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Principles of Locally Conformally Kähler Geometry

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Principles of Locally Conformally Kähler Geometry

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Introduction

> *Writing long books is a laborious and impoverishing act of foolishness: expanding in five hundred pages an idea that could be perfectly explained in a few minutes. A better procedure is to pretend that those books already exist and to offer a summary, a commentary.*
>
> Jorge Luis Borges

It was at the beginning of 1970s when Lieven Vanhecke gave a talk in Iaşi (Romania) about conformal Kähler manifolds. Izu Vaisman, a member of the Faculty of Mathematics of the University of Iaşi at that time, asked what can be said about the local situation. Vanhecke replied: "That's for you to find out". And it is exactly what Vaisman did. In 1976, he published the paper [Va1] in which he introduced the "locally conformal (almost) Kähler manifolds". It was the birth of LCK geometry. In a long series of papers, Vaisman clarified the notion by comparing it with the Kähler manifolds, gave the first examples, and introduced the class of "LCK manifolds with parallel Lee form" (which now bears his name). Other examples were given at the beginning of the 1980s by Franco Tricerri, [Tr]. Starting in 1980, the notion was also considered by several Japanese mathematicians, important results being obtained by Toyoko Kashiwada, Yoshinobu Kamishima, Kazumi Tsukada, etc.

In the first 20 years, this new kind of Hermitian geometry was studied mainly using differential geometry methods. The focus was on Riemannian and conformal properties. For example, a great deal of work was dedicated to isometric submanifolds (totally geodesic, totally umbilical, minimal, totally real, CR etc.). The exception was the paper [Tr] in which the blow-up at points was proved to preserve the LCK class.

Most of these findings were gathered in the monograph [DO] written by Sorin Dragomir and the first author. The present book starts from where the previous one ends.

In our book we combine the methods of algebraic geometry, functional analysis and complex analysis to study the LCK manifolds, which properly belong to differential geometry. Similar to Kähler geometry, the subject often transcends the boundary of its domain, and we arrive at the point when no differential geometry is involved. Still, the background is always differential geometric: we mention several works where the authors study the p-adic versions of LCK manifolds ([Mus, Vos, Schol]), but we never pursue this direction.

We tried to be accessible to beginner students; this is one of the reasons why we cover several preliminary topics, such as foliations, Frobenius theorem, and Ehresmann connections.

We start with the definition of complex and Kähler manifolds, mainly to fix the notation. For an introduction to complex geometry, see [Dem4, GriHa, Mor2, Voi, Huy]. Note that the notation, especially the signs, vary from one author to another. Within this book, we try to be consistent, sometimes without success.

The preliminary requirements for this book vary from part to part. The first part, expanded from several lecture courses, is oriented toward advanced undergraduate and master students. We assume a working knowledge of differential geometry (Riemannian structures, connections, principal bundles, de Rham algebra), topology (de Rham cohomology, Poincaré duality, fundamental groups, local systems), Lie groups and algebras, basic algebraic geometry, basic complex analysis and basic functional analysis. We use Hodge theory, citing, without proof, several key results, such as the Hodge decomposition. We also use, without proofs, elements of the theory of Stein manifolds; Demailly in [Dem4] gives all the tools that are necessary for our use.

The second part treats more advanced subjects and the requirements are much higher. We try to give a basic introduction to several key notions, such as derived functors and the Grothendieck spectral sequence, but a working knowledge of algebraic geometry (in the scope of Demailly's textbook [Dem4]) and homological algebra (Grothendieck's Tohôku paper [Gro3]) is necessary. A graduate student studying algebraic geometry won't find it too specialized, and we tried to lower the requirements by introducing each subject within the text.

The third part is a survey of current research on LCK geometry. We give new proofs of most results, using the methods developed earlier in this book. The third part is oriented towards the researchers who operate with the knowledge of differential geometry required for the subject.

In the first two parts of this book, every chapter is augmented by a sequence of exercises. Originally, the exercises were used in the lecture courses, but we expanded them to include the whole new series, giving an introduction to some subjects (such as elliptic curves and Galois theory). Near the end of the writing, we put a moratorium on adding new chapters; at that moment, all new results we got went to the exercises. This is why some of the exercises are difficult theorems in themselves. There is little consistency or system in the exercises, but we hope that every reader would find something interesting to herself or himself.

The last chapter ("Open questions") is a logical conclusion of the same approach: some of the questions we mention are pretty much impossible to solve, but most of them could be used by an early or advanced graduate student as a part of her or his diploma work. We avoided using general ideas such as "develop a geometric flow" in favour of concrete conjectures, and described the context whenever possible.

Every chapter of the first two parts starts with an introduction, which is more or less independent of the body of the chapter. Its purpose is to set forth the narrative of the main body, explain the history, and place the subject in a wider mathematical context; often, it tells the same story, with more flourish and less mathematical rigour. Most of the definitions given in the introductions are repeated in the respective chapters, in more formal fashion.

With painful trepidation, we avoid several important subjects, most notably, stable bundles, stable coherent sheaves, and Yang–Mills connections: the book is too big as it is. For an introduction to Yang–Mills geometry, see [LT]. Also, most of the concepts relating hyperkähler and hypercomplex geometry to LCK geometry is relegated to "open questions". Finally, we did little justice to Sasakian geometry, which we view as an integral part of our subject. Almost everything related to things Sasakian is beautifully explained in the book [BoG1], by Charles P. Boyer and Krzysztof Galicki, and we did not want to repeat their work.

Thanks

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Bucharest – Rio de Janeiro, August 2022.

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Part I

Lectures in locally conformally Kähler geometry

The Dao gives birth to unity,
Unity gives birth to duality,
Duality gives birth to trinity,
Trinity gives birth to a myriad of things.

Tao Te Ching, by Laozi

Chapter 1

Kähler manifolds

The shift penetrates verses throughout (especially contemporary verses); it is one of the most important parts of the verse. It modifies the word, the stanza, the sounds.

One must establish a special "shift police force" for the prompt capture of shifts, which would leave their authors gaping in amazement.

The shift conveys movement and space.
The shift conveys multiplicity of meanings and images.
The shift is the style of our contemporary life.
The shift is a new discovery of America!..

SHIFTOLOGY OF RUSSIAN VERSE: AN OFFENSIVE AND EDUCATIONAL TREATISE, BY A. KRUCHENYKH

1.1 Complex manifolds

Definition 1.1: Let M be a smooth manifold. An **almost complex structure** is a section $I \in \operatorname{End}(TM)$ that satisfies $I^2 = -\operatorname{Id}_{TM}$.

The couple (M, I) is called an **almost complex manifold**.

Slightly abusing the language, we denote the extension of I to $TM_{\mathbb{C}}$ by I as well. The eigenvalues of this operator are $\pm\sqrt{-1}$. Let $TM_{\mathbb{C}} = T^{0,1}M \oplus T^{1,0}M$ be the corresponding eigenvalue decomposition.

Remark 1.2: In algebraic geometry, one should always make clear that the vector bundle is distinct from its space of sections. One usually uses the notation $\Gamma(B)$ or $H^0(M, B)$ for the space of sections of a vector bundle B. However, in differential geometry one could avoid this distinction (see Section 2.1). Throughout this book, we often use the same letter for the bundle over a smooth manifold and its space of sections.

Definition 1.3: An almost complex structure is **integrable** if $[X, Y] \in T^{1,0}M$ for all $X, Y \in T^{1,0}M$. In this case, I is called **a complex structure**.

L. Ornea, M. Verbitsky, *Principles of Locally Conformally Kähler Geometry*,
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A manifold with an integrable almost complex structure is called **a complex manifold**.

Theorem 1.4: (Newlander–Nirenberg)
An almost complex structure is integrable if and only if the manifold admits an atlas with charts taking values in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$, and with **holomorphic** changes of coordinates.

Remark 1.5: The $C^\infty(M)$-linear map $N: \Lambda^2(T^{1,0}M) \longrightarrow T^{0,1}M$ defined by the commutator is called **the Nijenhuis tensor** of I. One can represent N as a section of $\Lambda^{2,0}M \otimes T^{0,1}M$.

1.2 Holomorphic vector fields

Definition 1.6: A real vector field $Z \in TM$ is called **real holomorphic** if $Z(f)$ is holomorphic for any holomorphic function f defined on some open subset of M.

Remark 1.7: Let $X \in TM$ be a real vector field. Then

$$X = \frac{1}{2}(X - \sqrt{-1}\,IX) + \frac{1}{2}(X + \sqrt{-1}\,IX).$$

Clearly, $X - \sqrt{-1}\,IX \in T^{1,0}M$ (and is called the (1,0) part of X), whereas $X + \sqrt{-1}\,IX \in T^{0,1}M$ (and is called the (0,1) part of X).

Definition 1.8: A (1,0)-vector field $X \in T^{1,0}M$ is called **holomorphic** if its real part is real holomorphic in the sense of Definition 1.6.

Remark 1.9: A (1,0)-vector field $X \in T^{1,0}M$ is holomorphic if and only if $Z(f)$ is holomorphic for any local holomorphic function f defined on some open subset of M. Moreover, taking the (1,0)-part gives a bijective correspondence between real holomorphic vector fields and (1,0)-holomorphic vector fields.

The next proposition is clear.

Proposition 1.10: Let X be a real vector field. The following are equivalent:

(i) X is real holomorphic.

(ii) $\mathrm{Lie}_X I = 0$, where Lie_X denotes the Lie derivative along X.

(iii) The flow generated by X consists in biholomorphic transformations.

Remark 1.11: Let (M, I) be a compact complex manifold. Then the group of biholomorphisms of M, $\mathrm{Aut}(M, I)$, is a Lie group whose Lie algebra is the space of real holomorphic vector fields. See also Exercise 1.2.

Remark 1.12: For any holomorphic vector field X, the fields X and $X^c := I(X)$ commute. Indeed,

$$[X, X^c] = \mathrm{Lie}_X(X^c) = \mathrm{Lie}_X(IX) = \mathrm{Lie}_X(I)(X) + I(\mathrm{Lie}_X(X)) = 0.$$

1.3 Hermitian manifolds

Definition 1.13: A Riemannian metric g on an almost complex manifold (M, I) is called **Hermitian** if $g(IX, IY) = g(X, Y)$. In this case

$$g(IX, Y) = g(I^2X, IY) = -g(X, IY) = -g(IY, X)$$

and hence $\omega(X, Y) := g(IX, Y)$ is skew-symmetric.

Definition 1.14: The differential form $\omega \in \Lambda^2(M)$ is called **the Hermitian form** of (M, I, g).

Remark 1.15: ω is $\mathsf{U}(1)$-invariant, and hence **of Hodge type** (1,1).

Definition 1.16: Let g, g' be two Riemannian metrics on the same manifold M. They are said **conformal** if there exists $f \in C^\infty(M)$ such that $g' = e^f g$. The set $[g] = \{e^f g\,;\ f \in C^\infty(M)\}$ is called **the conformal class** of g. If (M, I) is a complex manifold and one metric in a conformal class is Hermitian with respect to I, then all metrics in the conformal class are so.

1.4 Kähler manifolds

Definition 1.17: A Hermitian manifold (M, I, g, ω) is called **Kähler** if $d\omega = 0$. ω is then called the Kähler form, and its cohomology class $[\omega] \in H^2(M, \mathbb{R})$ is called the **Kähler class**.

1.4.1 Examples of Kähler manifolds

Example 1.18: $\mathbb{C}^n$ with the flat metric $g = \operatorname{Re}\left(\sum dz_i \otimes d\overline{z}_i\right)$, and with Kähler form $\omega = \sqrt{-1}\,\sum dz_i \wedge d\overline{z}_i$.

Example 1.19: The complex projective space. Let $\mathbb{C}P^n$ be the complex projective space, and g a $\mathsf{U}(n+1)$-invariant Riemannian metric. It is called **the Fubini–Study metric on** $\mathbb{C}P^n$. Note that the Fubini–Study metric is defined up to a constant factor. In some textbooks this constant is fixed by requiring that the Hopf fibration $h\ S^{2n+1} \longrightarrow \mathbb{C}P^n$ takes a tangent vector $v \in T_xS^{2n+1}$ orthogonal to a fibre of h to a vector $dh(v) \in T_{h(x)}\mathbb{C}P^n$ of the same length.[1]

The Fubini–Study metric can be obtained by taking an arbitrary Riemannian metric, then averaging it with $\mathrm{U}(n+1)$ using the Haar measure on $\mathrm{U}(n+1)$.

Remark 1.20: For any $x \in \mathbb{C}P^n$, the stabilizer for the $\mathrm{U}(n+1)$ action $\mathrm{St}(x)$ is isomorphic to $\mathsf{U}(n)$. The Fubini–Study metric on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $\mathsf{U}(n)$-invariant, and hence unique up to a constant. As $d\omega\big|_x$ is a $\mathsf{U}(n)$-invariant 3-form on $\mathbb{C}^n$, it has to vanish, because $-\operatorname{Id} \in \mathsf{U}(n)$, and hence $d\omega = 0$.

[1] This is equivalent to asking that the sectional curvature $K(x, y)$ of the Fubini–Study metric satisfies $1 \leqslant K(x, y) \leqslant 4$ for any pair of orthogonal unit tangent vectors; see [Bes].

Remark 1.21: With the same argument, Hermitian symmetric spaces are Kähler.

Example 1.22: The product of two Kähler manifolds is Kähler with respect to the product metric.

Example 1.23: The blow-up at points and along submanifolds of a Kähler manifold is again Kähler.

Example 1.24: Complex submanifolds.

Definition 1.25: An almost complex submanifold $X \subset M$ of an almost complex manifold (M, I) is a smooth submanifold that satisfies $I(TX) = TX$.
If I is integrable, then X is called **a complex submanifold** of M.

Remark 1.26: Let $X \subset M$ be an almost complex submanifold of (M, I), where I is integrable. Then $(X, I\big|_{TX})$ is a complex manifold.

Since exterior differentiation commutes with pullback, the restriction of the Kähler form is closed on each complex submanifold of a Kähler manifold.
In particular, *every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.*

1.4.2 Menagerie of complex geometry

Usually, in algebraic geometry one deals with projective manifolds. There are two wider classes one should consider when studying projective ones.

Example 1.27: A complex manifold that is birational to a projective manifold is called **Moishezon**.
The transcendence degree $a(M)$ of the field $k(M)$ of global meromorphic functions on a compact complex manifold M satisfies $a(M) \leqslant \dim_{\mathbb{C}} M$, as shown by Moishezon, [Moi]; equality here means that M is Moishezon. The number $a(M)$ is also called **the algebraic dimension** of M.

Theorem 1.28: (**Moishezon**, [Moi]) Any Kähler Moishezon manifold is projective.

Example 1.29: Small deformations of Kähler manifolds often result in non-projective Kähler ones (even for a torus and a K3 surface).

The class That includes Moishezon and Kähler manifolds is called **Fujiki class C**, [Fuj]. A manifold is **of Fujiki class C** if it is bimeromorphic to a Kähler manifold. As shown in *loc. cit.*, Fujiki class C is closed under all natural operations that occur in algebraic geometry (such as taking moduli spaces or images).

Remark 1.30: The Kähler minimal model program [HoPe], [CDV], would imply that any Kähler manifold admits a sequence of bimeromorphic fibrations with fibres that are either projective, hyperkähler or tori, and hence the class of Kähler manifolds is probably very restricted. By contrast, the class of complex manifolds is huge.

The fundamental group of a Kähler manifold is very special. On the other hand:

Theorem 1.31: (Taubes [Tau1], **Panov–Petrunin** [PP]) For any finitely generated, finitely presented group Γ, there exists a compact, complex 3-dimensional manifold M with $\pi_1(M) = \Gamma$.

Conjecture 1.32: (Yau [Yau2]) Let (M, I) be a compact almost complex manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Then I can be deformed to a complex structure.

Remark 1.33: The following important result of Gromov ([Gr2, p. 103]) can be cited to support this conjecture. Let M be a non-compact almost complex manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Then M admits a complex structure.

Remark 1.34: The (non-)existence of a complex structure is highly non-trivial even in the simplest cases, such as S^6. See [HKP], [CDV]. On the other hand, S^6 has a natural almost complex structure constructed out of octonions, see [Bal] for example, that admits a compatible **nearly-Kähler** metric, which means that ∇I is antisymmetric, see [BFGK], [Bry].

Remark 1.35: It is known that non-Kähler complex manifolds are much more abundant than Kähler ones, except in complex dimension 2, where non-Kähler manifolds are few and much better understood than projective ones. However, it is very hard to come up with new examples of compact, non-Kähler complex manifolds.

1.5 Exercises

1.5.1 Kähler geometry and holomorphic vector fields

1.1. Let G be a compact, complex, connected Lie group. Prove that G is abelian.

1.2. Let X be a holomorphic vector field on a complex manifold, that is, one that satisfies $\operatorname{Lie}_X I = 0$. Prove that IX is also holomorphic.

Hint: $\operatorname{Lie}_X I = A(I)$, where $A = \nabla(X)$ acts by the formula $A(I)(v) = A(Iv) - IA(v)$. Therefore, *X is holomorphic if and only if $\nabla(X)$ is complex linear.* Since $\nabla(I) = 0$, one has $\nabla(IX) = I(\nabla(X))$, and hence $\nabla(X)$ is complex linear, implying that $\nabla(IX)$ is complex linear.

1.3. Let (M, I, g, ω) be an almost complex Hermitian manifold, ∇ a torsion-free connection. Assume $\nabla\omega = 0$ and $\nabla I = 0$. Then (M, I, g, ω) is Kähler.

Hint: Use $[X, Y] = \nabla_X Y - \nabla_Y X$, to show that $T^{1,0}$ is involutive, then use $d\omega = \operatorname{Alt}(\nabla\omega)$, which holds for torsion-free connections.

1.4. Let (M, I) be an almost complex manifold, and $d^c := IdI^{-1}$. Prove that I is integrable if and only if $dd^c = -d^c d$.

1.5. Let ω be a non-degenerate 2-form on a Riemannian manifold, and ∇ its Levi–Civita connection. Assume that $\nabla(\omega) = 0$. Prove that M admits a complex structure I such that $\nabla(I) = 0$.

1.6. Let (M, I) be an almost complex manifold, $\dim_{\mathbb{C}} M = n$, $U \subset M$ a dense, open subset, and $\Omega \in \Lambda^{n,0}(U)$ a non-degenerate $(n, 0)$-form. Assume that $d\Omega = 0$. Prove that the almost complex structure I is integrable.

1.7. A **holomorphic differential** on an almost complex manifold is a closed $(1, 0)$-form. Let G be a finite group acting on $M = \mathbb{C}^n$ by holomorphic maps. Prove that M admits a non-zero G-invariant holomorphic differential.

1.8. Let $M = \mathbb{C}P^{2m} \times \mathbb{C}P^{2n}$, for some $m, n \in \mathbb{Z}^{>0}$. Prove that M does not admit a Kähler structure with non-standard orientation.

Hint: Prove that all complex manifolds have a canonical orientation. Prove that this orientation can be given by the top power of the Kähler form, if the manifold is Kähler.

1.9. Let $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ be the so-called linear Hopf manifold, with A an invertible linear operator with operator norm $\|A\| < 1$. Prove that M admits no symplectic structures.

1.10. Let M be a complex manifold, admitting a non-zero holomorphic vector field ξ with zero set Z. Suppose that Z is non-empty and zero-dimensional. Prove that the topological Euler characteristic of M is non-negative.

1.11. Let M be a compact Kähler manifold, and $\mathrm{Alb}(M) := \frac{H^0(\Omega^1(M))^*}{\Lambda}$, where $H^0(\Omega^1(M))$ is the space of holomorphic differentials, and Λ the group generated by all integrals over integer homology classes. The group $\mathrm{Alb}(M)$ is called **the Albanese variety** of M, denoted by $\mathrm{Alb}(M)$. Prove that:

(a) The lattice $\Lambda \subset H^0(\Omega^1(M))^*$ is discrete and cocompact, and hence $\mathrm{Alb}(M)$ is a compact complex torus.

(b) Fix $m \in M$. Consider the map $\Psi : M \longrightarrow \mathrm{Alb}(M)$, taking $x \in M$ to the map $\theta \mapsto \int_\gamma \theta$, where γ is any path-connecting m to x. Prove that Ψ is defined unambiguously and is holomorphic.

1.5.2 The Lie algebra of holomorphic Hamiltonian Killing fields

The following exercises are not elementary; we advise the less prepared reader to skip them. We will sometimes refer to these statements later in this book. The reader can find these results in [Bes, Chapter 2.H].

1.12. Let (M, ω) be a compact Kähler manifold, $G(M)$ the group of its holomorphic diffeomorphisms, $\mathfrak{g} \subset TM$ the Lie algebra of vector fields satisfying $\nabla(X) = 0$, where ∇ is the Levi–Civita connection, and G its Lie group.

(a) Prove that $G \subset G(M)$.

(b) Prove that all its orbits are compact complex tori in M.

(c) Prove that $i_X\omega$ is the real part of a holomorphic 1-form for any $X \in \mathfrak{g}$.

(d) Let $G_1 \subset G(M)$ be the group of holomorphic diffeomorphisms acting trivially on $\mathrm{Alb}(M)$. Prove that for $X \in \mathfrak{g}$ there is a holomorphic 1-form η such that $\langle \eta, X\rangle \neq 0$. Use this to prove that $G_1 \cap G = \{\mathrm{Id}\}$.

1.13. Let X be a vector field on a Riemannian manifold (M, g). Prove that X is Killing if and only if the operator $A := \nabla(X) \in \mathrm{End}(TM)$ taking Y to $\nabla_Y(X)$ is antisymmetric: $g(A(x), y) = -g(x, A(y))$ for all $x, y \in T_mM$.

1.14. Consider the standard Euclidean form g and the Hermitian form ω on a vector space $V = \mathbb{R}^{2n} = \mathbb{C}^n$. Let $A \in \mathrm{End}_{\mathbb{R}}(V)$ be a matrix that satisfies $\omega(Ax, y) = -\omega(x, Ay)$ and $g(Ax, y) = -g(x, Ay)$, and $\omega(IAx, y) = -\omega(x, IAy)$ for all $x, y \in V$. Prove that $A = 0$.

1.15. Let $X, Y \in TM$ be holomorphic vector fields on a Kähler manifold, and ∇ the Levi–Civita connection.

(a) Prove that $\nabla_{IX}Y = I\nabla_X Y$.

(b) Let $A(\cdot) := \nabla_X\cdot$. Assume that $\nabla_X\omega = \nabla_{IX}\omega = 0$. Prove that $\omega(IA\cdot,\cdot) = -\omega(\cdot, IA\cdot)$ is equivalent to $\mathrm{Lie}_{IX}\,\omega = 0$.

Hint: Prove that $\nabla_{IX} - \mathrm{Lie}_{IX} = IA$, and show that $\nabla_{IX}\omega - \mathrm{Lie}_{IX}\,\omega = \omega(IA\cdot,\cdot) + \omega(\cdot, IA\cdot)$.

1.16. Let (M, I, ω) be a Kähler manifold and let $\mathfrak{g} \subset TM$ be the subalgebra formed by vector fields $X \in TM$ such that $\mathrm{Lie}_X I = 0$, and $\mathrm{Lie}_X\,\omega = \mathrm{Lie}_{IX}\,\omega = 0$.

(a) Prove that for all $X \in \mathfrak{g}$, one has $\nabla(X) = 0$.

(b) Let Y be a holomorphic vector field that satisfies $\nabla(Y) = 0$. Prove that $Y \in \mathfrak{g}$.

(c) Prove that $\mathfrak{g}$ is abelian, if M is compact.

Hint: Use Exercise 1.13, Exercise 1.15 and Exercise 1.14 for the first part.

1.17. Let ∇ be a torsion-free connection on a complex manifold (M, I), satisfying $\nabla(I) = 0$. Prove that a vector field V is holomorphic if and only if $\nabla(V) \in \mathrm{End}(TM)$ commutes with $I \in \mathrm{End}(TM)$.

Hint: Prove that $\nabla_{IZ}(V) = I(\nabla_V(Z)) + [V, IZ]$ for any vector fields Z, V on M. Use $\nabla_V(Z) + [V, Z] = \nabla_Z V$.

1.18. Let v be a vector field on a Kähler manifold (M, I, ω), satisfying $\mathrm{Lie}_v\,\omega = 0$, and $\eta := i_v(\omega)$ the corresponding 1-form, that is closed by Cartan formula. Let ∇ be the Levi–Civita connection. Prove that $\nabla\eta \in \mathsf{Sym}^2(T^*M)$ for any closed 1-form η. Prove that v is holomorphic if and only if the 1-form $\eta := i_v(\omega)$ satisfies $\nabla\eta \in \mathsf{Sym}^{1,1}(T^*M)$, that is, $\nabla\eta$ is of type (1,1) with respect to the Hodge decomposition on $\mathsf{Sym}^*(T^*M)$.

Hint: Use the previous exercise.

1.19. Let η be a closed real 1-form on a compact Kähler manifold that satisfies $\nabla\eta \in \mathsf{Sym}^{1,1}(T^*M)$. Consider the decomposition $\eta = \alpha + \eta'$ where η' is exact and α the real part of a holomorphic 1-form. Prove that $\nabla\eta' \in \mathsf{Sym}^{1,1}(T^*M)$ and $\nabla\alpha = 0$.

Hint: Prove that the harmonic decomposition commutes with the connection and use this to show that $\nabla\alpha \in \mathsf{Sym}^{1,1}(T^*M)$. Show that $\nabla\alpha \in \mathsf{Sym}^{2,0}(T^*M)$ when α is a closed holomorphic 1-form.

Definition 1.36: Let (M, ω) be a symplectic manifold. We denote the contraction of a differential form η with a vector field v by $i_v(\eta)$. A vector field v on M is called **Hamiltonian** if $\operatorname{Lie}_v \omega = 0$ and the 1-form $i_v(\omega)$ is exact.[2]

1.20. Let v be a holomorphic Killing vector field on a compact Kähler manifold M. Prove that there exists a unique Hamiltonian holomorphic vector field v' such that $\alpha := i_{v+v'}(\omega)$ is the real part of a holomorphic 1-form. Prove that v is Hamiltonian if and only if $\alpha = 0$.

Hint: Use the previous exercise.

1.21. Let $\mathfrak{s}$ be the space of closed 1-forms on a compact Kähler manifold that satisfy $\nabla\eta \in \mathsf{Sym}^{1,1}(T^*M)$, $\mathfrak{g} \subset \mathfrak{s}$ the space of parallel 1-forms in $\mathfrak{s}$, and $\mathfrak{h}$ the space of all exact 1-forms in $\mathfrak{s}$.

(a) Show that any exact 1-form on a compact manifold M vanishes somewhere in M.

(b) Prove that $\mathfrak{g} \cap \mathfrak{h} = 0$.

(c) Use Exercise 1.20 to show that $\mathfrak{g} \oplus \mathfrak{h} = \mathfrak{s}$.

1.22. Denote by $\mathfrak{h}$ the algebra of Hamiltonian holomorphic vector fields (they are a posteriori Killing) on a compact Kähler manifold (M, I, ω), and by $\mathfrak{s}$ the Lie algebra of holomorphic Killing vector fields. Prove that $\mathfrak{h}$ is normal in $\mathfrak{s}$, and the quotient $\mathfrak{s}/\mathfrak{h}$ is isomorphic to the Lie algebra $\mathfrak{g}$ of holomorphic parallel vector fields (Exercise 1.16). Prove that $\mathfrak{s}$ can be obtained as a semi-direct product of $\mathfrak{g}$ and $\mathfrak{h}$.

1.23. Prove that a holomorphic Killing vector field on a compact Kähler manifold is Hamiltonian if and only if it has a fixed point.

Hint: Use the decomposition $\mathfrak{g} \oplus \mathfrak{h} = \mathfrak{s}$.

[2]The form $i_v(\omega)$ is closed, because $\operatorname{Lie}_v \omega = d(i_v(\omega)) = 0$.