a MNC in $\mathfrak{H}$ if $\nu(\overline{co}(\mathfrak{D})) = \nu(\mathfrak{D})$ for every $\mathfrak{D} \in \mathfrak{P}(\mathfrak{H})$, where $\mathfrak{P}(\mathfrak{H})$ is the power set of $\mathfrak{H}$ and $\overline{co}(\mathfrak{D})$ is the convex hull of $\mathfrak{D}$.

Definition 1.3.6 The MNC of Hausdorff $\nu(\cdot)$ is defined for a bounded subset $\mathfrak{D}$ of $\mathfrak{H}$ as,

$$\nu(\mathfrak{D}) = inf\{\epsilon > 0 \mid \mathfrak{D} \text{ has a finite } \epsilon - \text{net in } \mathfrak{H}\}.$$

Lemma 1.3.1 *[16] The following properties of MNC are very crucial to prove our results.*

I. If $\mathfrak{D} \subseteq \mathcal{C}(\mathcal{J}, \mathfrak{H})$ is bounded, then, for given arbitrary $\epsilon > 0, \exists\ \{\upsilon_n\}_{n=1}^{\infty} \subset \mathfrak{D} \ni \nu(\mathfrak{D}) \leq 2\nu\big(\{\upsilon_n\}_{n=1}^{\infty}\big) + \epsilon$.

II. Let $\mathfrak{D} = \big\{\upsilon_n : \mathcal{J} \to \mathfrak{H} \big| \upsilon_n$ is Bochner integrable for every $n \in \mathbb{N}\big\}$, then

  a. $\mathfrak{D}$ is a countable and bounded.

  b. $\nu(\mathfrak{D}(\zeta))$ is Lebesgue integrable on $\mathfrak{H}$ with $\nu\Big(\int_0^{\zeta}\upsilon_n(\aleph)d\aleph : n \in \mathbb{N}\Big) \leq 2\int_0^{\zeta}\nu(\mathfrak{D}(\aleph))d\aleph$.

Lemma 1.3.2 *[17] Let $\mathfrak{R}_r \subset \mathfrak{Y}$ be a bounded convex subset with $0 \in \mathfrak{R}_r$. Let $\mathcal{F} : \mathfrak{R}_r \subset \mathfrak{Y} \to \mathfrak{Y}$ be a continuous map $\ni$ for a countable subset $\mathfrak{D} \subset \mathfrak{R}_r$ and $\mathfrak{D} \subseteq \overline{co}(\{0\} \cup \mathcal{F}(\mathfrak{D})) \implies \overline{\mathfrak{D}}$ is compact. Then, $\mathcal{F}$ has at least one fixed point in $\mathfrak{Y}$.*

1.4 Derivation of Mild Solution

Definition 1.4.1 [18] A one-parameter family of bounded linear operators $\{\mathfrak{C}(\zeta)\}_{\zeta \in \mathbb{R}}$ mapping $\mathfrak{H}$ to $\mathfrak{H}$ is called a strongly continuous cosine family if

i. $\mathfrak{C}(0) = I$, the identity operator in $\mathfrak{L}(\mathfrak{H})$,
ii. $\mathfrak{C}(\aleph + \zeta) - \mathfrak{C}(\aleph - \zeta) = 2\mathfrak{C}(\aleph)\mathfrak{C}(\zeta) \ \forall \ \aleph, \zeta \in \mathbb{R}$,
iii. $\zeta \longmapsto \mathfrak{C}(\zeta)v$ is continuous on $\mathbb{R}$ for any $v \in \mathfrak{H}$.

The associated sine family $\{\mathfrak{S}(\zeta)\}_{\zeta \in \mathbb{R}}$ of $\{\mathfrak{C}(\zeta)\}_{\zeta \in \mathbb{R}}$ is defined by

$$\mathfrak{S}(\zeta)v = \int_0^{\zeta} \mathfrak{C}(\aleph)v d\aleph, \ \ v \in \mathfrak{H}, \ \ \zeta \in \mathbb{R}.$$

Lemma 1.4.1 *If $\{\mathfrak{C}(\zeta)\}_{\zeta \in \mathbb{R}}$ denotes a strongly continuous cosine family in $\mathfrak{H}$. Then,*

i. $\exists \, \mathfrak{M} \geq 1, \ \omega \geq 0 \ \ni: \|\mathfrak{C}(\zeta)\|_{\mathfrak{L}(\mathfrak{H})} \leq \mathfrak{M} e^{\omega|\zeta|} \ \forall \ \zeta \in \mathbb{R}$.
ii. $\|\mathfrak{S}(\zeta) - \mathfrak{S}(\aleph)\|_{\mathfrak{L}(\mathfrak{H})} \leq \mathfrak{M} \left| \int_{\aleph}^{\zeta} e^{\omega|\tau|} d\tau \right| \ \forall \ \zeta, \in \mathbb{R}$.
iii. If $v \in \mathfrak{H}$, then $\frac{d}{d\zeta}\mathfrak{C}(\zeta)v = A\mathfrak{S}(\zeta)v$ and $\mathfrak{S}(\zeta)v \in \mathfrak{D}(\mathcal{A})$.

Remark 1.4.1 The operators $\mathfrak{C}(\zeta)$ and $\mathfrak{S}(\zeta)$ are compact in the uniform operator topology $\forall \ \zeta \in \mathbb{R}$.

Now, we describe the Mittag–Leffler function $E_{q,p}(\mathfrak{z})$ and the Mainardi's Wright-type function $M_q(\mathfrak{z})$, respectively. For info [19, 20].

$$E_{q,p}(\mathfrak{z}) \ = \sum\nolimits_{n=0}^{\infty} \frac{\mathfrak{z}^n}{\Gamma(qn + p)}, \ \ p, q > 0, \ \mathfrak{z} \in \mathbb{C}, \tag{1.6}$$

$$M_q(\mathfrak{z}) \ = \sum\nolimits_{n=0}^{\infty} \frac{(-\mathfrak{z})^n}{n!\Gamma(1 - q(n + 1))}, \ \ p \in (0, 1), \ \mathfrak{z} \in \mathbb{C}. \tag{1.7}$$

Lemma 1.4.2 *For every $\zeta \geq 0$, the Mainardi's Wright-type function has the following properties:*

$$M_q(\zeta) \geq 0, \int_0^{\infty} \theta^{\delta} M_q(\theta) d\theta = \frac{\Gamma(1 + \delta)}{\Gamma(1 + q\delta)} \, for \ -1 < \delta < \infty$$

and for $\mathfrak{z} \in \mathbb{C},\ q \in (0,1)$,

$$E_{q,1}(-\mathfrak{z}) = \int_0^\infty M_q(\theta) e^{-\mathfrak{z}\theta} d\theta,\ E_{q,q}(-\mathfrak{z}) = \int_0^\infty q\theta M_q(\theta) e^{-\mathfrak{z}\theta} d\theta.$$

Next, we begin with the linear non-homogeneous fractional evolution system given below

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha,\beta} v(\zeta) = \mathcal{A}v(\zeta) + f(\zeta),\ \zeta \in \mathcal{J}_1,\ \alpha \in (1,2),\ \beta \in [0,1], \\ \mathcal{I}_{0^+}^{\kappa} v(\zeta)\Big|_{\zeta=0} = v_0,\ \dfrac{d}{d\zeta}\mathcal{I}_{0^+}^{\kappa} v(\zeta)\Big|_{\zeta=0} = v_1. \end{cases} \tag{1.8}$$

The following lemma is necessary in solving the above initial value problem.

Lemma 1.4.3 *[21] Let* $v \in \mathfrak{L}(a,b),\ n-1 < \alpha \le n,\ 0 \le \beta \le 1,\ \mathcal{I}_{0^+}^{(1-\beta)(n-\alpha)} v \in \mathfrak{AC}^k\ [a,b],\ 0 \le k \le n-1$. *Then, the fractional integral* $\mathcal{I}_{0^+}^{\alpha}$ *and the GHFD* $\mathcal{D}_{0^+}^{\alpha,\beta}$ *are linked by the equation:*

$$\mathcal{I}_{0^+}^{\alpha} \mathcal{D}_{0^+}^{\alpha,\beta} v(\zeta) = v(\zeta) - v_{\alpha,\beta}(\zeta), \zeta > 0 \tag{1.9}$$

where

$$v_{\alpha,\beta}(\zeta) = \sum\nolimits_{m=0}^{n-1} \frac{\zeta^{m-(1-\beta)(n-\alpha)}}{\Gamma(m-(1-\beta)(n-\alpha)+1)} \lim_{t\to 0^+} \frac{d^m}{d\zeta^m} \mathcal{I}_{0^+}^{(1-\beta)(n-\alpha)} v(\zeta).$$

Theorem 1.4.1 *Suppose* $v_0 \in \mathfrak{D}(\mathcal{A}), v_1 \in \mathfrak{H}, f \in \mathfrak{L}(\mathcal{J}, \mathfrak{H})$. *Let* $q = \frac{\alpha}{2}$. *If* $v(\zeta)$ *denotes a solution of* (1.8)*, then the following integral equation is satisfied:*

$$v(\zeta) = \mathfrak{C}_{\alpha,\beta}(\zeta) v_0 + \mathfrak{K}_{\alpha,\beta}(\zeta) v_1 + \int_0^\zeta \mathfrak{P}_q(\zeta - \aleph) f(\aleph) d\aleph,$$

where

$$\mathfrak{C}_{\alpha,\beta}(\zeta) = \mathcal{I}_{0^+}^{(1-\beta)(\alpha-2)} \int_0^\infty M_q(\theta) \mathfrak{C}(\zeta^q \theta) d\theta;$$

$$\mathfrak{K}_{\alpha,\beta}(\zeta) = \int_0^\zeta \mathfrak{C}_{\alpha,\beta}(\aleph) d\aleph;\ \mathfrak{P}_q(\zeta) = \zeta^{q-1} \int_0^\infty q\theta M_q(\theta) \mathfrak{S}(\zeta^q \theta) d\theta.$$

For proof of the above result, see Appendix 1.8.

Remark 1.4.2 Since $\mathfrak{C}(\zeta)$ and $\mathfrak{S}(\zeta)$ are linear operators for every $\zeta \ge 0$, it is obvious that the operators $\mathfrak{C}_{\alpha,\beta}(\zeta), \mathfrak{K}_{\alpha,\beta}(\zeta), \mathfrak{P}_q(\zeta)$ are linear.

Remark 1.4.3 The result proved in Theorem 1.4.1 is more generalized than the result in Zhou et al. [22], i.e., if $\beta = 1$, we have the result of Theorem 3.1 in [22].

The following results are provided without proof as those are easy to do.

Lemma 1.4.4 *The operator* $\mathfrak{P}_q(\zeta)$ *is compact for every* $\zeta \geq 0$.

Lemma 1.4.5 *The following estimates are valid for any* $\zeta \geq 0$ *and any* $v \in \mathfrak{H}$.

i. $\|\mathfrak{C}_{\alpha,\beta}(\zeta)v\| \leq \dfrac{\mathfrak{M}\zeta^{(1-\beta)(\alpha-2)}}{\Gamma((1-\beta)(\alpha-2)+1)}\|v\|.$

ii. $\|\mathfrak{K}_{\alpha,\beta}(\zeta)v\| \leq \dfrac{\mathfrak{M}\zeta^{(1-\beta)(\alpha-2)+1}}{\Gamma((1-\beta)(\alpha-2)+2)}\|v\|.$

iii. $\|\mathfrak{P}_q(\zeta)v\| \leq \dfrac{\mathfrak{M}\zeta^{\alpha-1}}{\Gamma(\alpha)}.$

Lemma 1.4.6 *The operators* $\{\mathfrak{C}_{\alpha,\beta}(\zeta)\}_{\zeta\geq 0}, \{\mathfrak{K}_{\alpha,\beta}(\zeta)\}_{\zeta\geq 0}, \{\mathfrak{P}_q(\zeta)\}_{\zeta\geq 0}$ *are strongly continuous.*

1.5 Main Results

The control function is defined in this section, along with its continuity and boundedness properties. The existence theorem will be proven by using suitable assumptions via MFPT. Furthermore, the approximate controllability of (1.1) is proven by assuming that the equivalent linear system is approximately controllable.

Definition 1.5.1 A function $v(\zeta) \in \mathfrak{Y}$ is said to be a mild solution of (1.1) if $v_0 \in \mathfrak{D}(\mathcal{A}),\ v_1 \in \mathfrak{H}$ and for each $y \in \mathfrak{L}^2(\mathcal{J}, \mathfrak{U})$, the integral equation is satisfied.

$$v(\zeta) = \mathfrak{C}_{\alpha,\beta}(\zeta)v_0 + \mathfrak{K}_{\alpha,\beta}(\zeta)v_1 + \int_0^\zeta \mathfrak{P}_q(\zeta-\aleph)\big[\mathcal{B}y(\aleph) + f(\aleph, v(\aleph))\big]d\aleph$$
$$+ \int_0^\zeta \mathfrak{P}_q(\zeta-\aleph)\int_0^\aleph a(\aleph,\varsigma)g(\varsigma, v(\varsigma))d\varsigma d\aleph.$$

Note 2 Hereafter, unless specified $\mathfrak{D}$ represents a bounded countable set.

For convention, we present some terminology.

$c_0 = \dfrac{\mathfrak{M}\mathfrak{M}_\mathcal{B}}{\lambda\Gamma(\alpha)};\ c_1 = \dfrac{\mathfrak{M}b^{(1-\beta)(2-\alpha)}}{\Gamma(\alpha)};\ c_2 = \dfrac{\mathfrak{M}\mathfrak{M}_\mathcal{B}b^\alpha}{\Gamma(\alpha+1)};$

$c(\alpha, q_i) = \left(\dfrac{1-q_i}{\alpha-q_i}\right)^{1-q_i},\ i = 1, 2;\ \mathfrak{M}_1 = \|v_b\|_\mathfrak{Y} + \mathfrak{M}_2;$

$\mathfrak{M}_2 = \dfrac{\mathfrak{M}}{\Gamma((1-\beta)(\alpha-2)+1)}\|v_0\| + \dfrac{\mathfrak{M}b}{\Gamma((1-\beta)(\alpha-2)+2)}\|v_1\|;$

$$\mathfrak{M}_y = c_0\bigg(\mathfrak{M}_1 + c_1 c(\alpha, q_1) b^{\alpha-q_1} \|\xi_f\|_{\mathfrak{L}^{\frac{1}{q_1}}} + c_1 \frac{b^\alpha}{\alpha}\, \tilde{a} \|\xi_g\|_{\mathfrak{L}^\infty}\bigg);$$

$$\Pi := c_1\bigg(1 + c_0 c_2\bigg)\bigg(c(\alpha, q_1) b^{\alpha-q_1} \varLambda_1 + \frac{b^\alpha}{\alpha} \tilde{a} \varLambda_2\bigg).$$

Now, let us presume the following to prove the upcoming results.

(Hf) The function $f : \mathcal{J} \times \mathfrak{H} \to \mathfrak{H}$ satisfies,

(1) $f(\zeta, v)$ is continuous $\forall\, \zeta \in \mathcal{J}$ and is strongly measurable $\forall\, v \in \mathfrak{H}$.
(2) For some $r \in \mathbb{R}^+, \exists\ q_1 \in (0, \alpha)$ and a function $\xi_f \in \mathfrak{L}^{\frac{1}{q_1}}(\mathcal{J}, \mathbb{R}^+) \ni$: $\sup_{\|v\| \le r} \|f(\zeta, v(\zeta))\| \le \xi_f(\zeta)$ for a.e $\zeta \in \mathcal{J}$ and $\liminf_{r\to\infty} \frac{\|\xi_f\|_{\mathfrak{L}^{\frac{1}{q_1}}}}{r} \triangleq \varLambda_1 < \infty$.
(3) For $\mathfrak{D} \subset \mathfrak{Y}$, $\exists\, q_2 \in (0, \alpha)$ and a function $\psi_f \in \mathfrak{L}^{\frac{1}{q_2}}(\mathcal{J}, \mathbb{R}^+) \ni$: $\nu(f(\zeta, \mathfrak{D}(\zeta))) \le \psi_f(\zeta)\nu(\mathfrak{D})$, $\zeta \in \mathcal{J}$.

(Hg) The function $g : \mathcal{J} \times \mathfrak{H} \to \mathfrak{H}$ satisfies,

(1) $g(\zeta, v)$ is continuous $\forall\, \zeta \in \mathcal{J}$ and is strongly measurable $\forall\, v \in \mathfrak{H}$.
(2) For some $r \in \mathbb{R}^+, \exists$ and a function $\xi_g \in \mathfrak{L}^\infty(\mathcal{J}, \mathbb{R}^+) \ni$: $\sup_{\|v\| \le r} \|g(\zeta, v(\zeta))\| \le \xi_g(\zeta)$ for a.e $\zeta \in \mathcal{J}$ and $\liminf_{r\to\infty} \frac{\|\xi_g\|_{\mathfrak{L}^\infty}}{r} \triangleq \varLambda_2 < \infty$.
(3) For $\mathfrak{D} \subset \mathfrak{Y}$, $\exists$ and a function $\psi_g \in \mathfrak{L}^\infty(\mathcal{J}, \mathbb{R}^+) \ni$: $\nu(g(\zeta, \mathfrak{D}(\zeta))) \le \psi_g(\zeta)\nu(\mathfrak{D})$, $\zeta \in \mathcal{J}$.

(Hm) $$\rho := \big(1 + 2c_0 c_2\big)\bigg\{\frac{2\mathfrak{M} b^{\alpha-q_2}}{\Gamma(\alpha)} c(\alpha, q_2) \|\psi_f\|_{\mathfrak{L}^{\frac{1}{q_2}}} + \frac{4\mathfrak{M} b^\alpha \tilde{a}}{\Gamma(\alpha+1)} \|\psi_g\|_{\mathfrak{L}^\infty}\bigg\} < 1.$$

Definition 1.5.2 For given $\lambda > 0$ and $v_b \in \mathfrak{H}$, define the control function as follows:

$$\begin{aligned} y(\zeta) &= (b-\zeta)^{1-\alpha} \mathcal{B}^* \mathfrak{P}_q^*(b-\zeta) R\big(\lambda, \Psi_0^b\big)\bigg[v_b - \mathfrak{C}_{\alpha,\beta}(b) v_0 - \mathfrak{K}_{\alpha,\beta}(b) v_1 \\ &\quad + \int_0^b \mathfrak{P}_q(b-\aleph) f(\aleph, v(\aleph)) d\aleph + \int_0^b \mathfrak{P}_q(b-\aleph) \int_0^\aleph a(\aleph, \varsigma) g(\varsigma, v(\varsigma)) d\varsigma d\aleph\bigg], \end{aligned}$$

where $R\big(\lambda, \Psi_0^b\big)$ is the resolvent operator given by $R\big(\lambda, \Psi_0^b\big) = \big(\lambda I + \Psi_0^b\big)^{-1}$.

Note 3 $\mathcal{B}^*, \mathfrak{P}_q^*(\zeta)$, respectively, denotes the adjoint of $\mathcal{B}, \mathfrak{P}_q(\zeta)$.

Note 4 Ψ_0^b is the controllability Grammian operator defined by $\Psi_0^b = \int_0^b \mathfrak{P}_q(b-\aleph)\mathcal{B}\mathcal{B}^*\mathfrak{P}_q^*(b-\aleph)d\aleph$.

Theorem 1.5.1 *The control function is bounded and continuous on* $\mathfrak{Y}$.

The proof is given in Appendix 1.9.

Theorem 1.5.2 *Suppose that $\mathcal{A}$ generates a strongly continuous cosine family $\{\mathfrak{C}(\zeta)\}_{\zeta\in\mathbb{R}}$ in $\mathfrak{H}$. If the assumptions (Hf),(Hg),(Hm), are satisfied. Then, the system* (1.1) *has at least one mild solution provided that* $\Pi \leq 1$.

Proof Define an operator $\mathcal{F} : \mathfrak{Y} \to \mathfrak{Y}$ as follows:

$$(\mathcal{F}v)(\zeta) = \mathfrak{L}_{\alpha,\beta}(\zeta)v_0 + \mathfrak{K}_{\alpha,\beta}(\zeta)v_1 + \int_0^\zeta \mathfrak{P}_q(\zeta - \aleph)\big[\mathcal{B}y(\aleph) + f(\aleph, v(\aleph))\big]d\aleph$$
$$+ \int_0^\zeta \mathfrak{P}_q(\zeta - \aleph)\int_0^\aleph a(\aleph, \varsigma)g(\varsigma, v(\varsigma))d\varsigma d\aleph.$$

It is interesting to note that the mild solution of (1.1) is equivalent to the fixed point of $\mathcal{F}$. We exhibit this theorem in the following four phases.

Step 1. $\exists\, r > 0 \ni: \mathcal{F}(\mathfrak{R}_r) \subseteq \mathfrak{R}_r$.

Suppose the contrary, that implies $\forall r > 0, \exists\ v \in \mathfrak{R}_r \ni: \|(\mathcal{F}v)(\zeta)\|_{\mathfrak{Y}} > r, \zeta \in \mathcal{J}$. Then,

$$\begin{aligned}
r &< \sup_{\zeta\in\mathcal{J}} \zeta^{(1-\beta)(2-\alpha)}\|(\mathcal{F}v)(\zeta)\| \\
&\leq \mathfrak{M}_2 + \sup_{\zeta\in\mathcal{J}} \zeta^{(1-\beta)(2-\alpha)}\Bigg\{\frac{\mathfrak{M}\mathfrak{M}_{\mathcal{B}}}{\Gamma(\alpha)}\int_0^\zeta (\zeta-\aleph)^{\alpha-1}\|y(\aleph)\|d\aleph \\
&+\frac{\mathfrak{M}}{\Gamma(\alpha)}\int_0^\zeta (\zeta-\aleph)^{\alpha-1}\|f(\aleph, v(\aleph))\|d\aleph \\
&\quad+\frac{\mathfrak{M}}{\Gamma(\alpha)}\int_0^\zeta (\zeta-\aleph)^{\alpha-1}d\aleph\left\|\int_0^\zeta a(\zeta,\aleph)g(\aleph, v(\aleph))d\aleph\right\|\Bigg\} \\
&\leq \mathfrak{M}_2 + c_0c_2\left(\mathfrak{M}_1 + c_1c(\alpha, q_1)b^{\alpha-q_1}\|\xi_f\|_{\mathfrak{L}^{\frac{1}{q_1}}} + c_1\frac{b^\alpha}{\alpha}\tilde{a}\|\xi_g\|_{\mathfrak{L}^\infty}\right) \\
&\quad+c_1c(\alpha, q_1)b^{\alpha-q_1}\|\xi_f\|_{\mathfrak{L}^{\frac{1}{q_1}}} + c_1\frac{b^\alpha}{\alpha}\tilde{a}\|\xi_g\|_{\mathfrak{L}^\infty}.
\end{aligned}$$

Dividing by r and letting $r \to \infty$, we have

$$1 \leq c_0c_2\left(c_1c(\alpha, q_1)b^{\alpha-q_1}\Lambda_1 + c_1\frac{b^\alpha}{\alpha}\tilde{a}\Lambda_2\right) + c_1c(\alpha, q_1)b^{\alpha-q_1}\Lambda_1 + c_1\frac{b^\alpha}{\alpha}\tilde{a}\Lambda_2$$

which is a contradiction. Hence $\mathcal{F}(\mathfrak{R}_r) \subseteq \mathfrak{R}_r$.

Step 2. $\mathcal{F} : \mathfrak{R}_r \to \mathfrak{R}_r$ is continuous $\forall\ \zeta \in \mathcal{J}$.