Probability Theory and Stochastic Modelling 106

Yves Le Jan

Random Walks and Physical Fields

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Preface

C'est un chemin fâcheux borné de peu d'espace,
Tracé de peu de gens que la ronce pava,
Où le chardon poignant ses testes esleva,
Pren courage pourtant, et ne quitte la place.

The aim of this book is to introduce some basic objects and properties related to random walks on graphs and to present some fundamental relations between these objects and random fields used in mathematical physics.

These objects include *Markov loops, spanning forests, random holonomies and covers*.

Those random fields arise in mathematical models of statistical physics (e.g. Ising model, percolation) and in constructive quantum field theory.

Quantum field theory (particle physics) uses three kinds of fields: *Bose fields, Fermi fields and Gauge fields* which are operator-valued fields indexed by space-time and satisfying Poincaré invariance. Performing a "Wick rotation", i.e. considering a purely imaginary time, produces a set of fields satisfying Euclidean invariance in which Bose fields and gauge fields commute and can be viewed as random fields while Fermi fields anticommute as 1-forms or Grassmann variables. Besides, this set of fields satisfies a reflection positivity (or Osterwalder-Schrader positivity) property that allows reconstruction of the quantum fields.

However, interactions between the fields and the subsequent renormalization are sources of great mathematical difficulties, especially in dimension higher than two. These difficulties have been overcome, to a certain extent, in dimension two and three only. One easy way to overcome them, at least from a practical point of view, is to replace the continuous space-time by a discrete lattice. This has been used extensively in physics as a non-perturbative approach to quantum field theory. The lecture notes of Ehrard Seiler [72] are representative of the mathematical background of this approach. It was suggested there that the construction of a continuum limit might appear more feasible if one considers the extended objects, such as loop holonomies, associated with the fields, rather than the fields themselves. In particular, random loops and bridges had been introduced as useful tools in constructive field theory and statistical physics models (e.g. in [81] and [6]).

One purpose of this book is to develop this idea, taking into account more recent developments within probability theory. These fields and the random objects mentioned hereinabove are indeed related: random loops to Bose fields, spanning forests to Fermi fields, holonomies and covers to gauge fields.

We will introduce Poissonian ensembles of Markov loops, their occupation fields and their holonomies, uniform spanning trees, Fermi fields and gauge fields, i.e. connections. Some important properties of these objects will be derived, as well as the relations of Markov loop ensembles with other statistical models such as Ising model, random flows, configuration models and combinatorial maps. The ambition here is not to propose an encyclopedia on these topics (it would require several volumes) but to introduce them as aspects of the same mathematical scenery. An interesting feature is that interactions with gauge fields, in the case of discrete gauge groups, are interpreted in terms of lift to cover spaces. However, many essential aspects of lattice field theory are not mentioned here and have still to be interpreted in this context (e.g. the use of different fermionic actions, confinement properties, thermodynamic limit and phase transition, renormalization and scaling limits of correlation functions etc.). This work remains in many respects woefully incomplete. Nevertheless, one can hope this introduction may encourage those with a sense of adventure and an interest in mathematical physics to follow this demanding path.

Several sections propose an improved presentation of a large part of the results published in [34, 35] where the main emphasis was put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These ensembles appeared informally already in [81], and were defined in [29] for planar Brownian motion in relation with SLE processes. Note however that topics related to the Brownian case, in particular renormalization results given in Chapter 10 of [35] for two-dimensional Brownian loops and in [45] for Lévy processes, are not included here. Our framework is essentially discrete, which allows to avoid heavy technicalities.

New material includes results published in [37–39, 41, 44] [5, 40, 48, 50] and [42]. A mini course on this topic was given at NYU-Shanghai in 2018.

The text is essentially self-contained, but the reader is assumed to be familiar with basic notions of probability, Poisson point processes and discrete Markov chains.

Sections marked by a star * are not referred to in the subsequent sections.

Let us give an overview of the most notable results presented in the book.

In the first chapter, we review some basic notions of Markovian potential theory in the simple context of a finite or countable graph $\mathcal{G} = (X, E)$, equipped with conductances C and killing rates κ. These notions include the energy, the Green function, the heat semigroup, the continuous time Markov chain and Feynman-Kac formula. They are the discrete analogues of classical potential theoretical notions whose probabilistic interpretations involve Brownian motion.

The next six chapters are dedicated to the study of loop ensembles and some related statistical physical models. The second chapter introduces the Markov loop measure μ and the related Poisson processes of loops $\mathcal{L}_\alpha$ of intensity $\alpha\mu$, often referred to as loop soups. The third chapter deals with decompositions induced by splitting the set X of graph vertices in two parts, D and F, the energy and the Markov chain being decomposed into its restriction to D and its trace on F. Excursion decomposition is applied to loop ensembles. In Chap. 4, occupation fields for vertices and oriented edges are introduced and their distributions are computed. Chapter 5 focuses on loop clusters distribution and the related notion of Markovian percolation, which generalizes the well-known i.i.d. percolation. The Gaussian free field ϕ is defined in Chap. 6, and the identity in law between the vertex occupation field of the loop ensemble $\mathcal{L}_{\frac{1}{2}}$ and $\frac{1}{2}\phi^2$ is proved . This formula, which appeared in [43] and [34], is related to an identity combining bridge local times with the Gaussian free field which is known as Dynkin's (or BFS Dynkin's) isomorphism (cf [6, 13, 32]). In Chap. 7, we show that edge occupation fields have remarkable distributions for intensity 1 (considering oriented edges) and $1/2$ (considering non-oriented edges). Moreover, after conditioning by the vertex occupation field, the loop ensemble of intensity 1 defines a remarkable distribution on flows with integral intensity defined on the graph. We also show, following [48] and [50], that after conditioning by the vertex occupation field, the clusters of the loop ensemble of intensity $1/2$ can be used to construct the F-K Ising model, which provides a coupling between this loop ensemble and the real free field. A relation is also established between these loops ensembles and a configuration model.

Chapters 8 and 9 focus on spanning trees and Fermi fields. In Chap. 8, we introduce loop-erased random walks and present an extended version of Wilson's algorithm which yields a loop ensemble of intensity 1 and a spanning forest of the graph. They are independent. We then show how a remarkable distribution on combinatorial maps can be derived from the configuration model. Discrete loops of $\mathcal{L}_1$ can be constructed as face contours of this random combinatorial map. In Chap. 9, we define fermionic fields from creation and annihilation operators on skew symmetric Fock space, and show how they can be used to prove two versions of the transfer current theorem for spanning trees. We then apply these results to complete graphs and get some asymptotic results for their spanning forests as the number of vertices increases to infinity. We also establish the supersymmetry relation with the corresponding bosonic fields, which are identified to the Gaussian free fields. We finally give an example of an interaction between trees and loops, which can be represented by a local interaction between bosonic and fermionic fields.

Chapter 10 focuses on topological properties of loops and graphs. Notions of universal cover and fundamental group are introduced. We show there is a one-to-one correspondence between geodesic (i.e. non-backtracking) loops and conjugacy classes of the fundamental groups. Distributions of loop homotopy classes and of homologies are studied. Galois covers, which are intermediate between the graph and its universal cover, are introduced.

In Chap. 11, given a group N, we introduce N-connections on a graph, loop holonomies and associated bosonic and fermionic field. When the group is discrete, connections induce Galois covers. Loops on the cover correspond to loops with trivial holonomy. Loops and spanning forests on the cover are related to bosonic and fermionic fields which can be decomposed using group representation theory into fields interacting with the connection.

We introduce Yang–Mills measure on discrete tori and, on any graph, the measure on connections given by the expectation of the product of holonomies of a loop ensemble. We observe that for high intensity and high killing rate this measure can approximate the Yang–Mills measure.

Then we prove that trace of holonomies determine an intertwining relation between merge-and-split generators on loop ensembles (which were introduced in Chap. 7) and Casimir operators on connections. By adding a deformation part to the generator on loops, this result is extended to the Casimir operator modified in order to be self-adjoint with respect to Yang-Mills measure. This relation contains the Schwinger-Dyson equation previously obtained as an essential step in the proof of the t'Hooft expansion for large $d = n$. In continuous spaces, such equations, which originate from physics, are often referred to as Markeenko-Migdal equations.

We conclude with Chap. 12 in which reflection positivity properties are established for all these fields, allowing to construct a physical Hilbert space, on which quantum observables can operate.

The (short) list of citations is mostly a list of works from which we remember that we collected some elements presented in this text. It is certainly very imperfect and does not pretend to be exhaustive, nor to describe the "history" of the different topics introduced in this book.

Orsay, France Yves Le Jan
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Contents

Chapter 1
Markov Chains and Potential Theory on Graphs

In this first chapter, we review briefly the basic notions of Markovian potential theory in the context of a countable graph $\mathcal{G} = (X, E)$, equipped with conductances C and damping (or killing) rates κ. These notions include the energy, the Green function, the Dirichlet space, capacitary potentials, the heat semigroup, Fokker–Planck equations, the continuous time Markov chain and Feynman–Kac formula. They are the discrete analogues of classical potential theoretical notions whose probabilistic interpretations involve Brownian motion (see for example [3, 11, 12, 17, 68]).

1.1 Graphs and Markov Chains

Our basic object will be a finite or countable space X and a set of *nonnegative conductances* $C_{x,y} = C_{y,x}$, indexed by pairs of distinct points of X. We say that $\{x, y\}$, for $x \neq y$ belonging to X, is an edge iff $C_{x,y} > 0$. The points of X together with the set of non oriented edges E define a graph $\mathcal{G} = (X, E)$. We say that X is the set of vertices of $\mathcal{G}$.

An oriented edge (x, y) is defined by the choice of an ordering in an edge. We set $-(x, y) = (y, x)$ and if $e = (x, y)$, we denote it also by (e^-, e^+). The edge $\{x, y\}$ is also denoted by $\pm(x, y)$. The degree d_x of a vertex x is by definition the number of edges incident at x.

Given two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of , a bijection j from X_1 onto X_2 is a graph isomorphism iff $j \times j$ maps E_1 onto E_2.

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We Generally Assume this Graph Is Connected The set of oriented edges is denoted by E^o. It will always be viewed as a subset of X^2, without reference to any imbedding. We say the graph is regular if all vertices have the same degree.

An important example is the case in which the conductances of the edges are equal to 1. Then the conductance matrix is the adjacency matrix of the graph: $A_{x,y} = 1_{\{x,y\} \in E}$.

If a graph is finite, its characteristic polynomial, eigenvalues and eigenspaces are the ones associated with its adjacency matrix.

A complete graph is defined by taking all conductances equal to one.

The complete graph with n vertices is denoted by K_n. The complete graph K_4 is the graph defined by the tetrahedron. K_4 is planar (i.e. can be imbedded in a plane), but K_5 is not.

Other common examples are, for $d > 2$ the d-regular tree, the lattice $\mathbb{Z}^d$ and the discrete tori $(\mathbb{Z}/p\mathbb{Z})^d$, for $p > 2$.

Together with the conductances, *we suppose given a damping (or killing) rate, i.e. a nonnegative function κ on X. Define $\lambda_x = \kappa_x + \sum_y C_{x,y}$. We will assume λ everywhere finite.*

The standard examples are the cases where (X, E) is a regular graph with uniform degree d, and we have unit conductances and κ a nonnegative constant.

Setting $P_{x,y} = \frac{C_{x,y}}{\lambda_x}$ we associate to the pair (C, κ) a stochastic or substochastic transition matrix P which is λ-symmetric i.e. such that

$$\lambda_x P_{x,y} = \lambda_y P_{y,x}$$

(this property is also referred to as detailed balance) with $P_{x,x} = 0$ for all x in X.

It defines an irreducible discrete time Markov chain on X. If κ does not vanish we add as usual an extra point Δ in which the chain is absorbed. We set $P_{x,\Delta} = \frac{\kappa_x}{\lambda_x}$ and $P_{\Delta,\Delta} = 1$. The graph can be extended into the graph $\mathcal{G}_\Delta$ obtained by adding to $\mathcal{G}$ the vertex Δ and edges $\{x, \Delta\}$ for vertices such that $\kappa_x > 0$.

In the standard examples, $P_{x,y} = \frac{1}{d+\kappa} A_{x,y}$.

In what follows, unless the converse is explicitly mentioned, we will assume that the transience property holds. It is well known it can be formulated in three equivalent ways:

(a) The Markov chain defined by P visits every vertex at most finitely many times.
(b) For any $x \in X$, $\sum_{n=0}^{\infty} [P^n]_{x,x} < \infty$.
(c) For any $x, y \in X$, $\sum_{n=0}^{\infty} [P^n]_{x,y} < \infty$.

Note that when the graph is finite and connected, this transience hypothesis is equivalent to assuming that κ does not vanish everywhere.

1.2 Green Matrices and Hitting Distributions

With this transience assumption, we can now define the *Green matrix* (or Green function) G on X^2:

$$G_{x,y} = G_{y,x} = \frac{1}{\lambda_y} \sum_{n=0}^{\infty} [P^n]_{x,y}.$$

Note that setting $M(\lambda)_{x,y} = \lambda_x \delta_{x,y}$, we have $M(\lambda)P = C$ and therefore:

$$M(\lambda)G = CG + I. \tag{1.1}$$

The diagonal matrix $M(\lambda)$ will sometimes be denoted by M_λ. Sometimes, it can be convenient to denote the Green matrix as a function: $G_{x,y} = G(x, y)$.

The Green matrices defined by conductances equal to 1 and all positive constants $\kappa = u$ define the resolvent of the graph.

Example 1.1 The Green matrix of the complete graph K_n with unit conductances and uniform killing measure of intensity $\kappa > 0$ is given by the matrix

$$\frac{1}{n+\kappa}(I + \frac{1}{\kappa}J)$$

where J denotes the (n, n) matrix with all entries equal to 1.

Proof Note first that $[\lambda I - C]G = I$, and that $\lambda I - C = (n+\kappa)I - J$. Hence $G = [(n+\kappa)I - J]^{-1}$. □

Example 1.2 The Green matrix of the d-regular tree with uniform killing measure of intensity $\kappa > 0$ is $\frac{u^{d(x,y)}}{\kappa+d(1-u)}$, with $u = \left(\frac{d+\kappa-\sqrt{(d+\kappa)^2-4(d-1)}}{2(d-1)}\right)$, $d(x, y)$ denoting the graph distance between x and y.

Proof From its definition, $G_{x,y}$ is clearly bounded by $\frac{1}{d+\kappa}\frac{1}{1-\frac{d}{d+\kappa}} = \frac{1}{\kappa}$. It depends only on $d(x, y)$, so we can set $G_{x,y} = g_k$ if $d(x, y) = k$. For $k \geq 1$, g_k solves the equation: $(d+\kappa)g_k = (d-1)g_{k+1} + g_{k-1}$ so that $g_k = u^k g_0$. Moreover, $(d+\kappa)g_0 = dg_1 + 1 = dug_0 + 1$. □

We denote by $\mathbb{P}_x$ or by $\mathbb{P}(\,.\,|\xi_0 = x)$ the law of the Markov chain ξ_n defined by P starting at x, and for any subset F of X, by T_F, the first hitting time of F by the path. Set $D = F^c$. $P^D = P|_{D\times D}$ is the transition matrix of the Markov chain killed at T_F. It is defined on the restricted graph $(D, E \cap (D \times D)$ by the same conductances and by the killing rate $\kappa_x^D = \kappa_x + \sum_{y\in F} C_{x,y}$. λ^D is the restriction of λ to D.

We denote by G^D the associated Green matrix. The hitting distributions define a submarkovian matrix H^F:

$$[H^F]_{x,y} = \mathbb{P}_x(\xi_{T_F} = y)$$

(H^F is called the *balayage or Poisson matrix* in potential theory).

Hitting distributions can be expressed in terms of Green matrices. The following proposition follows directly from these definitions.

Proposition 1.1

(a) For $y \in F$ and $x \in X$, we have

$$[H^F]_{x,y} = 1_{\{x=y\}} + 1_{\{x\in D\}} \sum_{z\in D} G^D_{x,z} C_{z,y}.$$

(b) The Green matrix admits the following decomposition

$$G = G^D + H^F G.$$

(c) Denoting by $G_{|F\times F}$ the restriction of the Green matrix to $F \times F$ and by $\tilde{H}^F$ the transposed of H^F, we have

$$G = G^D + H^F G_{|F\times F} \tilde{H}^F.$$

Note that as G and G^D are symmetric, it follows from (b) that

$$[H^F G]_{x,y} = [H^F G]_{y,x}.$$

1.3 Energy

Definition 1.1 For any complex function z on X, its energy is defined as:

$$e(z) = \frac{1}{2} \sum_{x,y} C_{x,y}(z(x) - z(y))(\overline{z}(x) - \overline{z}(y)) + \sum_x \kappa_x z(x)\overline{z}(x).$$

The space of functions of finite energy is equipped with the scalar product:

$$e(f, \overline{g}) = \frac{1}{2} \sum_{x,y} C_{x,y}(f(x) - f(y))(\overline{g}(x) - \overline{g}(y)) + \sum_x \kappa_x f(x)\overline{g}(x).$$

It contains finitely supported functions.

Note that $e(z)$ will also be denoted by $e(z, \overline{z})$. These definitions hold in all cases. From now on, we assume transience holds.

Theorem 1.1

(a) For any vertex x_0, define the function G^{x_0} by $G^{x_0}(x) = G_{x,x_0}$. G^{x_0} has finite energy and for any $x, y \in X$, $e(G^x, G^y) = G_{x,y}$.
(b) For any finitely supported function f, $e(f, G^{x_0}) = f(x_0)$.

Proof If X is finite, every function has finite energy. (a) and (b) follow directly from the fact that

$$e(f, \overline{g}) = \sum_x \lambda_x f(x)\overline{g}(x) - \sum_{x,y} C_{x,y} f(x)\overline{g}(y)$$

(we can simply write $e(f, \overline{g}) = \langle (M_\lambda - C)f, \overline{g}\rangle$) and use the expression of G as $[M(\lambda) - C]^{-1}$. We get in the same way that $e(f, G^{x_0}) = f(x_0)$ which implies the positive definiteness of e. Note also that $e(Gf, g) = \sum_x f(x)g(x)$, denoted by $\langle f, g\rangle$.

If X is infinite, any function with finite support has finite energy. Letting a finite set D increase to X, then, as P^D increases to P and $\lambda^D = \lambda$ on D, G^D increases to G. By Fatou's lemma, $e(G^{x_0}) \leq \liminf e([G^D]^{x_0})$. But for any function f supported in D, the definition of κ^D implies that $e(f) = e^D(f)$, hence, $e([G^D]^{x_0}) = G^D_{x_0,x_0}$ if $x_0 \in D$, and therefore if D large enough. Hence $e(G^{x_0}) \leq G_{x_0,x_0}$.

To prove the reverse inequality note that:
$e(G_0^x) = e([G^D]^{x_0}) + e(G^{x_0} - [G^D]^{x_0})) + 2e([G^D]^{x_0}, G^{x_0} - [G^D]^{x_0})$
$\geq G^D_{x_0,x_0} + 2e([G^D]^{x_0}, H^F G^{x_0})$. We conclude the proof of by checking that for any function f supported by D, $e(f, H^F G^{x_0})$ vanishes. This expression is given by a countable absolutely converging sum:
$\sum_{x\in D} \lambda_x f(x) H^F G^{x_0}(x) - \sum_{x\in D}\sum_y C_{x,y} f(x) G^{x_0}(y)$
$= \sum_{x\in D} \lambda_x f(x)(H^F G^{x_0}(x) - \sum_y P_{x,y} H^F G^{x_0}(y))$. Note finally that for any $(x, y) \in D \times F$, $H^F_{x,y} = \sum_z P_{x,z} H^F_{z,y}$. Hence, $e(G^{x_0}) = G_{x_0,x_0}$.

The same argument works for $G^x + G^y$ so we can conclude the proof of (a). We then get that for any finitely supported function f, taking D large enough to include its support, $e(f, G^{x_0}) = e(f, [G^D]^{x_0}) + e(f, H^F G^{x_0}) = e^D(f, [G^D]^{x_0}) = f(x_0)$. □

The Dirichlet space $\mathcal{D}$ (also known as the extended Dirichlet space (Cf [17])) is defined as the closure of the space $\mathcal{D}_0$ of finitely supported real functions equipped with the energy scalar product. It is clear from the above that G^{x_0} belongs to $\mathcal{D}$. The following theorem identifies it with a space of functions.

Theorem 1.2 *Let f_n be a Cauchy sequence in $\mathcal{D}_0$. Then f_n converges pointwise towards a function of finite energy f_∞ and $e(f_\infty - f_n)$ converges to 0. Moreover G^{x_0} belongs to $\mathcal{D}$ and for any f in $\mathcal{D}$, $e(f, G^{x_0}) = f(x_0)$.*