Jin Liang
Bei Hu

Credit Rating Migration Risks in Structure Models

Springer

Credit Rating Migration Risks in Structure Models

Jin Liang • Bei Hu

Credit Rating Migration Risks in Structure Models

Springer

Jin Liang
School of Mathematical Science
Tongji University
Shanghai, Shanghai, China

Bei Hu
Applied and Computational Mathematics
and Statistics
University of Notre Dame
NOTRE DAME, IN, USA

ISBN 978-981-97-2178-8 ISBN 978-981-97-2179-5 (eBook)
https://doi.org/10.1007/978-981-97-2179-5

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

With the broad acceptance of the Black-Scholes Theory by the financial industry, mathematics played more and more important role in pricing financial products, measuring risks, computing values, optimizing investments, etc. As the randomness is an essential feature of the financial market, the stochastic process is the main tool in financial mathematics. Another powerful tool is the partial differential equation (PDE). A famous example of a PDE model is the Black-Scholes model. The advantage of the PDE approach is the availability of the representation of the free boundaries. Study on the financial implications of free boundaries, usually corresponding to optimal exercises, credit rating migration, etc. is the strength for the PDE method.

In the past decade, the authors focused on a systematic study of credit risks models by partial differential equations. These works were motivated from financial credit problems; these problems are of independent mathematical interests. After the 2007–2008 financial crisis, more attentions have been paid to this area of research, and its significance became self-evident. This book collects the relevant researches on measuring credit risks, specializing in structure models on credit rating migration risks by the authors, their colleagues and students, using PDE models and methods. Of course, it is far from "complete" in this area; the goal is to "get the ball rolling" from here.

It is always a challenge to cross different fields, and this book intersects the fields of finance and mathematics. The authors will try their best to make the book readable for readers in both disciplines. Some chapters only deal with modeling, while some others are on mathematical theory. Readers are expected to be familiar with the basic mathematical analysis theory and mathematical modeling knowledge, though they can choose the chapters they are interested in. The book could be used as a text book for students in studying financial credit risks, and/or a reference for researchers in relevant areas. There are 10 chapters in this book. The structure of the book is as follows:

Chapter 1 Financial Background. In this chapter, the financial background, such as financial jargons, credit risks in finance, and some relevant concepts are explained.

Chapter 2 Preliminary Mathematical Theory. In this chapter, some basic mathematical knowledge, such as Markov chain, PDE, are provided; these tools will used in later chapters.

Chapter 3 Mathematical Models for Measuring Default Risks. There are two mainstream models used in measuring default, reduced form model and structure one; these are introduced in this chapter.

Chapter 4 Markov Chain Approach for Measuring Credit Rating Migration Risks. Reduced form models applied in measuring credit rating migration are presented in this chapter.

Chapter 5 Application of Reduced Form/Markov Chain Credit Rating Migration Model. Some examples for applying reduced form models are shown here.

Chapter 6 Structure Models for Measuring Credit Rating Migration Risks. This chapter includes the authors' contribution on both the modeling and the mathematical analysis of the structure models for credit rating migration.

Chapter 7 Theoretical Results in the Structural Credit Rating Migration Models. Using PDE techniques, the existence, uniqueness, regularities, asymptotic behavior, traveling wave and other properties of the solutions of the model presented in Chap. 6 are rigorously established in this section.

Chapter 8 Extensions for Structural Credit Rating Migration Models. The model introduced in Chap. 6 is extended to more general case, such as stochastic interest rate, multiple ratings, region switch and so on.

Chapter 9 Credit Derivatives Related to Rating Migrations. Some credit derivatives are discussed in this chapter.

Chapter 10 Numerical Simulation, Calibration and Recovery of Credit Rating Boundary. Numerical analysis, parameter calibration and estimate of the migration boundary of the models are given in this chapter.

The authors are grateful to Prof. Lishang Jiang, who gave many valuable suggestions during the preparation of the book.

Shanghai, China Jin Liang
Notre Dame, IN, USA Bei Hu

Contents

Chapter 1
Financial Background

To participate in a financial market, one should understand finance in some way. Mathematical model is a powerful tool not only to understand it deeper, but also provide the knowledge of the market. In this chapter, some relative financial concepts and jargons are explained.

1.1 Financial Risks, Credit and Ratings

Uncertainty is the hallmark of the financial market. This uncertainty brings about risks for all participants in the market. Credit risk has always been regarded as the biggest risk faced by the banking industry. The generation and development of credit derivatives aiming at reducing the serious losses caused by credit risk and transferring credit risk is a revolution of credit risk management in banking industry.

1.1.1 Financial Risks

Financial risk means uncertainty relative to financial activities, where usually associated with financial losses. So it is the risk for all participants in financial circle. It is a term that can apply to businesses, government entities, the financial market as a whole, and the individual.

There are several specific risk factors that can be categorized as a financial risk. Any risk is a hazard that produces damaging or unwanted results. Some more common are as follows:

- Market risk: the risk of unexpected changes in prices or rates of financial products in markets.

J. Liang, B. Hu, *Credit Rating Migration Risks in Structure Models*,
https://doi.org/10.1007/978-981-97-2179-5_1

- Credit risk: the risk of changes in value associated with unexpected changes in credit quality and the results of credit events.
- Liquidity risk: the risk that the costs of adjusting financial positions will increase substantially or that an investment will lose access to financing.
- Operational risk: the risk of fraud, systems failures, trading errors, such as deal mispricing, program bugs etc., and many other internal organizational risks.
- Systemic risk: the risk of breakdowns in marketwide liquidity, chain- reaction default and/or losses contagions etc..

1.1.2 Credit Risks

Credit risk is the one of main financial risks, which is caused by the counterparty of a trade not being able to meet its financial obligations. For example, when a lender offers a mortgage, or a credit card, there is a risk that the borrower may not make the repayment. Similarly, if a company sells a product or service to a customer in advance, there is a risk that the customer may not pay its invoice; a bond issuer may fail to make payment when requested. See also [2].

Credit risks are measured on the counterparty of a contract overall ability of repayment a contract according to its original terms. In recent years, credit risks draw more and more attentions. To assess credit risk on a financial institute, its credit history, capacity to repay, capital, conditions, and associated collateral are evaluated. Roughly to say, the credit risk can be calculated as the propensity and/or probability of the default. The theoretical researches on measure default risks are relatively rich and advanced. Usually it is described in several ratings. For a long time, the main credit risks are believed to be default risks, recently, the credit rating migration risks also show their significance.

The credit rating plays an important role in financial market. For example, if an investor considers buying a bond, the credit rating of the bond will be reviewed. If it has a low rating (B or C), the issuer has a higher risk of default, so that the bond price will be cheaper and higher return. Conversely, if it has a high rating (AAA, AA, or A), it's considered to be a safe investment, and the bond price will be more expensive and it has a lower return.

1.1.3 Credit Rating

In its simplest form, a credit rating is an assessment of a company's capability to meet its debt obligations. The majority of bond ratings are publicly disclosed and are used as important information by debt investors in their investment appraisal process, although they are also used by creditors and other parties for understanding an entity's credit profile. Investors also use a broad categorization of issuers as investment grade, which is usually high-quality ratings. However, even a high-quality rating cannot ensure the safety of the investment, a high-quality rating firm might also default be broke. It is a good example that Lemmon Brother with rating A was bankrupted in 2008 financial crisis. Rating look is associated not just at with “probability of default”, but also “loss given default”. This is particularly important for noninvestment grade issues, where the presence of credit enhancements (asset backing, security, covenants, priority ranking) or weaknesses (contractual or structural subordination, absence of security or covenants) can lead to individual issues being “ up” or “down” relative to other issues by the same borrowing group or overall corporate credit rating to reflect a lower expectation of recovery in the event of a default.

Credit rating migration not only shows the financial health of a company, but also reflects macro-economic conditionals, the methods of measuring rating migration can be extended to measure the risks of regime switch of the economic environment. For corporate bonds, on one hand, their credit ratings are effected by financial factors, such as interest rate, the firms values etc.. On the other hand, the information of credit rating change can be used to analyze the global market. In short, credit rating migration risks not only provide a wider investment scope but also effectively impact the markets. Therefore, studying the credit rating migration can be a reference for understanding the switching of the globally economical states such as bull or bear market, and it offers the great value in the credit rating changes across economic cycles. The value can be reflected in the different reactions of market participants to rating changes and how the reaction changes across economic expansion or contraction. Therefore, the equity market reacts more sharply to credit downgrades during contractions that contain a larger likelihood of default. As the credit rating changes are so important, we also have the interest in studying the credit default swap considering the rating migration risk of reference bonds.

There has been considerable interest in the application of credit models to the measurement, analysis and valuation of credit risk. The research on credit risk and its derivatives is a hot topic in current financial research. Credit rating models are allowed to play a role in the estimation of the risk parameters, where human judgments should not be involved. The financial institutes must also satisfy the supervisor that the data used to establish these models are representative of its exposures. Using these models, there should be no distortion in the calculation of regulatory capital and no special interests. The models are also required to be stable and able to predict default in real life. Some important parameters are

- PD: Probability of default.
- EAD: Exposure at default. It is the total value a bank is exposed to when a loan defaults. Using the internal ratings-based (IRB) approach, financial institutions calculate their risk. Banks often use internal risk management default models to estimate respective EAD systems. Outside of the banking industry, EAD is known as credit exposure.
- LGD: Loss given default. It is the amount of money a bank or other financial institution loses when a borrower defaults on a loan, depicted as a percentage of total exposure at the time of default. A financial institution’s total LGD is

calculated after a review of all outstanding loans using cumulative losses and exposure.

There are several mathematical models to estimate risk parameters. A measuring risk model, sometimes, is also built through pricing a financial product with such risk. Then better understanding of risks is achieved by analyzing the model.

Credit ratings are predominantly provided by three main independent rating agencies, although there are others. These main agencies are Standard & Poor's (S&P), Moody's Investor Services (Moody's), and Fitch IBCA (Fitch). These agencies use similar but not identical rating scales while the financial market often converts their ratings from one to another.

Though the methodologies are not open, the rating agencies may use broadly similar ones in their credit rating determination independently. That is, the differences in rating outcome may exist. For certain sectors or products, the agencies provide an overview of their notwithstanding identical information, in general the analysis will focus on business and financial risks areas. Business risk evaluation includes strengths/weaknesses of the operations of the entity, such as market cyclicality, geographic diversification, sector position, and competitive dynamics. This approach allows businesses to be compared against each other and relative strength/weakness to be identified. Financial risk valuation includes the financial flexibility of the entity, such as total sales and profitability measures, margins, growth expectations, liquidity, funding diversity and financial forecasts. The area of financial risk analysis is often distilled down to the analysis of a certain number of key credit ratios.

The ratings by different agencies are different, but similar. e.g.

Moody's ratings are: Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, Ba1, Ba2, Ba3, B1, B2, B3, Caa1, Caa2, Caa3, Ca, and C;

S&P ratings are: AAA, AA+, AA, AA-, A+, A,A-, BBB+, BBB, BBB-, BB+, BB, BB-, B+, B,B-, CCC+, CCC, CCC-, CC, and C.

1.1.4 Rating System

Rating system refers to the entire mathematical and technological infrastructure a financial institute has put in place to quantify and assign the risk parameters. The financial institutes are allowed to use multiple ratings systems for different exposures, though they are not allowed to use a particular rating system to minimize regulatory capital requirements. Rating systems must be well documented. They must enable a third party, like independent reviewers, to replicate the assignment of ratings and their appropriateness. All relevant up to date information can be used for ratings. All data relevant to assignment of ratings must be collected and maintained. The data collected is not only beneficial for improving the credit risk management process, but also required for necessary supervisory reporting. For a loan approval process, the requirements state that for corporate, sovereign or bank exposures all borrowers and guarantors must be assigned a rating, the rating process must be reviewed periodically by a body independent body, at least once a year.

The rating systems are also required to be regularly stress tested, considering economic downturn scenarios, financial crisis etc. These stress tests should not only consider the relevant internal data of the bank, but also macro-economic factors that might affect the accuracy of the rating system.

1.1.5 Financial Crisis

A financial crisis is any of a broad variety of situations in which some financial assets suddenly lose a large part of their nominal value. In the 19th and early 20th centuries, many financial crises were associated with banking panics, and many recessions coincided with these panics. Other situations that are often called financial crises include stock market crashes and the bursting of other financial bubbles, currency crises, and sovereign defaults. Financial crises directly result in a loss of paper wealth but do not necessarily result in significant changes in the real economy (e.g. the crisis resulting from the famous tulip mania bubble in the seventeenth century). Many economists have offered theories about how financial crises develop and how they could be prevented. There is no consensus, however, and financial crises continue to occur from time to time.

1.1.6 Basel Accord

The Basel Accords refer to the banking supervision Accords (recommendations on banking regulations) Basel I, Basel II and Basel III- issued by the Basel Committee on Banking Supervision (BCBS). They are called the Basel Accords as the BCBS maintains its secretariat at the Bank for International Settlements in Basel, Switzerland and the committee normally meets there. The Basel Accords is a set of recommendations for regulations in the banking industry. See [1].

Basel I is the round of deliberations by central bankers from around the world, and in 1988, the Basel Committee on Banking Supervision (BCBS) in Basel, Switzerland, published a set of minimum capital requirements for a bank. This is also known as the 1988 Basel Accord, and was enforced by law in the Group of Ten (G-10) countries in 1992.

Basel II was published initially in June 2004 and was intended to amend international banking standards that controlled how much capital banks were required to hold to guard against the financial and operational risks banks face. These regulations aimed to ensure that the more significant the risk a bank is exposed to, the greater the amount of capital the bank needs to hold to safeguard its solvency and overall economic stability. Basel II attempted to accomplish this by establishing risk and capital management requirements to ensure that a bank has adequate capital for the risk the bank exposes itself to through its lending, investment and trading activities. One focus was to maintain sufficient consistency of regulations so to limit competitive inequality amongst internationally active banks.

Basel II was implemented in the years prior to 2008, and was only to be implemented in early 2008 in most major economies; that year's Financial crisis occured before Basel II could become fully effective.

Basel III is a global, voluntary regulatory framework on bank capital adequacy, stress testing, and market liquidity risk. This third installment of the Basel Accords was developed in response to the deficiencies in financial regulation revealed by the financial crisis of 2007–08. It is intended to strengthen bank capital requirements by increasing bank liquidity and decreasing bank leverage.

Basel III was agreed upon by the members of the Basel Committee on Banking Supervision in November 2010, and was scheduled to be introduced from 2013 until 2015; however, implementation was extended repeatedly to 31 March 2019

1.2 Some Financial Market Concepts

In this section, we list some financial concepts that are essential to the following chapters. For more details of the content, readers are referred to the relative literatures, e.g. [4, 5].

1.2.1 Bond

The bond is a debt security, under which the issuer owes the holders a debt. Depending on the terms of the bond, The issuer is obliged to pay the holder interest at fixed intervals and to repay the principal at a so-called maturity date. Very often the bond is negotiable, that is, the ownership of the instrument can be traded in the secondary market. This means that once the transfer agents at the bank medallion stamp the bond, it is highly liquid on the secondary market.

The bonds can be divided into government bonds, financial bonds and corporate bonds. Government bonds are issued by government, so that it has the least risky but the least profitable. Corporate bonds are riskier and more likely profitable. Bonds are traded form bond markets. Bonds may also play roles as underlyings of derivatives.

A bond has following features:

Principal It also called nominal, par, or face amount. It is the amount on which the issuer pays interest, and which, most commonly, has to be repaid at the end of the term. Some structured bonds can have a redemption amount which is different from the face amount and can be linked to the performance of particular assets.

Maturity It is the date agreed in the contract when the life of the bond ends. The issuer has to repay the nominal amount on that date. As long as all due payments have been made, the issuer has no further obligations to the bond holders after then. In the market for United States Treasury securities, there are three categories of bond maturities:
short term (bills): maturities between 1 and 5 years;
medium term (notes): maturities between 6 and 12 years;
long term (bonds): maturities longer than 12 years.

Coupon It is the interest that the issuer pays to the holder. Mathematically it is used in a continuous rate fixed or vary throughout the life of the bond. The rate can be even more exotic. The name "coupon" arose because in the past, paper bond certificates were issued which had coupons attached to them, one for each interest payment. On the due dates the bondholder would hand in the coupon to a bank in exchange for the interest payment. Interest can be paid at different frequencies: generally semi-annual, i.e. every 6 months, or annual, sometime monthly.

Yield It is the rate of return received from investing in the bond, which usually refers either to:
The current yield, or running yield: It is simply the annual interest payment divided by the current market price of the bond (often the clean price).
The yield to maturity, or redemption yield: It is a more useful measure of the return of the bond. This takes into account the current market price, and the amount and timing of all remaining coupon payments and of the repayment due on maturity. It is equivalent to the internal rate of return of a bond.

Credit quality The quality of the issue refers to the probability that the bond-holders will receive the amounts promised at the due dates. This will depend on a wide range of factors. High-yield bonds are bonds that are rated below investment grade by the credit rating agencies. As these bonds are riskier than investment grade bonds, investors expect to earn a higher yield. These bonds are also called junk bonds.

Market price As the different features above are influenced, a bond of the market price varies over time. The price can be quoted as clean or dirty. "Dirty" includes the present value of all future cash flows, including accrued interest, and is most often used in Europe. "Clean" does not include accrued interest.

According to the contract, a bond also can have some other terms as follows

Callable bond is a bond that the issuer may redeem before it reaches the stated maturity date according to the contract. That is, a callable bond allows the issuing company to pay off their debt early for its favorable reason. It is also called redeemable bond.

Convertible bond is a fixed-income debt security that yields interest payments, but can be converted into a predetermined number of common stock or equity shares at the time during the bond's life, usually at the discretion of the bondholder.

There is a special bond called zero coupon bond. It is a bond where the face value is repaid at the time of maturity. Note that this definition assumes a positive time value of money. It does not make periodic interest payments, or have so-called coupons, hence the term zero-coupon bond. When the bond reaches maturity, its investor receives its par (or face) value. A zero-coupon bond is particularly interested in theoretical study, as the price shows directly the information of interest and credit level etc.

1.2.2 Options

An option is a contract which gives the buyer a right, but not the obligation, who can buy or sell an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the contract. The seller has the corresponding obligation to fulfill the transaction—to sell or buy—if the buyer (owner) “exercises” the option.

An option has following types:

According to the Option Rights

Call Optopn An option that conveys to the owner the right to buy at a specific price is referred to as a call.

Put Optopn An option that conveys the right of the owner to sell at a specific price is referred to as a put.

According to the Underlying Assets

Equity option, Bond option, Future option, Index option, Commodity option, Currency option etc..

Option Styles

European option An option which can only be exercised on expiry.

American option An option that may be exercised on any trading day on or before expiration.

Asian option An option whose payoff is determined by the average underlying price over some preset time period.

Bermudan option An option that may be exercised only on specified dates on or before expiration.

Barrier option Any option with the general characteristic that the underlying security's price must pass a certain level or "barrier" before it can be exercised.

Binary option An all-or-nothing option that pays the full amount if the underlying security meets the defined condition on expiration otherwise it expires.

Exotic option Any of a broad category of options that may include complex financial structures.

European call or put option is usually called a vanilla option.

1.2.3 Interest Rate Swap

A swap is a financial derivative in which two counterparties exchange cash flows. An interest rate swap (IRS) is such a derivative contract for specifying the nature of an exchange of payments benchmarked against an interest rate index. The most common IRS is a fixed for floating swap, whereby one party will make payments to the other based on an initially agreed fixed rate of interest, to receive back payments based on a floating interest rate index. Each of these series of payments is termed a 'leg', so a typical IRS has both a fixed and a floating leg. The floating index is commonly an interbank offered rate (IBOR) of specific tenor in the appropriate currency of the IRS, for example London Interbank Offered Rate (LIBOR) in USD.

In addition to the most common fixed leg versus floating leg IRS, in some cases, there are also fixed leg versus fixed leg and float leg versus float leg ones.

1.3 Credit Derivatives

To manage credit risks, credit derivatives are designed and applied as tools to manage the risks. The most common one is CDS. Others include CDO etc.

1.3.1 CDS

A credit default swap (CDS) is a financial swap agreement that the seller of the CDS will compensate the buyer in the event of a credit event of the reference, such as debt default, credit rating migration etc.. In another word, the seller of the CDS insures the buyer against the its reference asset. The buyer of the CDS makes a series of payments (the CDS "fee" or "spread") to the seller and, in return, may expect to receive a payoff if the asset happened lost. The most CDS is for defaulting event, for example a loan, the buyer of the CDS receives compensation (usually the face value of the loan), and the seller of the CDS takes possession of the defaulted loan or its market value in cash. However, any person in market can purchase a CDS, even buyers who do not hold the loan instrument and who have no direct insurable interest in the loan (these are called "naked" CDSs).

1.3.2 LCDS

A Loan-only CDS (LCDS) is similar to a standard-form CDS, which acts as a risk-ratio measure. The principal difference between a standard-form CDS and an LCDS is that the reference obligation for an LCDS is a syndicated loan, as opposed to a bond.

A LCDS is an over-the-counter credit derivative. One counterparty—the protection buyer—pays a periodical fixed spread (quoted on an annual basis) to the other one—the protection seller—for the life of the contract. The contract responds to two classes of events that result in early termination of the contract, credit and cancellation events. The set of credit events is defined by the documentation clause of the contract and can include bankruptcy, failure to pay, etc.. If a credit event occurs before the maturity date of the contract, then the deal terminates early and no further spread payments are due. Since the buyer has received protection for a partial period, he or she must make a payment to cover this time; were far to this as the accrued premium. Simultaneously, the seller compensates the buyer to cover his or her losses as a result of the credit event. If a cancellation event occurs, a particular loan is prepaid. The response of the deal to such an event before maturity is similar to a credit event: the deal terminates and no further spread payments are made. However, the seller makes no payment to the buyer.

1.3.3 CDO

A collateralized debt obligation (CDO) is a structured financial product that pools together cash flow-generating assets and repackages this asset pool into discrete tranches that can be sold to investors, who take the corresponding risk and returns of the tranches. A collateralized debt obligation is named for the pooled assets, such as mortgages, bonds, loans, insurance, and even CDS. The tranches in a CDO vary substantially in their risk profiles. For a CDO referenced bonds, the senior tranches are generally safer because they have first priority on payback from the collateral in the event of default. As a result, the senior tranches of a CDO generally have a higher credit rating and offer lower coupon rates than the junior tranches, which offer higher coupon rates to compensate for their higher default risk. One of pricing model can be seen in [6].

1.3.4 CPDO

A constant proportion debt obligation (CPDO) aims at paying high coupons and returning the principal to the investors by putting the capital into a bank and leveraging a nominal credit exposure to indices. The leverage needs to be adapted dynamically to generate high coupon payments (usually, 100–200 basis points above London interbank offered rate (LIBOR)) for investors. Cash-out and cash-in terms are included in a CPDO contract in order to avoid substantial losses and reduce the risk exposure of the portfolio. The cash-out term is a minimal return guarantee to the CPDO investors, while the cash-in term sets the maximal payoff to them.

In a CPDO contract, an investor provides some principal to a CPDO manager (special purpose vehicle (SPV)) by buying a CPDO, and this capital is the initial investment. The CPDO manager then builds a portfolio by putting the capital into a bank account and keeping a position in credit indices (for example, iTraxx or CDX) with the bank account as a nominal to obtain high returns. The manager adjusts the leverage of the CDS dynamically to pay coupons of LIBOR plus a constant spread to the investor, and return the principal at termination. The performance of a CPDO can be characterized by three states: cash-in, cash-out and failure to return the principal at the termination. If the asset value is high enough to cover the present value of all future coupon payments and principal redemption at expiration, then the SPV reduces the exposure to credit indices (the risky exposure) to zero, and puts all the asset in the bank account to receive a risk-free return. This case is called cash-in. When the value of the capital falls below a determined lower boundary (smaller than initial capital), the manager stops the contract and returns all what is left to the investor. This case is called cash-out, which is similar to the one of default. A model for CPDO can be seen in [8].

1.3.5 CCIRS

A Credit Contingent Interest Rate Swap (CCIRS) is a contract which provides protection to the fixed rate payer for avoiding the default risk of the floating rate payer in an IRS contract. The fixed rate payer purchases a CCIRS contract from the credit protection seller at the initial time and the protection seller will compensate the credit loss of the protection buyer in the IRS contract if the default incident happens during the life of the deal. This credit derivative offers a new way to deal with the counterparty risk of IRS. It is similar to CDS, though the value of the its underling IRS can be positive or negative. One of pricing model can be seen [3].

1.3.6 *Credit Spread Options*

Credit spreads are the difference between the yields of borrowers' debts (in the loan and/or bond markets) and those of Treasury bonds, where they have the same maturity date. Because treasury bonds do not default, the credit spreads are the excess yields that investors demand to compensate for the default risk. Using this risk premium provides an effective method to estimate the default risk. As credit default information is various during bond life time, the spread is various as well. A credit spread option protect such risk to a buyer who has a bond with various spread. It is with a particular borrower's credit spread as its underlying asset. The option offers a right to a buyer to gain the exceed spread benefits at a specific time in the future if the borrower's credit spread crosses the threshold stipulated in the contract, if the buyer pay some premium. The special time and the threshold spread are called maturity date and strike spread respectively. The buyer's income is the difference between the market and strike spreads multiplied by the nominal principal on the maturity date. One of pricing model can be seen in [7].

References

1. Basel Accords https://en.wikipedia.org/wiki/Basel_Accords
2. Duffie, D and K. J. Singleton, Credit Risk: Pricing, Measurement, and Management, Princeton University Press 2003
3. Huaying Guo, Jin Liang, The Valuation of CCIRS with a New Design, Proceding of ICCS2018.(2017)
4. Hall, J., OPTIONS, FUTURES, AND OTHER DERIVATIVES, Prentice-Hall, Inc., New Jersey, 1989.
5. Jiang, L., MATHEMATICAL MODELING AND METHODS FOR OPTION PRICING, World Scientific, 2005.
6. Liang, J. and YJ. Zhou, Calibration of Expected Loss from Maket Data of CDO Tranches, Mathematics in Practice and Theory, 41, 2011,1–9. Journal of Economics and Business. 92, 1–9.
7. Xiao, CZ,Valuation of Zero-Coupon Bonds and Credit Spread Options with Credit Rating Migration Risks, Tongji University Thesis,2016
8. Yang, X., J Liang and Y Wu, CPDO with Finite Termination: Maximal Return under Cash-in and Cash-out Conditions, ANZIAM Journal, 57(3), (2016), pp 207–221

Chapter 2
Preliminary Mathematical Theory

In this chapter, we collect some basic mathematical theorems and concepts which will be useful in later chapters. Of course, it is impossible to present these concepts and formulas in details in this short chapter. Our goal is to provide convenience to the readers. For further studies, details can be found in the references provided in this chapter.

2.1 Some Theorems in Functional Analysis

2.1.1 Sobolev Spaces and Embedding Theorems

We begin with some definitions with conventional notations. Let Ω be a domain in $\mathbb{R}^n$, $\Omega_T = \Omega \times [0, T)$. We say

- $u \in L^p(\Omega_T)$, if

$$\int_{\Omega_T} |u|^p dxdt < \infty;$$

- $u \in W^{2,1,p}(\Omega_T)$, if

$$\int_{\Omega_T} (|u_t|^p + |u_{xx}|^p + |u|^p)dxdt < \infty;$$

- $u \in C^{\alpha,\alpha/2}(\overline{\Omega}_T)$, for $0 < \alpha < 1$, if

$$\sup_{(x,t),(x_0,t_0)\in\overline{\Omega}_T,(x,t)\neq(x_0,t_0)} \left(|u(x,t)| + \frac{|u(x,t) - u(x_0,t_0)|}{|x - x_0|^\alpha + |t - t_0|^{\alpha/2}}\right) < \infty;$$

J. Liang, B. Hu, *Credit Rating Migration Risks in Structure Models*,
https://doi.org/10.1007/978-981-97-2179-5_2

- $u \in C^{1+\alpha}(\overline{\Omega})$, for $0 < \alpha < 1$, if

$$\sup_{x,x_0\in\overline{\Omega},(x,t)\neq(x_0,t_0)} \Big(|u(x)| + \frac{|u(x)-u(x_0)|}{|x-x_0|^\alpha} + \frac{|u_x(x)-u_x(x_0)|}{|x-x_0|^\alpha}\Big) < \infty.$$

The parabolic version of the embedding theorems is very similar to the elliptic version when the t derivatives is considered "half the order" of x-derivatives (namely, one t derivative should roughly appear in places of two x-derivatives). There are also embedding theorems with different weights in x and t directions. We only list one here.

In this theorem, we only need to remember that t-direction "takes two-dimensions" when compared with the elliptic version.

Theorem 2.1.1 *Let $u \in W^{2,1,p}(\Omega_T)$, $\partial\Omega \in C^2$. Then*

(1). ([14, p. 80, Lemma 3.3 with $r = 0, s = 1, l = 1$ in (3.15)]), or ([3, p. 29, Theorem 2.3]). *If $1 \le p < n+2$, then for $q = \dfrac{(n+2)p}{n+2-p}$,*

$$\|\nabla_x u\|_{L^q(\Omega_T)} \le C\|u\|_{W^{2,1,p}(\Omega_T)};$$

(2). ([14, p. 80, Lemma 3.3 with $r = 0, s = 0, l = 1$ in (3.15)]), or ([3, p. 29, Theorem 2.3]). *If $1 \le p < (n+2)/2$, then for $q_1 = \dfrac{(n+2)p}{n+2-2p}$,*

$$\|u\|_{L^{q_1}(\Omega_T)} \le C\|u\|_{W^{2,1,p}(\Omega_T)};$$

(3). ([14, p. 80, Lemma 3.3 with $r = 0, s = 0$ and $s = 1, l = 1$ in (3.16)]), or ([3, p. 38, Theorem 3.4]). *If $p > n+2$, then for $\alpha = 1 - \dfrac{n+2}{p}$,*

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} \le C\|u\|_{W^{2,1,p}(\Omega_T)},$$

where C depends on n, p, Ω and lower bounds of T.

Since one x derivative is approximately "half t derivative", it may be necessary in some situations to use fractional derivatives in the study of parabolic equations. The fractional α $(0 < \alpha < 1)$ derivative in t direction as follows: A function u is said to have its fractional α derivative in t in L^p space if and only if the following

$$\Big\{\int_\Omega\int_0^T\int_0^T \Big(\frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^\alpha}\Big)^p \frac{dt d\tau}{|t-\tau|} dx\Big\}^{1/p}$$

is finite. If one allows fractional derivatives, then the embedding theorems can also be extended to these spaces.

For more details about Sobolev spaces, see [1].

2.1.2 Fixed Point Theorems

We collect here several fixed point theorems, which are useful for proving existence of solutions to nonlinear equations and systems.

Theorem 2.1.2 (Contraction Mapping Principle ([9, p. 74, Theorem 5.1])) *Let X be a Banach space and let K be a closed convex set in X. If M is a mapping defined on K such that*

$$Mx \in K \quad \forall x \in K, \tag{2.1.1}$$

$$\sup_{x,y \in K,\, x \neq y} \frac{\|Mx - My\|}{\|x - y\|} < 1, \tag{2.1.2}$$

then M has a unique fixed point in K.

Contraction mapping principle is powerful for solving evolution problems.

Theorem 2.1.3 (Schauder Fixed Point Theorem ([9, p. 280, Corollary 11.2])) *Let X be a Banach space and let K be a* bounded *closed convex set in X. If M is a mapping on K such that*

$$Mx \in K \quad \forall x \in K, \tag{2.1.3}$$

$$M \text{ is continuous}, \tag{2.1.4}$$

$$\overline{MK} \text{ is compact}. \tag{2.1.5}$$

Then M has at least one fixed point in K.

Theorem 2.1.4 (Leray–Schauder Fixed Point Theorem ([9, p. 280, Theorem 11.3])) *Let X be a Banach space and M a mapping on X such that*

$$M : X \to X \text{ is continuous}, \tag{2.1.6}$$

$$M \text{ is compact, i.e., for any bounded set } B,\ \overline{MB} \text{ is a compact set}, \tag{2.1.7}$$

$$\text{the set } \{x \mid x = \lambda Mx \text{ for } \lambda \in [0, 1]\} \text{ is bounded in } X. \tag{2.1.8}$$

Then M has at least one fixed point in X.

2.2 Differential Equations

2.2.1 ODE

An ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and the derivatives of those functions.

In general, a linear ODE of the following form

$$a_0(t) + a_1(t)y + a_2(t)y' + ... + a_n(t)y^{(n)} = 0, \tag{2.2.1}$$

where $a_i(t),\ i = 0, 1, ...n$ are given continuous functions. For initial value problems, $n - 1$ initial conditions $y(0) = y_0, y'(0) = y_1, \cdots, y^{(n-1)}(0) = y_{n-1}$ should be provided.

The simplest example initial value problem is

$$a_0(t) + a_2(t)y' = 0, \quad y(0) = y_0, \qquad (a_2 \neq 0),$$

its solution is given by

$$y(t) = y_0 - \int_0^t \frac{a_0(s)}{a_2(s)} ds;$$

another example is

$$a_1(t)y + a_2(t)y' = 0, \quad y(0) = y_0, \qquad (a_2 \neq 0)$$

which is solved explicitly:

$$y(t) = y_0 e^{-\int_0^t \frac{a_1(s)}{a_2(s)} ds},$$

For more results about ordinary differential equations, see [22].

2.2.2 PDE

If a differential equation involves derivatives of more than one independent variables, it is called a partial differential equation (PDE). Partial differential equations are classified into different types. Among them there are three classical types: hyperbolic, elliptic and parabolic ones. We should emphasize that there are PDEs which do not belong to any of the three types. The books [3, 4] and [12] are excellent textbooks for beginning graduate students.

A typical linear second order PDE of parabolic type can be written as

$$Lu = u_t - a^{ij} D_{ij} u + b^i D_i u + cu = f, \quad \text{in } \Omega_T, \tag{2.2.2}$$

it is assumed to be uniformly parabolic if there exist positive constants λ, Λ such that

$$\lambda|\xi|^2 \leq a^{ij}(x, t)\xi_i \xi_j \leq \Lambda|\xi|^2, \quad \forall (x, t) \in \Omega_T,\ \xi \in \mathbb{R}^n. \tag{2.2.3}$$

Here, we call L a parabolic operator.

The simplest case of a parabolic equation is a heat equation:

$$u_t = a^2 u_{xx}, \tag{2.2.4}$$

where u is the temperature function of position x and time t, a is a constant, called diffusivity of the medium.

2.2.3 Fundamental Solution

A fundamental solution for a linear partial differential operator L is a formulation in the language of distribution theory.

Denote by $\delta(x)$ the Dirac delta "function", a fundamental solution Φ is the solution of the equation

$$L\Phi = 0, \quad x \in R, \ t > 0, \qquad \Phi|_{t=0} = \delta(x).$$

The fundamental solution of the heat Eq. (2.2.4) is

$$\Phi(x,t) = \frac{1}{a\sqrt{4\pi t}} \exp\Big(-\frac{x^2}{4a^2 t}\Big). \tag{2.2.5}$$

If at initial time $t = 0$ the temperature $u(x, 0)$ is a given function $u_0(x)$, then the problem of the heat Eq. (2.2.4) with the initial condition u_0 is called a Cauchy problem, with its solution given by

$$u(x,t) = \int_R \Phi(x-y,t) u_0(y) dy,$$

where Φ is defined in (2.2.5).

Remark 2.2.1 Black-Scholes equation (2.4.4) can be reduced to a heat equation after some transformation.

2.2.4 Weak Solutions and a Weak Maximum Principle

Unless otherwise indicated, in the remainder of this chapter we consider our problems in a bounded domain only. Consider a parabolic equations of divergence form in $\Omega_T \equiv \Omega \times (0, T]$:

$$Lu \equiv u_t - D_j(a^{ij} D_i u + d^j u) + (b^i D_i u + cu) = f + \sum_i D_i f^i, \tag{2.2.6}$$

where (2.2.3) is assumed.

Definition 2.2.1 For $f, c, f^i, d^j, b^i \in L^2(\Omega_T)$ and $g \in L^2[0, T; H^1(\Omega)]$, $u_0 \in H^1(\Omega)$, we say that $u \in L^2[0, T; H^1(\Omega)] \cap L^\infty[0, T; L^2(\Omega)]$ (this space is also known as $V_2(\Omega_T)$) is a *weak solution* of the Dirichlet problem

$$\begin{cases} Lu = f + \sum_i D_i f^i \text{ in } \Omega_T, \\ u = g \qquad\qquad \text{on } \Gamma_T \equiv \partial\Omega \times [0, T], \\ u\Big|_{t=0} = u_0(x), \end{cases} \tag{2.2.7}$$

if u satisfies

$$\begin{cases} \int_0^T\int_\Omega \Big\{ -uv_t + (a^{ij} D_i u + d^j u) D_j v + (b^i D_i u + cu)v \Big\} dxdt - \int_\Omega u_0 v\Big|_{t=0} dx \\ = \int_0^T\int_\Omega fvdxdt - \int_0^T\int_\Omega f^i D_i v dxdt, \ \forall v \in C^1(\overline{\Omega}_T), \ v = 0, \text{ on } \Gamma_T \cup \{t = T\}, \\ u - g \in L^2[0, T; H_0^1(\Omega)]. \end{cases} \tag{2.2.8}$$

Definition 2.2.2 We define *weak subsolution* and *weak supersolution* by replacing the equality above with "$\le$" and "$\ge$" respectively, and further requiring the test function v to be non-negative.

Theorem 2.2.1 (Weak Maximum Principle) *Let the assumption* (2.2.3) *be in force, and* $f, f^i \in L^2(\Omega_T)$*,* $c, b^i, d^i \in L^\infty(\Omega_T)$ *and* $g \in L^2[0, T; H^1(\Omega)]$*,* $u_0 \in H^1(\Omega)$*, and*

$$\int_0^T\int_\Omega (c\phi + d^i D_i\phi) dxdt \ge 0, \quad \forall \phi \in C^1(\overline{\Omega}_T), \phi = 0 \textit{ on } \Gamma_T, \ \phi \ge 0,$$

$$\int_0^T\int_\Omega (f\phi - f^i D_i\phi) dxdt \le 0, \quad \forall \phi \in C^1(\overline{\Omega}_T), \phi = 0 \textit{ on } \Gamma_T, \ \phi \ge 0,$$

$$g \le 0 \quad \textit{on } \Gamma_T, \qquad u_0 \le 0 \quad \textit{on } \Omega.$$

If $u \in L^2[0, T; H^1(\Omega)] \cap L^\infty[0, T; L^2(\Omega)]$ *is a subsolution, then*

$$u \le 0 \quad \textit{in } \Omega_T.$$

This theorem can be derived directly from [16, p. 128, Theorem 6.25].

Remark 2.2.2 The existence of a weak solution requires higher regularity on the coefficients.

2.2.5 Schauder Theory

The Schauder theory is also a powerful tool for the classical solution for parabolic equations.

Consider the Dirichlet problem (2.2.2) with

$$u = g \quad \text{on } \Gamma_T, \tag{2.2.9}$$

$$u\Big|_{t=0} = u_0(x) \quad \text{for } x \in \Omega, \tag{2.2.10}$$

satisfying (2.2.3) and $a^{ij}, b^i, c \in C^{\alpha,\alpha/2}(\overline{\Omega}_T)$ $(0 < \alpha < 1)$ and

$$\frac{1}{\lambda}\left\{\sum_{i,j} |a^{ij}|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + \sum_i |b^i|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + |c|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)}\right\} \le \Lambda_\alpha. \tag{2.2.11}$$

Theorem 2.2.2 (Existence and Uniqueness, Dirichlet ([16, p. 94, Theorem 5.14])) *Let $\partial\Omega \in C^{2+\alpha}$ $(0 < \alpha < 1)$. Suppose that the coefficients in the Eq. (2.2.2) satisfy (2.2.3) and (2.2.11), $f \in C^{\alpha,\alpha/2}(\overline{\Omega}_T)$, $g \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$, and $u_0 \in C^{2+\alpha}(\overline{\Omega})$ satisfies the second order compatibility conditions. Then the problem (2.2.2)–(2.2.10) admits a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$.*

2.2.6 A Strong Maximum Principle

Theorem 2.2.3 ([16, p. 10, Lemma 2.6].) *Suppose that Ω satisfies the interior sphere condition at $x = x_0 \in \partial\Omega$ and $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies*

$$Lu \equiv u_t - a^{ij}D_{ij}u + b^iD_iu + cu \le 0 \quad \text{in } \Omega_T, \tag{2.2.12}$$

where $a^{ij}, b^i, c \in C(\overline{\Omega}_T)$ satisfy the ellipticity condition (2.2.3) and $c \ge 0$. If $u \not\equiv \text{const}$. and takes a non-negative maximum at (x_0, t_0), then

$$\liminf_{\sigma\to 0} \frac{u(x_0 + \sigma\eta, t_0) - u(x_0, t_0)}{\sigma} > 0 \tag{2.2.13}$$

for any direction η such that $\eta \cdot n > 0$, where n is the exterior normal vector.

In the case $c \equiv 0$, the phrase "non-negative maximum" may be replaced by "maximum".

Remark 2.2.3 Ω_T may be replaced by a general set $Q \subset \mathbb{R}^n \times [0, T]$ which satisfies the interior ellipsoid condition (see [16]).