

SpringerBriefs on PDEs and Data Science

Nik Cunniffe · Frédéric Hamelin ·
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Identifiability and Observability in Epidemiological Models

A Primer

 Springer

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*The World requires at least ten years to
understand a new idea, however important or
simple it may be.
Ronald Ross (1902 Nobel Prize)*

Preface

In Mathematical Epidemiology, many papers have the following structure:

- A model is proposed.
- Some parameters are given, extracted from the literature.
- Remaining unknown parameters are estimated by fitting the model to some observed data.

Fitting is done usually by using an optimization algorithm with the use, for example, of a least square method or a maximum likelihood estimation. To validate the estimation of parameters, one can use noisy synthetic simulated data obtained from the model for given values of the parameters, to check that the algorithm is able to reconstruct from the data the values of these parameters with accuracy.

One objective of this book is to show that this procedure is not always safe and that an examination of the identifiability of parameters is a prerequisite before a numerical determination of parameters. We will review different methods to study identifiability and observability and then consider the problem of numerical identifiability. Our touchstone will be the most famous, but simple, model in Mathematical Epidemiology, the SIR model of Kermack and Mckendrick [73]. This model received renewed attention with the COVID-19 pandemic [106]. Parameter identifiability analysis addresses the problem of which unknown parameters of an ODE model can uniquely be recovered from observed data. We will show that, even for very simple models, identifiability is far from being guaranteed.

The problem of identifiability for epidemiological models is relatively rarely addressed. For instance, a search in the Mathematical Reviews of the American Mathematical Society¹ for 2020 with `epid*` AND `identifiability` gives only 4 papers, while `epidem*` AND `parameter` returns 68 publications. Only a small subset of the later publications addresses the problem of identifiability. In particular, the following publications consider the problem of identifiability

¹ <https://mathscinet.ams.org/mathscinet>.

in epidemiological models: [19, 33, 50, 51, 68, 72, 81, 87, 91, 99, 107, 119–121, 130, 132]. However, the majority of these papers were published elsewhere than in Biomathematics journals. Note that we make a distinction between publications that address directly the parameter estimation problem in epidemiological models (such as in references [13, 20, 21, 27–29, 37, 56, 57, 63, 101, 113, 132] for instance) and works that study explicitly the identifiability property of models. As explained in this book, this is an intrinsic property to be studied prior to determination of parameters values.

The question of observability, i.e. the ability to reconstruct state variables of the model from measurements, is often considered separately from the problem of identifiability. Either model parameters are known, or an identifiability analysis is performed prior to the study of observability. Indeed, the concepts of identifiability and observability are closely related, as we show in this book. However, for certain models, it is possible to reconstruct state variables with observers, while the model is not identifiable. In other situations, we show that considering jointly identifiability and observability with observers can be a way to solve the identifiability problem. This is another illustration of the utility of the concept of observers. This is why we shall dedicate a fair part of this monograph to reviewing the concept of observers and their practical constructions in epidemiology.

This book is aimed at scientists, researchers and graduate students, who use or develop mathematical models for epidemiology, and who are not yet familiar with the concepts of control science (detectability, observability, observers) applied to this field.

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Contents

1	Introduction	1
1.1	Definitions	1
1.2	Historical Notes	7
1.3	Identifiability in Mathematical Epidemiology	8
1.4	The Concept of Observers	8
2	Mathematical Foundations	9
2.1	Preliminaries	9
2.2	Observability	10
2.3	About Identifiability	20
2.4	Identifiability Does Not Necessarily Require Observability	22
2.5	Identifiability via Decoupled Variables	23
3	Analysis of the Kermack-McKendrick Model	27
3.1	History	27
3.2	The Different Forms of the SIR Model	28
3.3	Observability and Identifiability of the SIR Model	28
3.3.1	The SIR Model When Observing a Ratio of the Infected Population	29
3.3.2	The SIR Model When Observing the Incidence	34
4	Observers Synthesis	39
4.1	Introduction	39
4.2	Observers with Linear Error Dynamics	42
4.3	Observers for Systems with Lipschitz Non-linearity	44
4.4	Observers via Decoupled Variables	48
4.5	Reduced-Order Observers	49
4.6	The High-Gain Observer for Nonlinear Systems	51
4.7	Discussion	56
5	Practical and Numerical Considerations	59
5.1	Practical Identifiability	59
5.1.1	Rationale for Using Sensitivity Analysis	60

5.1.2	Observed System	60
5.1.3	Sensitivity Analysis	60
5.1.4	Ordinary Least Squares	61
5.1.5	Confidence Intervals	62
5.1.6	Computing the Sensitivity Matrix	63
5.1.7	Some Case Studies	64
5.1.8	Discussion	67
5.2	Observers in Practice	68
5.2.1	Observers with Linear Assignable Error Dynamics	68
5.2.2	About Observers with Lipchitz Non-linearity	70
5.2.3	Observers with Asymptotic Convergence	71
5.2.4	Observers with Partially Assignable Error Dynamics	71
5.2.5	High Gain Observer	73
5.3	A Case Study : An Observer to Estimate State and Parameter	74
A	Proofs of Some Useful Lemmas	81
A.1	Proof of Lemma 4.1	81
A.2	Proof of Lemma 4.2	84
A.3	Proof of Theorem 4.1	86
B	Implementation of the “Boarding School” Example	89
B.1	Derivation of the Fisher’s Information Matrix	89
B.2	Numerical Implementation	91
C	Implementation of the “Plague in Bombay” Example	95
C.1	Derivation of the Fisher Information Matrix	95
C.2	Numerical Implementation	97
D	Generalized Least Squares	101
	References	103