Frontiers in Elliptic and Parabolic Problems

Arnaud Rougirel

# Unified Theory for Fractional and Entire Differential Operators

## An Approach via Differential Quadruplets and Boundary Restriction Operators

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Frontiers in Mathematics

# Frontiers in Elliptic and Parabolic Problems



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## An Approach via Differential Quadruplets and Boundary Restriction Operators

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# Introduction

# 1

## 1.1 Motivations

It appears that the methods for solving entire order and fractional ordinary differential equations are similar [Die10, Chap. 5 & 6]. Regarding partial differential equations, there are also analog results for usual evolutionary partial differential equations and for the corresponding time-fractional equations. In addition, the solvability methods may be similar for equations with a first-order time derivative and for the corresponding fractional equations involving a Riemann–Liouville derivative [ORB21]. When we consider fractional equations involving a Caputo derivative, the solvability methods are more involved than in standard cases, but remain quite similar: see [Zac09, Zac12, FRW22, FKR+21]. Hence, we may expect to set up *a unified theory* for the analysis of entire and fractional derivatives. This is our first motivation.

Let us be a little more explicit about such a theory. Being given a positive real number $b$ and a function $h : (0, b) \to \mathbb{R}$, consider the following three basic differential equations.

$$\mathrm{D}^1 u = u + h \tag{1.1.1}$$

$${}^{\mathrm{RL}}\mathrm{D}^{\alpha} v = v + h \tag{1.1.2}$$

$${}^{\mathrm{C}}\mathrm{D}^{\alpha} w = w + h \tag{1.1.3}$$

where the unknowns $u$, $v$ and $w$ are scalar functions defined on $(0, b)$. Here $\mathrm{D}^1 u$ denotes the first-order derivative of $u$, and ${}^{\mathrm{RL}}\mathrm{D}^{\alpha} v$, ${}^{\mathrm{C}}\mathrm{D}^{\alpha}_{L^p} w$ are respectively, the Riemann–Liouville derivative of $v$ and the Caputo derivative of $w$. We seek an abstract framework in which the latter equations can be rewritten under the form



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$$\mathrm{D}u = u + h, \tag{1.1.4}$$

and (1.1.4) is solvable. Otherwise said, we are interested in a framework allowing to solve the three Eqs. (1.1.1)–(1.1.3) in one shot. Of course, an issue is to define the symbol D appearing in (1.1.4). In general, the cost to pay for an unified theory is an increase of the level of abstraction.

Let us give a second motivation. Starting from the heat equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u$$

where the unknown $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ is a function of the time variable $t$ and the space variable $x$, we consider the following more general evolution equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}u = Lu \tag{1.1.5}$$

where $L$ is an operator on an abstract space $\mathcal{X}$, and $u : (0, \infty) \to \mathcal{X}$. In order to solve (1.1.5), it is assumed that $L$ keeps some properties of the Laplace operator $\Delta$. For example, it is assumed that $L$ is (at least) sectorial [Hen81] or dissipative [Paz83] or self-adjoint [Bre11].

Now our aim is to go one step further by introducing the equation

$$Au = Lu \tag{1.1.6}$$

where $A$ and $L$ are operators acting on another abstract space $\mathcal{Z}$ and $u$ lies in the domains of $A$ and $L$. Proceeding for $A$ as we have done for $L$, we would like that $A$ keeps some properties of the operator $\frac{\mathrm{d}}{\mathrm{d}t}$. There are many works in that direction (see for instance [dPG75, Sho97]). However, we would like that $A$ keeps also some properties of fractional operators. Moreover, we look for an framework allowing to solve boundary value problems at an abstract level.

## 1.2 Some Issues in Fractional Calculus

How many initial conditions? If we consider a first-order initial value problem, it is clear that only one initial condition is needed for well posedness. Also, a second-order initial value problem requires two initial conditions. Now consider a fractional initial value problem of order $\alpha$ where $\alpha$ lies in $(0, 1)$. Then, the issue is to find the suitable number of initial conditions to get a well-posed problem. As we will see in Chap. 7, this number is equal to the dimension of the kernel of the corresponding fractional operator $\mathrm{D}^\alpha$.

What are the boundary values of functions in the domain of $\mathrm{D}^\alpha$? Since we consider only operators acting on functions of the one variable $x$, if $\alpha = 1$ and $x$ lies in the interval

$(0, b)$, then a function $u$ in the domain of $\mathrm{D}^1$ has two boundary values, namely $u(0)$ and $u(b)$. The issue is to define and compute the boundary values in the case where $\alpha$ is not a integer.

This issue is central in this book and is addressed mainly in Chap. 6. Altough the precise representation of these boundary values depends of course on the specific fractional differential operator under consideration, there are some invariants. More precisely, there are always $2n$ boundary values where $n$ is the finite dimension of the kernel of $\mathrm{D}^\alpha$. Furthermore, these boundary values may be splitted into two parts of equal size $n$. A first part is obtained by evaluations of some functions at $x = 0$, whereas the second part is obtained by evaluations at $x = b$.

Another issue is to classify the dozens of fractional time derivatives available in the literature [DGGS20, Sil20]. We have started this task by classifying derivatives built upon the usual first-order derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ and the Riemann–Liouville derivative (see Sect. 6.4).

## 1.3 Unified Theories

### 1.3.1 How to Construct a Unified Theory?

The starting point is to gather well-chosen significant examples. Secondly, identify *the invariants* among these examples, and define classes of objects from these invariants. At this stage, we proceed by induction in the sense that we go from *particular* situations to a *general* case. Notice that there is nothing to prove here.

Then, deduce the properties of these objects, to setup a mathematical theory. Finally, apply the abstract results of the theory to any suitable situations and in particular to the previous examples.

Unified theories are very common in mathematics. As a first example, let us quote *linear algebra*. Abstract vector spaces constitute a class of objects defined from the invariants. Linear maps between vector spaces form another class. Basically, computations in $\mathbb{R}^n$, $\mathbb{C}^n$, or in the space of polynomials are replaced by computations in an abstract vector space. *The theory of symmetric operators* and *the groups theory* are also unified theories.

### 1.3.2 Outline of Our Unified Theory

This theory offers to unify the calculus of standard and fractional derivatives by means of an abstract operator-theoretic approach. Significant examples involve usual first-order derivatives, Riemann–Liouville and Caputo derivatives. We consider the *operators* induced by these derivatives, denoted respectively by $\mathrm{D}^1$, ${}^{\mathrm{RL}}\mathrm{D}^\alpha$ and ${}^{\mathrm{C}}\mathrm{D}^\alpha$. This basic change of point of view (i.e., operators versus derivatives) is of fundamental importance in our theory.

Then, the analysis of these operators (in particular the computation of their adjoint) allows to identify some invariants. The first invariant is that $\mathrm{D}^1$, ${}^{\mathrm{RL}}\mathrm{D}^\alpha$, and ${}^{\mathrm{C}}\mathrm{D}^\alpha$ possess a right inverse. For $\mathrm{D}^1$, this property is well known and is called *the fundamental theorem of calculus*. Then, we consider these operators as instances of an unbounded operator acting on some abstract space. Let us call $\mathbb{A}$ this operator, and suppose for simplicity that $\mathbb{A}$ acts on a (abstract) Hilbert space $\mathcal{H}$. We assume that $\mathbb{A}$ possesses a right inverse, which means that $\mathbb{A}\mathbb{B} = \mathrm{id}_{\mathcal{H}}$ for some operator $\mathbb{B} : \mathcal{H} \to \mathcal{H}$. This statement, which was seen as property in concrete examples, now becomes an axiom of the theory.

The second invariant turns out to concern $\mathbb{B}$ and is the following. The adjoint $\mathbb{B}^*$ of $\mathbb{B}$ is conjugate to $\mathbb{B}$ in the sense that

$$\mathbb{B}^* = \mathbb{S}\mathbb{B}\mathbb{S},$$

where $\mathbb{S}$ is a basic operator on $\mathcal{H}$, namely *a symmetry* (see Sect. 4.2.1). These two invariants led us to introduce the so-called class of *differential triplets* $(\mathbb{A}, \mathbb{B}, \mathbb{S})$ on the Hilbert space $\mathcal{H}$.

When the underlying abstract space is a reflexive Banach space, we have to consider *a differential quadruplet* (see Chap. 5). Then, things are more involved, but the fundamental ideas remain the same.

Let us give two topics developed in this theory. Let us go back for simplicity to the Hilbertian setting where $(\mathbb{A}, \mathbb{B}, \mathbb{S})$ is a differential triplet acting on a Hilbert space $\mathcal{H}$. The first aim is to define abstract boundary values, which we call *(abstract) endogenous boundary values*. Let us first explain the meaning of the adjective *endogenous* in the present context. If we consider a partial differential equations set on an open ball $\Omega$ of $\mathbb{R}^3$, then its solutions live typically in a subspace of $L^2(\Omega)$. However, the boundary values of these solutions lie in general in (a subspace of) $L^2(\partial\Omega)$, which is disjoint from $L^2(\Omega)$. So, in order to define boundary values at an abstract level, it is natural to introduce extra spaces [Cal39]. In this sense, usual boundary values may be described as *heterogeneous*. On the contrary, we introduce a kind of boundary values directly linked to the ambient space $\mathcal{H}$. That is why they are called *endogenous*.

In the case where the kernel of $\mathbb{A}$ has positive (finite) dimension (let us say $n$), it appears that any element in the domain of $\mathbb{A}$ possesses $2n$ *(abstract) endogenous boundary values*, which are coordinates on a subspace of $D(A)$ isomorphic to $\ker \mathbb{A} \times \ker \mathbb{A}$. See Sect. 4.4 and in particular Definition 4.4.2. Moreover, these $2n$ endogenous boundary values may be splitted in two parts. A first part is obtained by projection on the kernel of $\mathbb{A}$ (Sect. 7.2.3), and in applications correspond to initial values. See (4.1.3) and Sect. 7.3. The other $n$ endogenous boundary values of the second part are a little bit more difficult to detect even in a Hilbertian setting. They correspond in applications to "final" boundary values. We refer the reader to Sects. 4.4.2, 5.7.1.3, and 6.4 for more details.

This splitting reveals a main difference between *linear initial value problems* and *boundary values problems*. The former are *unconditionally solvable* (Definition 7.2.9) and use only endogenous boundary values of the first part. On the other hand, the latter are in

general *not unconditionally solvable* and use endogenous boundary values of the second part. There results that roughly speaking only *sublinear initial value problems* are solvable by iterative methods (Sect. 7.6).

A second topic is to establish a quantitative theory of linear endogenous boundary value problems. That is, we will give existence and uniqueness results for linear systems of abstract differential equations supplemented with inhomogeneous endogenous boundary conditions. The equations have the form

$$\mathbb{A}u = Lu + h$$

where $L$ is a bounded operator on $\mathcal{H}$ commuting with $\mathbb{B}$, $h$ lies in $\mathcal{H}$, and the unknown $u$ belongs to the domain of $\mathbb{A}$. It should be emphasized that there is no time variable here since $\mathcal{H}$ is a generic Hilbert space. This quantitative theory is featured in Chap. 7.

The abstract results of these two topics apply to the first-order derivative and to the Riemann–Liouville and Caputo derivatives of order $\alpha$. For example, Eqs. (1.1.1)–(1.1.3) are particular cases of the latter equation. Moreover, these results apply also to more general derivatives built upon *Sonine kernels*. See for instance, Chap. 6 and Sects. 7.3–7.5.

## 1.4 About This Book

### 1.4.1 Preliminary Remarks

The author has tried to ensure that the skills allowing a self-contained reading of this book are that of a postgraduate diploma in mathematical analysis. That said, any additional information should be freely accessible online.

Since this book establishes the axiomatic foundations of a certain branch of the fractional calculus, there results that in principle no a priori knowledge on fractional calculus is needed. However, acculturation to fractional calculus will make reading easier.

One of the main ideas of this book is to state and prove ready-to-use results at an abstract level and to apply these results to entire and fractional derivatives. That shows the deep unity between entire and fractional derivatives.

To make the new objects introduced in this book more easily accessible, we do not always go from general to particular cases. For instance, regarding the unified theory, the Hilbertian setting is covered first, although in many aspects, it is a particular case of the Banach space setting. This approach obviously impacts the way to read this book (see Sect. 1.4.3).

This book is organized into chapters, sections, subsections, paragraphs, and subparagraphs. For instance Sect. 5.7.2.1 refers to the first paragraph in the second subsection of the 7th section in Chap. 5. Moreover, Equation (1.2.17) is the 17th equation in the second section of Chap. 1. Proposition 2.3.5 takes place in the third section of Chap. 2.

### 1.4.2 Outline of the Book

The unified theory is featured in Chaps. 4 and 5. The former deals with differential triplets and boundary restriction operators in a Hilbertian setting. The latter is concerned with differential quadruplets and boundary restriction operators on Banach spaces. The next chapter deals with applications. It is a catalog of differential quadruplets. The boundary values of functions in the domain of various fractional differential operators are computed, and some (fractional) integration by parts formulae are given. Also, adjoints of fractional differential operators are computed.

Chapter 7 is concerned with fractional boundary values problems. That is, in applications, we consider systems of fractional differential equations supplemented with (endogenous) boundary conditions. In accordance with the precited main idea, almost all the existence and uniqueness results are proved at an abstract level. The exception concerns *non linear* problems (Sect. 7.6). The last chapter has a flavor of partial differential equations and deals with abstract and fractional Laplace operators.

Prerequisites are given in Chaps. 2 and 3. These chapters should contain no new material; however, some results are not usual.

### 1.4.3 How to Read This Book?

The answer depends of the aims of the reader. In the sequel, we list some aims and give a corresponding way to proceed.

(i) To get a quick overview of the unified theory and its applications, it is simpler to consider only the Hilbertian setting. So we suggest to read Chap. 4, which deals with the theory of abstract differential triplets and boundary restriction operators. The ideas underlying the formal definitions of these objects are explained. Also, we give some simple examples built upon the usual first-order derivative. There is no fractional calculus in this chapter. Basic applications are given in Sect. 6.2. Regarding differential equations, only the introductory example of Chap. 7.1 takes place on a Hilbert space. However, almost all the theory leading to *initial* values problems may be read without reference to differential quadruplets. It comprises Sects. 7.2.1–7.2.5 and 7.3, 7.4.

(ii) To get a comprehensive overview of the unified theory and its applications, we suggest to read Chaps. 5–8. Notice, however, that the underlying ideas of Chap. 4 are not repeated in Chap. 5. Also the latter contains less examples.

## 1.5 Notation and Conventions

### 1.5.1 Sets of Numbers

The set

$$\{0, 1, 2, 3, \dots\}$$

of non negative integers is denoted by $\mathbb{N}$. *An index* is a non-negative integer. For all integers $i$ and $j$, *the Kronecker symbol* is

$$\delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.1}$$

*The convention of repeated indexes* means that $\sum_{i=1}^{d} x_i y_i$ is abbreviate by $x_i y_i$. That is to say, we set

$$x_i y_i := \sum_{i=1}^{d} x_i y_i.$$

The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Real constants whose value is irrelevant for our purpose are generically denoted by $C$. Any complex number $z$ may be written under the form $z = a + \mathrm{i}b$ where $a$, $b$ lie in $\mathbb{R}$, and $\mathrm{i}^2 = -1$.

The symbol $\mathbb{K}$ denotes any element of the set $\{\mathbb{R}, \mathbb{C}\}$, that-is-to-say $\mathbb{K}$ is equal to $\mathbb{R}$ or $\mathbb{C}$.

Let $m$ and $n$ be integers. If $n < m$ then any sum of the form

$$\sum_{i=m}^{n} \cdots$$

is set to zero.

### 1.5.2 Miscellaneous Notation

Let $b$ be a positive real number and $X$ be a set. For any function $f : (0, b) \to X$, we denote by $f(b - \cdot)$ the function

$$(0, b) \to X, \qquad x \mapsto f(b - x).$$

The symbols := are used to introduce an affectation or a notation. For instance, if $x$ is any number in $\mathbb{K}$, then $x := 1$ means that the value of $x$ is set to 1. Also

$$E := \{x \in \mathbb{R} \mid x + 3 \geq 0\},$$

means that the latter set is denoted by $E$.

### 1.5.3 Matrices

Let $m$ and $n$ be a positive integers. For each index $i$ in $[1, m]$ and $j$ in $[1, n]$, let $a_{i,j}$ belong to $\mathbb{K}$. By

$$(a_{i,j})_{1\leq i\leq m,\, 1\leq j\leq n},$$

we mean the matrix with $m$ lines and $n$ columns, whose entry located at the intersection of the $i^{\text{th}}$ line and $j^{\text{th}}$ column is $a_{i,j}$. The space of such matrices is denoted by $\mathcal{M}(m \times n, \mathbb{K})$. We abbreviate $\mathcal{M}(n \times n, \mathbb{K})$ by $\mathcal{M}(n, \mathbb{K})$, and $(a_{i,j})_{1\leq i\leq n,\, 1\leq j\leq n}$ by $(a_{i,j})_{i,j=1,\ldots n}$.

If

$$M := (a_{i,j})_{1\leq i\leq m,\, 1\leq j\leq n}$$

is any element of $\mathcal{M}(m \times n, \mathbb{K})$ then *its transpose* denoted by $M^t$, is the matrix

$$(a_{j,i})_{1\leq i\leq n,\, 1\leq j\leq m}.$$

Notice that $M^t$ lies in $\mathcal{M}(n \times m, \mathbb{K})$. Moreover, $\mathbb{K}^n$ is identified with $\mathcal{M}(n \times 1, \mathbb{K})$.

### 1.5.4 Spaces

If $\mathcal{X}$ is a vector space and $x_1, \ldots, x_n$ lie in $\mathcal{X}$ then

$$\langle x_1, \ldots, x_n \rangle$$

denotes the subspace of $\mathcal{X}$ generated by $x_1, \ldots, x_n$.

Let $(\mathcal{X}, \|\cdot\|)$ be a normed space over $\mathbb{K}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{X}$ and $x$ lies in $\mathcal{X}$ then the notation

$$x_n \xrightarrow{\mathcal{X}} x$$

means that $(x_n)_{n\in\mathbb{N}}$ converges toward $x$ in $\mathcal{X}$, that-is-to-say

$$\|x_n - x\| \xrightarrow[n\to\infty]{} 0.$$

If no confusion may occur, we will write simply $x_n \to x$.

# Background on Functional Analysis

# 2

## 2.1 Vector Spaces

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be three sets. In general, a map defined on $\mathcal{X} \times \mathcal{Y}$ with values in $\mathcal{Z}$ is denominated by a letter, let us say, $f$. The image of any ordered pair $(x, y)$ of $\mathcal{X} \times \mathcal{Y}$ is denoted by $f(x, y)$. However, in some situations, this image is designated by $\langle x, y \rangle$ or by $(x, y)$. For the basic maps on *vector spaces*, the image of $(x, y)$ is $x + y$ or $xy$. These maps, which we will call *operations*, are denoted by $+$ and . (i.e. the dot symbol). Let $\mathbb{K}$ be equal to $\mathbb{R}$ or $\mathbb{C}$.

### 2.1.1 Basic Definitions

**Definition 2.1.1** Being given a set $\mathcal{X}$, let $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $. : \mathbb{K} \times \mathcal{X} \to \mathcal{X}$ be two maps. The triplet $(X, +, .)$ is called a *vector space over* $\mathbb{K}$ provided the following properties hold true.

(i) $(\mathcal{X}, +)$ is an Abelian group. That is to say, the map $+$ is commutative, associative and possesses an *identity element* denoted by $0_{\mathcal{X}}$ or simply be 0 if no confusion may occur. Moreover, for each $x$ in $\mathcal{X}$, there exists an element $x'$ in $\mathcal{X}$ such that $x + x' = 0_{\mathcal{X}}$.
(ii) For each $(\lambda, \mu)$ in $\mathbb{K} \times \mathbb{K}$ and each $x$ in $\mathcal{X}$, one has $(\lambda\mu)x = \lambda(\mu x)$. Moreover, $1x = x$.
(iii) For each $(\lambda, \mu)$ in $\mathbb{K} \times \mathbb{K}$ and each $(x, y)$ in $\mathcal{X} \times \mathcal{X}$, the following distributivity properties hold:

$$(\lambda + \mu)x = \lambda x + \mu x, \qquad \lambda(x + y) = \lambda x + \lambda y.$$



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A vector space over $\mathbb{K}$ is also called a $\mathbb{K}$-*vector space*. If $(X, +, .)$ is $\mathbb{K}$-*vector space* then the operations $+$ and . are called *(vector) addition* and *scalar multiplication*. □

It is customary to skip the dot symbol for scalar multiplication, that is, we have written and we will write $\lambda x$ instead of $\lambda . x$. Also, the identity element, which we will call *the zero vector of* $\mathcal{X}$, is unique. Most of the time, a $\mathbb{K}$-*vector space* $(\mathcal{X}, +, .)$ will be designated simply by $\mathcal{X}$.

**Definition 2.1.2** Let $\mathcal{X}$ be a $\mathbb{K}$-vector space. A subset $M$ of $\mathcal{X}$ is called a *subspace of* $\mathcal{X}$ if $M$ is non-empty and $M$ is closed under addition and scalar multiplication. That is,

$$x + y \in M, \qquad \lambda x \in M,$$

for each $(\lambda, x, y)$ in $\mathbb{K} \times M \times M$. □

A subspace of a vector space is a vector space. More precisely, if $(X, +, .)$ is a $\mathbb{K}$-vector space and $M$ is a subspace of $\mathcal{X}$, then the restriction of the addition $+$ to the set $M \times M$ induces a map, still labeled $+$, from $M \times M$ into $M$. Similarly, the restriction of the scalar multiplication to $\mathbb{K} \times M$ induces an operation, still labeled . with values into $M$. With these notations, $(M, +, .)$ is a $\mathbb{K}$-vector space.

By restricting the scalar multiplication of a complex vector space, we may pass easily from a complex space to a real space.

**Definition 2.1.3** Let $(\mathcal{X}, +, .)$ be a complex vector space. Denoting by $\circ$ the restriction of the scalar multiplication to $\mathbb{R} \times \mathcal{X}$, the triplet $(\mathcal{X}, +, \circ)$ is a real vector space, which will be designated by $\mathcal{X}_{\mathbb{R}}$. We will call it the *real space associated to* $\mathcal{X}$. □

Let us notice that $\mathcal{X}_{\mathbb{R}}$ contains the same vectors than $\mathcal{X}$. In particular, $\mathrm{i}x$ lies in $\mathcal{X}_{\mathbb{R}}$ for each $x$ in $\mathcal{X}$. However, if $x \neq 0$, then $\mathrm{i}x$ and $x$ are linearly independent in $\mathcal{X}_{\mathbb{R}}$.

The *power set of a set* $E$ is the set of all subsets of $E$. The power set of $E$ is denoted by $\mathcal{P}(E)$. The empty set is an element of $\mathcal{P}(E)$. If we want to remove this set from the power set, we will consider the set $\mathcal{P}(E) \setminus \{\emptyset\}$ instead of $\mathcal{P}(E)$.

Starting from a $\mathbb{K}$-vector space $\mathcal{X}$, operations on $\mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$ may be inferred from vector addition and scalar multiplication.

**Definition 2.1.4** Let $\mathcal{X}$ be a $\mathbb{K}$-vector space. The addition of sets denoted again by $+$ and defined on $\mathcal{P}(\mathcal{X}) \setminus \{\emptyset\} \times \mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$ with values in $\mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$ maps any ordered pair $(U, V)$ of non-empty subsets of $\mathcal{X}$ into the set

$$U + V := \{u + v \mid u \in U,\ v \in V\}.$$

The image $U + V$ is called the *sum of $U$ and $V$*. If $U$ is a singleton, namely if $U = \{x\}$ for some $x$ in $\mathcal{X}$, then we will write $x + U$ instead of $\{x\} + U$.

In a same way, the operation . defined on $\mathbb{K} \times \mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$ with values in $\mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$ maps any ordered pair $(\lambda, U)$ into the set

$$\lambda U := \{\lambda u \mid u \in U\}. \qquad \square$$

By using commutativity and associativity of the vector addition, and the stability of subspaces with respect to addition, this proposition is easily proved.

**Proposition 2.1.1** *Let $\mathcal{X}$ be a $\mathbb{K}$-vector space, and $U$, $V$, $W$ be non-empty subsets of $\mathcal{X}$. Then,*

(i) $U + V = V + U$;
(ii) $(U + V) + W = U + (V + W)$;
(iii) *If $U \subseteq V$ then $U + W \subseteq V + W$;*
(iv) *If $M$ is a subspace of $\mathcal{X}$ then $M + M = M$.*

In the same way, by using properties of scalar multiplication, the following results are easily proved.

**Proposition 2.1.2** *Being given a $\mathbb{K}$-vector space $\mathcal{X}$, let $U$, $V$ be non-empty subsets of $\mathcal{X}$, and $\lambda$, $\mu$ belong to $\mathbb{K}$. Then,*

(i) $\lambda(U + V) = \lambda U + \lambda V$;
(ii) $(\lambda + \mu)U = \lambda U + \mu U$;
(iii) *If $M$ is a subspace of $\mathcal{X}$, then $\lambda M = M$.*

### 2.1.2 Quotient Spaces

Quotient space is a important tool is this chapter since it will be useful for studying the product of operators (see Sect. 2.4.3). Let $\mathcal{X}$ be a $\mathbb{K}$-vector space.

**Definition 2.1.5** Let $M$ be a subspace of $\mathcal{X}$. *The quotient space* of $\mathcal{X}$ by $M$ is the set of all elements $\mathcal{A}$ of $\mathcal{P}(\mathcal{X})$ satisfying $\mathcal{A} = x + M$ for some $x$ in $\mathcal{X}$. Denoting by $\mathcal{X}/M$ this quotient space, we have in symbol

$$\mathcal{X}/M = \{\mathcal{A} \in \mathcal{P}(\mathcal{X}) \mid \exists\, x \in \mathcal{X},\ \mathcal{A} = x + M\}. \qquad \square$$

Clearly, $\mathcal{X}/M$ is a subset of $\mathcal{P}(\mathcal{X})$. If $M := \mathcal{X}$, then $\mathcal{X}/M = \mathcal{X}/\mathcal{X} = \{\mathcal{X}\}$, hence $\mathcal{X}/\mathcal{X}$ contains only one element. On the other hand, if $M = \{0_{\mathcal{X}}\}$, then

$$\mathcal{X}/\{0_{\mathcal{X}}\} = \big\{\{x\} \in \mathcal{P}(\mathcal{X}) \mid x \in \mathcal{X}\big\}.$$

Let us observe that there are one-to-one correspondences between $\mathcal{X}/\mathcal{X}$ and $\{0_{\mathcal{X}}\}$, and between $\mathcal{X}/\{0_{\mathcal{X}}\}$ and $\mathcal{X}$. Since $\{0_{\mathcal{X}}\}$ and $\mathcal{X}$ are vector spaces, we may equip $\mathcal{X}/\mathcal{X}$ and $\mathcal{X}/\{0_{\mathcal{X}}\}$ with a vector space structure. More generally, we will see in Theorem 2.1.5 that every quotient space may be turned into a vector space.

**Proposition 2.1.3** *Let $M$ be a subspace of a vector space $\mathcal{X}$, and let $x$, $x'$ be any vectors of $\mathcal{X}$. Then, the following assertions hold.*

(i) $x' \in x + M \iff x' + M = x + M$.
(ii) *For each $x$ in $\mathcal{X}$, there exists a unique $\mathcal{A}$ in $\mathcal{X}/M$ such that $x$ lies in $\mathcal{A}$. This unique element $\mathcal{A}$ in $\mathcal{X}/M$, denoted by $[x]_{\mathcal{X}/M}$ or simply by $[x]$, is called the* class of x in $\mathcal{X}/M$.
(iii) *For each $x$ in $\mathcal{X}$, $[x] = x + M$.*
(iv) *Let $\mathcal{A}$ belong to $\mathcal{X}/M$. Then for each $x$ in $\mathcal{A}$, one has $[x] = \mathcal{A}$.*

***Proof*** In order to prove (i), let us assume that $x' \in x + M$. Thus, Proposition 2.1.1 (iii) yields that $x' + M \subseteq (x + M) + M$. By Proposition 2.1.1 (ii) and (iv), we have $(x + M) + M = x + M$. Then, $x' + M \subseteq x + M$. By the definition of $x + M$, we infer that $x \in x' + M$, thus we have also $x + M \subseteq x' + M$, so that $x' + M = x + M$. Conversely, since $x' = x' + 0_{\mathcal{X}}$, we derive that $x' \in x' + M$. Assuming $x' + M = x + M$, we deduce $x' \in x + M$, which proves (i).

Since $x \in x + M$, we get the existence part for Item (ii) by choosing $\mathcal{A} := x + M$. If $\mathcal{B} \in \mathcal{X}/M$ contains also $x$, then writing $\mathcal{B}$ under the form $y + M$ for some $y \in \mathcal{X}$, we get that $x \in y + M$. Thus $\mathcal{A} = \mathcal{B}$ by (i). That shows the uniqueness of $\mathcal{A}$.

Item (iii) follows from the existence part in the proof of (ii). Finally, we write any $\mathcal{A}$ in $\mathcal{X}/M$ under the form $x' + M$ for some $x'$ in $\mathcal{X}$, and consider any element $x$ of $\mathcal{A}$. Then $x \in x' + M$. Thus $x + M = x' + M$ by (i), and $[x] = \mathcal{A}$ due to Item (iii). □

Notice that Proposition 2.1.3 may be proved by using *the equivalence relation*: $x \mathscr{R} x' \iff x' \in x + M$.

Let us now equip $\mathcal{X}/M$ with a vector space structure. For each $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{X}/M$ and $\lambda$ in $\mathbb{K}$, $\mathcal{A} + \mathcal{B}$ and $\lambda\mathcal{A}$ are defined through Definition 2.1.4. A priori, these sets belong to $\mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$, hence we have to check more specifically that they live in $\mathcal{X}/M$.

**Proposition 2.1.4** *Let $M$ be a subspace of a vector space $\mathcal{X}$. Then, for each $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{X}/M$ and $\lambda \in \mathbb{K}$, the sets $\mathcal{A} + \mathcal{B}$ and $\lambda\mathcal{A}$ belong to $\mathcal{X}/M$. Moreover,*

$$\mathcal{A} + \mathcal{B} = [x + x'], \qquad \lambda\mathcal{A} = [\lambda x],$$

*for each $(x, x')$ belonging to $\mathcal{A} \times \mathcal{B}$.*

***Proof*** It is enough to prove the two equalities stated in the proposition. In order to establish the first one, let us consider any $x$ in $\mathcal{A}$, and any $x'$ in $\mathcal{B}$. By Proposition 2.1.3 (iv) and (iii), one has $x + M = \mathcal{A}$, $x' + M = \mathcal{B}$. Whence

$$\begin{aligned} \mathcal{A} + \mathcal{B} &= x + x' + M && \text{(by Proposition 2.1.1)} \\ &= [x + x'] && \text{(by Proposition 2.1.3 (iii)).} \end{aligned}$$

In the same way, Propositions 2.1.2 and 2.1.3 entail

$$\lambda\mathcal{A} = \lambda x + \lambda M = \lambda x + M = [\lambda x].$$ □

Proposition 2.1.4 yields that the restriction of the addition to the set $\mathcal{X}/M \times \mathcal{X}/M$ induces a map, still labeled $+$, from $\mathcal{X}/M \times \mathcal{X}/M$ into $\mathcal{X}/M$. Similarly, the restriction of the scalar multiplication to $\mathbb{K} \times \mathcal{X}/M$ induces an operation, still labeled . with values in $\mathcal{X}/M$. This construction leads to the following result.

**Theorem 2.1.5** *Let $M$ be a subspace of a $\mathbb{K}$-vector space $\mathcal{X}$. Then with the above mentioned operations on $\mathcal{X}/M$, $(\mathcal{X}/M, +, .)$ is a $\mathbb{K}$-vector space. Besides, $0_{\mathcal{X}/M} = M$.*

***Proof*** By Proposition 2.1.1, the addition is commutative and associative on $\mathcal{X}/M$. Moreover, for each $\mathcal{A}$ in $\mathcal{X}/M$, one has $\mathcal{A} = x + M$ for some $x$ in $\mathcal{A}$. Thus, Proposition 2.1.1 yields

$$\mathcal{A} + M = (x + M) + M = x + M = \mathcal{A}.$$

Whence $M$ is the *identity element*, that is $0_{\mathcal{X}/M} = M$. Besides, $\mathcal{A} = [x]$ by Proposition 2.1.3 (iii), and $-\mathcal{A} = [-x]$ by Proposition 2.1.4. Thus

$$\begin{aligned} \mathcal{A} + (-\mathcal{A}) &= [0_{\mathcal{X}}] && \text{(by Proposition 2.1.4)} \\ &= M && \text{(by Proposition 2.1.3 (iii)).} \end{aligned}$$

There results that $(\mathcal{X}/M, +)$ is an Abelian group. Finally, the Items (ii) and (iii) in Definition 2.1.1 are easily obtained from Proposition 2.1.2. □

**Example 2.1.6** Let $\Omega$ be a non-empty set, $\mathscr{F}$ be a $\sigma$-algebra on $\Omega$, and $\mu : \mathscr{F} \to [0, \infty]$ be a measure on $\mathscr{F}$. We denote by $\mathcal{X} := \mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K})$ the $\mathbb{K}$-vector space of all *measurable* functions $f : \Omega \to \mathbb{K}$. By *measurable*, we mean that $f^{-1}(U)$ lies in $\mathscr{F}$, for each open subset $U$ of $\mathbb{K}$.

For any $f$, $g$ in $\mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K})$, we recall that $f = g$ *almost everywhere on* $\Omega$ (and abbreviate $f = g$ *a.e. on* $\Omega$) provided there exists some $N$ in $\mathscr{F}$ such that $\mu(N) = 0$ and

$$f = g \qquad \text{on } \Omega \setminus N.$$

Let us now consider the following subset of $\mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K})$

$$M := \left\{ f \in \mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K}) \mid f = 0 \text{ a.e. on } \Omega \right\}.$$

We claim that $M$ is a subspace of $\mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K})$. Indeed, it is clear that $M$ is non-empty and stable under scalar multiplication. In order to check the stability under addition, we notice that for each $f$, $g$ in $\mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K})$, there exist $N_f$ and $N_g$ in $\mathscr{F}$ of zero measure and such that

$$f = 0 \qquad \text{on } \Omega \setminus N_f, \qquad g = 0 \qquad \text{on } \Omega \setminus N_g.$$

Thus,

$$f + g = 0 \qquad \text{on } \Omega \setminus (N_f \cup N_g).$$

Since

$$\mu(N_f \cup N_g) \leq \mu(N_f) + \mu(N_g),$$

we deduce that $f + g$ lies in $M$, which completes the proof of the claim.

We put

$$L^0_\mu(\Omega, \mathbb{K}) := \mathscr{L}^0_{\mathscr{F}}(\Omega, \mathbb{K})/M.$$

Proposition 2.1.3 entails that $[f] = [g]$ if and only if $f = g$ a.e on $\Omega$. It is customary to write $f$ instead of $[f]$. Beside, by Theorem 2.1.5, $L^0_\mu(\Omega, \mathbb{K})$ is a $\mathbb{K}$-vector space.

In the sequel, we will only consider the case where $\Omega$ is an open subset of the Euclidean space $\mathbb{R}^d$ (where $d$ is a positive integer), $\mathscr{F} := \mathscr{F}_{\mathrm{L},\Omega}$ is the $\sigma$-algebra of *Lebesgue measurable subsets of* $\Omega$, and $\mu := \mu_{\mathrm{L},\Omega}$ is the *Lebesgue measure on* $\Omega$ (see, for instance, [Rud87] for more details). Then, the space $L^0_{\mu_{\mathrm{L},\Omega}}(\Omega, \mathbb{K})$ will be denoted by $L^0(\Omega)$. In the particular case where $d = 1$ and $\Omega$ is the interval $(a, b)$ of $\mathbb{R}$, we will write $L^0(\Omega)$ under the simplified form $L^0(a, b)$. □

Theorem 2.1.5 allows linear algebra between quotient spaces. We will give now a basic example of a linear map between some quotient spaces.

**Proposition 2.1.6** *Let $M$ and $V$ be subspaces of a $\mathbb{K}$-vector space $\mathcal{X}$. For each $\mathscr{A}$ in $V/(V \cap M)$, there exists a unique $\mathcal{A}$ in $\mathcal{X}/M$ such that $\mathscr{A} \subseteq \mathcal{A}$. Also, $\mathcal{A} = [x]_{\mathcal{X}/M}$ for each $x$ in $\mathscr{A}$.*

***Proof*** Let $\mathscr{A}$ be in $V/(V \cap M)$ and $x_0$ be in $\mathscr{A}$. In order to show uniqueness, let $\mathcal{A}$, $\mathcal{B}$ be elements of $\mathcal{X}/M$ containing $\mathscr{A}$. Then, $x_0$ lies in $\mathcal{A} \cap \mathcal{B}$, so that $\mathcal{A} = \mathcal{B}$ by Proposition 2.1.3 (ii). The existence goes as follows: using Proposition 2.1.3 once again, we get

$$\mathscr{A} = [x_0]_{V/(V\cap M)} = x_0 + V \cap M \subseteq x_0 + M = [x_0]_{\mathcal{X}/M}. \qquad \square$$

Under the assumptions and notation of Proposition 2.1.6, we define the map $i_V$ by

$$i_V : V/(V \cap M) \to \mathcal{X}/M, \qquad \mathscr{A} \mapsto \mathcal{A}, \tag{2.1.1}$$

where $\mathcal{A}$ is the unique element of $\mathcal{X}/M$ containing $\mathscr{A}$. Then, one has

$$i_V\big([x]_{V/(V\cap M)}\big) = [x]_{\mathcal{X}/M}, \qquad \forall\, x \in V. \tag{2.1.2}$$

Let us notice that (2.1.2) could be used for the definition of $i_V$. However, this mapping is a priory multivalued. Thus, we would have to prove a posteriori that $i_V$ is univoque, i.e., that $[x]_{\mathcal{X}/M}$ is independent of the choice of $x$ in $[x]_{V/(V\cap M)}$. In general, we prefer to avoid, if possible, such reasoning and thus, define maps in a direct way.

Let us now give some properties of $i_V$.

**Proposition 2.1.7** *Let $M$ and $V$ be subspaces of a vector space $\mathcal{X}$. Then, the map $i_V$ defined by* (2.1.1) *is linear and injective.*

***Proof*** Let us start to show that $i_V$ is additive. For each $x$ and $x'$ in $\mathcal{V}$,

$$\begin{aligned} &i_V\big([x]_{V/(V\cap M)} + [x']_{V/(V\cap M)}\big) \\ &\qquad = i_V\big([x + x']_{V/(V\cap M)}\big) \\ &\qquad = [x + x']_{\mathcal{X}/M} && \text{(by (2.1.2))} \\ &\qquad = [x]_{\mathcal{X}/M} + [x']_{\mathcal{X}/M} \\ &\qquad = i_V\big([x]_{V/(V\cap M)}\big) + i_V\big([x']_{V/(V\cap M)}\big) && \text{(by (2.1.2)).} \end{aligned}$$

The identity for the scalar multiplication goes in a same way, so its proof is skipped. Finally, let $x$ belong to $V$ and satisfy

$$i_V\big([x]_{V/(V\cap M)}\big) = 0_{\mathcal{X}/M}.$$

Since $0_{\mathcal{X}/M} = M$ by Theorem 2.1.5, we deduce with (2.1.2) that $x$ lies in $M$. Since $x$ belongs to $V$ as well, we derive

$$[x]_{V/(V\cap M)} = 0_{V/(V\cap M)}.$$

Hence $i_V$ is injective. □

Theorem 2.1.5 allows us to introduce the notion of *co-dimension of a subspace*, which turns out to be a purely algebraic object.

**Definition 2.1.7** Let $\mathcal{X}$ be a $\mathbb{K}$-vector space. We say that a subspace $M$ of $\mathcal{X}$ has *finite co-dimension in* $\mathcal{X}$ if the dimension of $\mathcal{X}/M$ is finite. In that case, the dimension of $\mathcal{X}/M$ is called the *co-dimension of* $M$ *in* $\mathcal{X}$ and is denoted by $\operatorname{codim}_{\mathcal{X}} M$. □

**Proposition 2.1.8** *Let $\mathcal{X}$ be a $\mathbb{K}$-vector space, and $M$, $V$ be subspaces of $\mathcal{X}$. We assume that $M$ is contained in $V$, and that $M$ has finite co-dimension in $\mathcal{X}$. Then, $V$ has also finite co-dimension in $\mathcal{X}$.*

***Proof*** Any $\mathcal{A}$ in $\mathcal{X}/V$ reads $\mathcal{A} = x + V$ for some $x$ in $\mathcal{X}$. By considering a basis $([e_1]_{\mathcal{X}/M}, \ldots, [e_n]_{\mathcal{X}/M})$ of $\mathcal{X}/M$, there exist $\lambda_1, \ldots, \lambda_n$ in $\mathbb{K}$ such that

$$[x]_{\mathcal{X}/M} = \sum_{i=1}^{n} \lambda_i [e_i]_{\mathcal{X}/M}.$$

Since $M \subseteq V$,

$$x \in \sum_{i=1}^{n} \lambda_i e_i + V,$$

so that Proposition 2.1.3 (iv) entails that

$$[x]_{\mathcal{X}/V} = \sum_{i=1}^{n} \lambda_i [e_i]_{\mathcal{X}/V}.$$

Recalling that $\mathcal{A} = [x]_{\mathcal{X}/V}$, we deduce that the vectors $[e_1]_{\mathcal{X}/V}, \ldots, [e_n]_{\mathcal{X}/V}$ generate $\mathcal{X}/V$. □

### 2.1.3 Direct Sums

*Direct sum* is a fundamental tool in the analysis of *differential triplets and quadruplets*. *A direct sum decomposition* generalizes the notion of basis and works in infinite dimensional spaces.

**Definition 2.1.8** Let $\mathcal{X}$ be a $\mathbb{K}$-vector space, $n$ be a positive integer, and $M_1, \ldots, M_n$ be subspaces of $\mathcal{X}$. If

$$M_i \bigcap \big( \sum_{j \neq i} M_j \big) = \{0\}, \qquad \forall\, i = 1, \ldots, n,$$

then, $M_1, \ldots, M_n$ are said to be *in direct sum*, and the set

$$M_1 \oplus \cdots \oplus M_n := \Big\{ \sum_{i=1}^{n} m_i \mid (m_1, \ldots, m_n) \in M_1 \times \cdots \times M_n \Big\}$$

is called the *direct sum of the $M_i$'s*. In the particular case where $n = 2$, $M_1$ and $M_2$ are in direct sum if $M_1 \cap M_2 = \{0\}$. □

**Proposition 2.1.9** *Let $\mathcal{X}$ be a $\mathbb{K}$-vector space, $n$ be a positive integer, and $M_1, \ldots, M_n$ be subspaces of $\mathcal{X}$. Then, the following assertions are equivalent.*

(i) *$M_1, \ldots, M_n$ are in direct sum.*
(ii) *For each $x$ in $\sum_{i=1}^{n} M_i$, there exists a unique $n$-tuple $(m_1, \ldots, m_n)$ in $M_1 \times \cdots \times M_n$ such that*

$$x = \sum_{i=1}^{n} m_i.$$

(iii) *If $(m_1, \ldots, m_n)$ lies in $M_1 \times \cdots \times M_n$ and*

$$\sum_{i=1}^{n} m_i = 0,$$

*then $m_1 = \cdots = m_n = 0$.*

***Proof*** By linearity, (ii) and (iii) are equivalent. Let us prove that (i) implies (iii). For, let $(m_1, \ldots, m_n)$ belong to $M_1 \times \cdots \times M_n$ and satisfy

$$\sum_{i=1}^{n} m_i = 0.$$

Then for each $i = 1, \ldots, n$, one has

$$m_i = \sum_{j \neq i} (-m_j).$$

Thus $m_i = 0$ since the $M_i$'s are in direct sum. Hence (iii) holds true. Conversely let $i$ be any index in $\{1, \ldots, n\}$. Any vector $m_i$ in

$$M_i \bigcap \left( \sum_{j \neq i} M_j \right)$$

reads

$$m_i = \sum_{j \neq i} (-m_j),$$

where $m_j$ lies in $M_j$ for each $j \neq i$. By (iii), there results that $m_i = 0$. □

**Proposition 2.1.10** *Let $\mathcal{X}$ be a $\mathbb{K}$-vector space, and $V$, $M_1$, and $M_2$ be subspaces of $\mathcal{X}$. We assume that*

(i) *$M_1$, $M_2$ are in direct sum;*
(ii) *$V$ and $M_1 \oplus M_2$ are in direct sum.*

*Then $V$, $M_1$, $M_2$ are in direct sum.*

***Proof*** Let $(v, m_1, m_2)$ in $V \times M_1 \times M_2$ be such that

$$v + m_1 + m_2 = 0.$$

By Assumption (ii), $v = m_1 + m_2 = 0$. Next, (i) implies that $m_1 = m_2 = 0$. We conclude thanks Proposition 2.1.9. □

Of course, the direct sum of subspaces is commutative, that is, $M \oplus V = V \oplus M$. An useful situation is when $M$ is given and we can find a subspace $V$ for which $V \oplus M$ is equal to the whole space $\mathcal{X}$. In that case, $V$ is called a *complementary subspace of $M$*. More precisely, we give the following definition.

**Definition 2.1.9** Let $M$ be a subspace of a vector space $\mathcal{X}$. A subspace $V$ of $\mathcal{X}$ is *a complementary subspace of $M$ in $\mathcal{X}$* if $M$ and $V$ are in direct sum and $\mathcal{X} = M \oplus V$. Besides, $M$ and $V$ are said to be *complementary subspaces in $\mathcal{X}$* if $V$ is a complementary subspace of $M$ in $\mathcal{X}$. □

The next result gives a sufficient condition for the existence of a complementary subspace.