Albert C. J. Luo

Two-dimensional Crossing and Product Cubic Systems, Vol. II

Crossing-linear and Self-quadratic Product Vector Field

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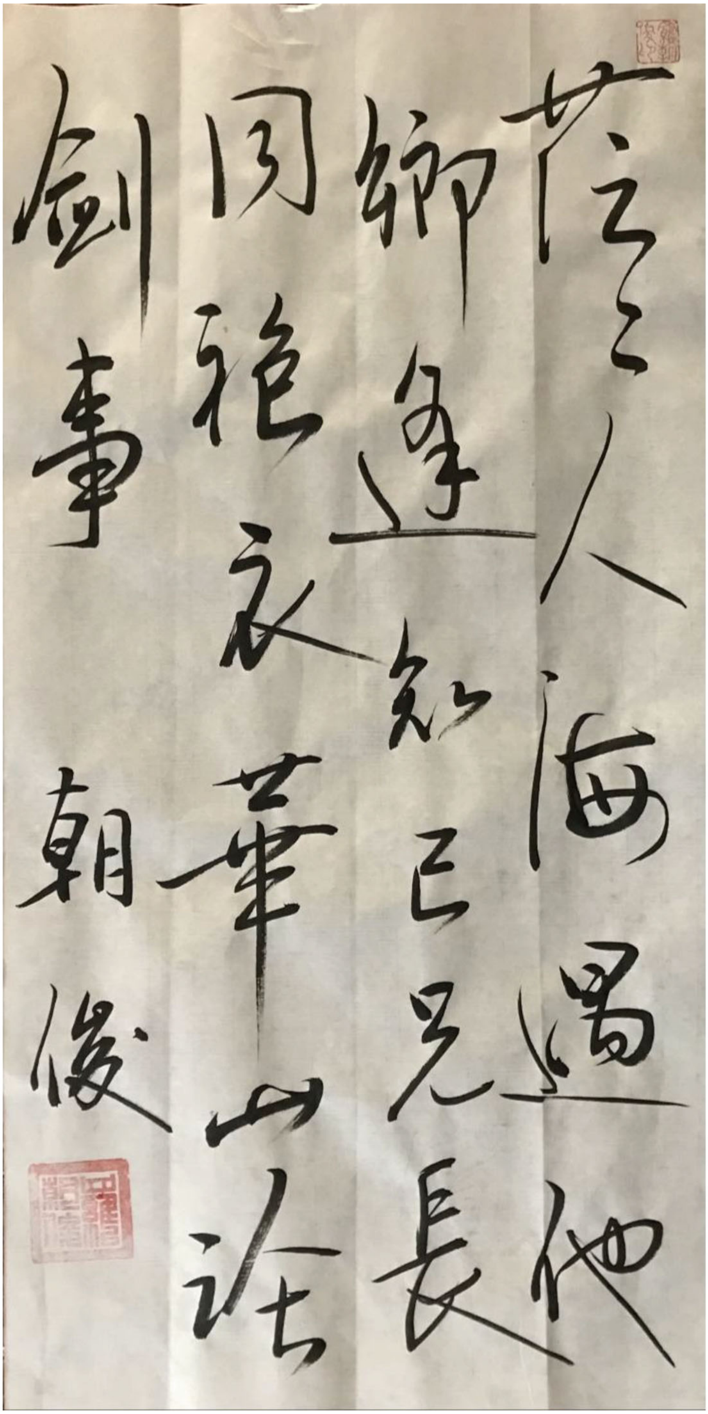

Preface

In this book, consider crossing and product cubic systems with a crossing-linear and self-quadratic product vector field. Nonlinear dynamics and singularity for such crossing and product cubic systems are presented. The parabola-saddles and third-order saddles and centers are discussed as the appearing bifurcations. The switching bifurcations are based on up-down hyperbolic upper-to-lower saddles on the up-down and down-up infinite-equilibriums, parabola-saddles on the inflection-saddle infinite-equilibriums, hyperbolic and circular upper-to-lower saddles on parabola-saddle infinite equilibriums, and parabola-saddles on the inflection-source and sink infinite-equilibriums. The materials in this book are scattered in five chapters.

In Chap. 1, nonlinear dynamics and singularity of a crossing and product cubic system with a crossing-linear and self-quadratic product vector field are discussed, and the corresponding switching dynamics are discussed through the infinite-equilibriums. A theory for nonlinear dynamics behaviors of such a cubic system is presented through a theorem. Such a cubic system possesses parabola-saddle, third-order saddles and centers, and hyperbolic singular flows for appearing bifurcations. The switching bifurcations are up-down hyperbolic upper-to-lower saddles, parabola-saddles on the inflection-saddle infinite-equilibriums, parabola-hyperbolic and circular upper-to-lower saddles, and parabola-saddle on the inflection-source and sink infinite-equilibriums.

In Chap. 2, the parabola-saddle, third-order centers and saddles, and hyperbolic singular flows in the crossing and product cubic systems are presented, and the corresponding switching dynamics are discussed through infinite-equilibriums in such cubic systems. There are three switching bifurcations on the infinite-equilibriums. The up-down hyperbolic upper-to-lower saddles are the switching bifurcations of hyperbolic singular flows with saddle and center. The parabola-saddles are the switching bifurcations of parabola-saddles with hyperbolic-to-hyperbolic-secant flows. The hyperbolic and circular upper-to-lower saddles are the switching bifurcations of a third-order saddle and a hyperbolic-to-hyperbolic-secant flow with a third-order center and a hyperbolic-secant-to hyperbolic flow.

In Chap. 3, series of centers and saddles with hyperbolic singular flows in the crossing and product cubic systems are presented, and the corresponding switching dynamics are discussed through infinite-equilibriums. The hyperbolic singular flows are the appearing bifurcations of hyperbolic and hyperbolic-secant flows. The up-down hyperbolic upper-to-lower saddles on the infinite-equilibriums are the switching bifurcations of parabola-saddle with saddle and center. The same directional saddles and centers are discovered in such cubic systems.

In Chap. 4, parabola-saddle and hyperbolic singular flows in the crossing and product cubic systems are presented, and the corresponding switching dynamics are discussed through infinite-equilibriums. The parabola-saddles on the inflection-source and sink infinite-equilibriums are the switching bifurcations of a saddle and a hyperbolic-secant flow with a center and hyperbolic flow. The parabola-hyperbolic sink-to-source on the parabola-sink and source infinite-equilibriums are the switching bifurcations of a parabola-saddle and a hyperbolic flow with another parabola-saddle with hyperbolic-secant flow. The third-order parabola-saddles are the switching bifurcations of a third-order saddle and a hyperbolic-secant flow with third-order center and hyperbolic flow.

In Chap. 5, simple equilibriums and hyperbolic flows forming a series in the crossing and product cubic systems are presented, and the corresponding switching dynamics are discussed through the inflection-source and sink infinite-equilibriums. The parabola-saddles on single and double inflection-source and sink infinite-equilibriums are discussed. The parabola-saddles are the switching bifurcations of a saddle and a hyperbolic-secant flow with a center and hyperbolic flow.

Finally, the author hopes the materials presented herein can provide a better understanding of cubic nonlinear systems in science and engineering.

Edwardsville, IL, USA Albert C. J. Luo

Contents

Chapter 1
Crossing and Product-Cubic Systems

In this chapter, the nonlinear dynamics and singularity of a crossing and product-cubic system with a crossing-linear and self-quadratic product vector field are discussed, and the corresponding switching dynamics are discussed through the infinite equilibriums. A theory for the nonlinear dynamic behaviors of such a cubic system is presented through a theorem. Such a cubic system possesses parabola-saddles, third-order saddles and centers. Parabola-saddles are the appearing bifurcations of saddles and centers. Third-order saddles are the appearing bifurcations from a saddle to saddle, center, and saddle. A third-order center is the appearing bifurcation from a center to center, saddle, and center. Up-down hyperbolic upper-to-lower saddles are the switching bifurcations of hyperbolic singular flows with a saddle and center. Parabola-saddles are the switching bifurcations of parabola-saddles with hyperbolic-to-hyperbolic-secant flows. Parabola-hyperbolic upper-to-lower saddles are the switching bifurcations of a third-order saddle and a hyperbolic-to-hyperbolic-secant flow. Parabola-circular upper-to-lower saddles are the switching bifurcations of a third-order center and a hyperbolic-secant-to-hyperbolic flow. Parabola-saddles on the inflection source and sink infinite equilibriums are the switching bifurcations of a saddle and a hyperbolic-secant flow with a center and hyperbolic flow.

1.1 Crossing-Linear and Self-Quadratic Product Vector Fields

In this section, a crossing and product cubic system with a crossing-linear and self-quadratic product vector field will be discussed. The corresponding dynamical behaviors will be presented through the following theorem.

Theorem 1.1 *Consider a two-dimensional, cubic dynamical system with a product-cubic vector field and a crossing-cubic vector field as*

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$$\begin{aligned} \dot{x}_{j_2} &= a_{j_2j_20}(x_{j_1} - b_{j_2j_11})(x_{j_1}^2 + B_{j_2j_1}x_{j_1} + C_{j_2j_1}),\\ \dot{x}_{j_1} &= a_{j_1j_10}(x_{j_2} - a_{j_1j_21})(x_{j_1}^2 + B_{j_1j_1}x_{j_1} + C_{j_1j_1}), \end{aligned} \tag{1.1}$$

with

$$\Delta_{j_1j_1} = B_{j_1j_1}^2 - 4C_{j_1j_1}, \Delta_{j_2j_1} = B_{j_2j_1}^2 - 4C_{j_2j_1}. \tag{1.2}$$

(i) For $\Delta_{j_1j_2} < 0$ and $\Delta_{j_2j_1} < 0$, the standard form is

$$\begin{aligned} \dot{x}_{j_2} &= a_{j_2j_20}(x_{j_1} - a_{j_2j_11})\left[(x_{j_1} - a_{j_2j_1})^2 + b_{j_2j_1}\right],\\ \dot{x}_{j_1} &= a_{j_1j_10}(x_{j_2} - a_{j_1j_21})\left[(x_{j_1} - a_{j_1j_1})^2 + b_{j_1j_1}\right] \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} &a_{j_1j_1} = -\frac{1}{2}B_{j_1j_1}, b_{j_1j_1} = \frac{1}{4}(-\Delta_{j_1j_1}),\\ &a_{j_2j_11} = b_{j_2j_11}, a_{j_2j_1} = -\frac{1}{2}B_{j_2j_1}, b_{j_2j_1} = \frac{1}{4}(-\Delta_{j_2j_1}). \end{aligned} \tag{1.4}$$

The first integral manifold is

$$\begin{aligned} &\frac{1}{2}\left[(x_{j_1} - a_{j_2j_11})^2 - (x_{j_10} - a_{j_2j_11})^2\right]\\ &+2(a_{j_1j_1} - a_{j_2j_1})(x_{j_1} - x_{j_10}) + \left\{(a_{j_1j_1} - a_{j_2j_1})(a_{j_1j_1} - a_{j_2j_11})\right.\\ &\left.+\frac{1}{2}\left[(a_{j_1j_1} - a_{j_2j_1})^2 + b_{j_2j_1} - b_{j_1j_1}\right]\right\} \ln\frac{|\,(x_{j_1} - a_{j_1j_1})^2 + b_{j_1j_1}\,|}{|\,(x_{j_10} - a_{j_1j_1})^2 + b_{j_1j_1}\,|}\\ &+\left\{(a_{j_1j_1} - a_{j_2j_11})\left[(a_{j_1j_1} - a_{j_2j_1})^2 + b_{j_2j_1} - b_{j_1j_1}\right]\right.\\ &\left.-2b_{j_1j_1}(a_{j_1j_1} - a_{j_2j_1})\right\}\frac{1}{\sqrt{b_{j_1j_1}}}(\arctan\frac{x_{j_1} - a_{j_1j_1}}{\sqrt{b_{j_1j_1}}} - \arctan\frac{x_{j_10} - a_{j_1j_1}}{\sqrt{b_{j_1j_1}}})\\ &= \frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2} - a_{j_1j_21})^2 - (x_{j_20} - a_{j_1j_21})^2\right]. \end{aligned} \tag{1.5}$$

The equilibrium of $(x_{j_2}^*, x_{j_1}^*) = (a_{j_1j_21}, a_{j_2j_11})$ has the following properties:

- *For $a_{j_1j_10} > 0$ and $a_{j_2j_20} > 0$,*

$$(a_{j_1j_21}, a_{j_2j_11}) = \underbrace{(\mathrm{UP}_+, \mathrm{UP}_+)}_{\text{positive saddle}}. \tag{1.6}$$

The equilibrium of $(x_{j_2}^*, x_{j_1}^*) = (a_{j_1j_21}, a_{j_2j_11})$ is a $(\mathrm{UP}_+,\mathrm{UP}_+)$*-positive saddle.*

- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}>0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_+,\mathrm{DP}_-)}_{\text{CCW center}}. \tag{1.7}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{DP}_+,\mathrm{DP}_-)$*-counter-clockwise center.*

- *For* $a_{j_1j_10}>0$ *and* $a_{j_2j_20}<0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_-,\mathrm{DP}_+)}_{\text{CW center}}. \tag{1.8}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{DP}_-,\mathrm{DP}_+)$*-clockwise center.*

- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}<0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{UP}_-,\mathrm{UP}_-)}_{\text{negative saddle}}. \tag{1.9}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{UP}_-,\mathrm{UP}_-)$*-negative saddle.*

(ii) For $\Delta_{j_1j_1}<0$ *and* $\Delta_{j_2j_1}=0$*, the standard form is*

$$\begin{aligned}\dot{x}_{j_2}&=a_{j_2j_20}(x_{j_1}-a_{j_2j_1s_1})(x_{j_1}-a_{j_2j_1s_2})^2,\\ \dot{x}_{j_1}&=a_{j_1j_10}(x_{j_2}-a_{j_1j_21})\left[(x_{j_1}-a_{j_1j_1})^2+b_{j_1j_1}\right]\end{aligned} \tag{1.10}$$

where

$$\begin{aligned}a_{j_1j_1}&=-\frac{1}{2}B_{j_1j_1},b_{j_1j_1}=\frac{1}{4}(-\Delta_{j_1j_1}),\\ a_{j_2j_1s_1}&=b_{j_2j_11},a_{j_2j_1s_2}=-\frac{1}{2}B_{j_2j_1}.\end{aligned} \tag{1.11}$$

(ii$_1$*) The first integral manifold is*

$$\begin{aligned}&\frac{1}{2}\left[(x_{j_1}-a_{j_2j_1s_1})^2-(x_{j_10}-a_{j_2j_1s_1})^2\right]\\ &+2(a_{j_1j_1}-a_{j_2j_1s_1})(x_{j_1}-x_{j_10})+\left\{(a_{j_1j_1}-a_{j_2j_1s_2})(a_{j_1j_1}-a_{j_2j_1s_1})\right.\\ &+\frac{1}{2}\left[(a_{j_1j_1}-a_{j_2j_1s_2})^2-b_{j_1j_1}\right]\Big\}\ln\frac{|\,(x_{j_1}-a_{j_1j_1})^2+b_{j_1j_1}\,|}{|\,(x_{j_10}-a_{j_1j_1})^2+b_{j_1j_1}\,|}\\ &+\Big\{(a_{j_1j_1}-a_{j_2j_1s_1})\left[(a_{j_1j_1}-a_{j_2j_1s_2})^2-b_{j_1j_1}\right]\\ &-2b_{j_1j_1}(a_{j_1j_1}-a_{j_2j_1s_1})\Big\}\frac{1}{\sqrt{b_{j_1j_1}}}(\arctan\frac{x_{j_1}-a_{j_1j_1}}{\sqrt{b_{j_1j_1}}}-\arctan\frac{x_{j_10}-a_{j_1j_1}}{\sqrt{b_{j_1j_1}}})\\ &=\frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2}-a_{j_1j_21})^2-(x_{j_20}-a_{j_1j_21})^2\right].\end{aligned} \tag{1.12}$$

(ii$_{1a}$) The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_1})$ *has the following properties:*

- *For* $a_{j_1 j_1 0} > 0$ *and* $a_{j_2 j_2 0}(a_{j_2 j_1 s_1} - a_{j_2 j_1 s_2})^2 > 0$,

$$(a_{j_1 j_2 1}, a_{j_2 j_1 s_1}) = \underbrace{(\mathrm{UP}_+, \mathrm{UP}_+)}_{\text{positive saddle}}. \tag{1.13}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_1})$ *is a* (UP$_+$,UP$_+$)-positive saddle.

- *For* $a_{j_1 j_1 0} < 0$ *and* $a_{j_2 j_2 0}(a_{j_2 j_1 s_1} - a_{j_2 j_1 s_2})^2 > 0$,

$$(a_{j_1 j_2 1}, a_{j_2 j_1 s_1}) = \underbrace{(\mathrm{DP}_+, \mathrm{DP}_-)}_{\text{CCW center}}. \tag{1.14}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_1})$ *is a* (DP$_+$,DP$_-$)*-counter-clockwise center.*

- *For* $a_{j_1 j_1 0} > 0$ *and* $a_{j_2 j_2 0}(a_{j_2 j_1 s_1} - a_{j_2 j_1 s_2})^2 < 0$,

$$(a_{j_1 j_2 1}, a_{j_2 j_1 s_1}) = \underbrace{(\mathrm{DP}_-, \mathrm{DP}_+)}_{\text{CW center}}. \tag{1.15}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_1})$ *is a* (DP$_-$,DP$_+$)*-clockwise center.*

- *For* $a_{j_1 j_1 0} < 0$ *and* $a_{j_2 j_2 0}(a_{j_2 j_1 s_1} - a_{j_2 j_1 s_2})^2 < 0$,

$$(a_{j_1 j_2 1}, a_{j_2 j_1 s_1}) = \underbrace{(\mathrm{UP}_-, \mathrm{UP}_-)}_{\text{negative saddle}}. \tag{1.16}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_1})$ *is a* (UP$_-$,UP$_-$)-negative saddle.

(ii$_{1b}$) The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_2})$ *has the following properties:*

- *For* $a_{j_1 j_1 0} > 0$ *and* $a_{j_2 j_2 0}(a_{j_2 j_1 s_2} - a_{j_2 j_1 s_1}) > 0$,

$$(a_{j_1 j_2 1}, a_{j_2 j_1 s_2}) = \underbrace{(\mathrm{UP}, \mathrm{US})}_{\text{up-parabola upper-saddle}}. \tag{1.17}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_2})$ *is a* (UP,US)*-up-parabola upper-saddle.*

- *For* $a_{j_1 j_1 0} < 0$ *and* $a_{j_2 j_2 0}(a_{j_2 j_1 s_2} - a_{j_2 j_1 s_1}) > 0$,

$$(a_{j_1 j_2 1}, a_{j_2 j_1 s_2}) = \underbrace{(\mathrm{DP}, \mathrm{US})}_{\text{down-parabola upper-saddle}}. \tag{1.18}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 s_2})$ *is a* (DP,US)*-down-parabola upper-saddle.*

- *For $a_{j_1j_10} > 0$ and $a_{j_2j_20}(a_{j_2j_1s_2} - a_{j_2j_1s_1}) < 0$,*

$$(a_{j_1j_21}, a_{j_2j_1s_2}) = \underbrace{(\mathrm{DP}, \mathrm{LS})}_{\text{down-parabola lower-saddle}}. \tag{1.19}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_1s_2})$ is a (DP,LS)*-down-parabola lower-saddle.*

- *For $a_{j_1j_10} < 0$ and $a_{j_2j_20}(a_{j_2j_1s_2} - a_{j_2j_1s_1}) < 0$,*

$$(a_{j_1j_21}, a_{j_2j_1s_2}) = \underbrace{(\mathrm{UP}, \mathrm{LS})}_{\text{up-parabola lower-saddle}}. \tag{1.20}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_1s_2})$ is a (UP,LS)*-up-parabola lower-saddle.*

(ii$_2$) For $a_{j_2j_1s_1} = a_{j_2j_1s_2} = a_{j_2j_11}$,

$$\begin{aligned}\dot{x}_{j_2} &= a_{j_2j_20}(x_{j_1} - a_{j_2j_11})^3,\\ \dot{x}_{j_1} &= a_{j_1j_10}(x_{j_2} - a_{j_1j_21})\left[(x_{j_1} - a_{j_1j_1})^2 + b_{j_1j_1}\right].\end{aligned} \tag{1.21}$$

The first integral manifold is

$$\begin{aligned}&\frac{1}{2}\left[(x_{j_1} - a_{j_2j_11})^2 - (x_{j_10} - a_{j_2j_11})^2\right] + 2(a_{j_1j_1} - a_{j_2j_11})(x_{j_1} - x_{j_10})\\ &+\frac{1}{2}\left[3(a_{j_1j_1} - a_{j_2j_11})^2 - b_{j_1j_1}\right] \ln \frac{|\ (x_{j_1} - a_{j_1j_1})^2 + b_{j_1j_1}\ |}{|\ (x_{j_10} - a_{j_1j_1})^2 + b_{j_1j_1}\ |}\\ &+(a_{j_1j_1} - a_{j_2j_11})\left[(a_{j_1j_1} - a_{j_2j_11})^2 - 3b_{j_1j_1}\right] \frac{1}{\sqrt{b_{j_1j_1}}}(\arctan \frac{x_{j_1} - a_{j_1j_1}}{\sqrt{b_{j_1j_1}}} - \arctan \frac{x_{j_10} - a_{j_1j_1}}{\sqrt{b_{j_1j_1}}})\\ &= \frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2} - a_{j_1j_21})^2 - (x_{j_20} - a_{j_1j_21})^2\right].\end{aligned} \tag{1.22}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_1s_2})$ has the following properties:

- *For $a_{j_1j_10} > 0$ and $a_{j_2j_20} > 0$,*

$$(a_{j_1j_21}, a_{j_2j_11}) = \underbrace{(\mathrm{UP}_+, 3^{\mathrm{rd}}\mathrm{UP}_+)}_{\text{third-order positive saddle}}. \tag{1.23}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_11})$ is a $(\mathrm{UP}_+, 3^{\mathrm{rd}}\mathrm{UP}_+)$*-third-order positive saddle.*

- *For* $a_{j_1j_1 0} < 0$ *and* $a_{j_2j_2 0} > 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1 1}) = \underbrace{(\mathrm{DP}_+, 3^{\mathrm{rd}}\mathrm{DP}_-)}_{\text{third-order CCW center}} . \tag{1.24}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1 1})$ *is a* $(\mathrm{DP}_+,3^{\mathrm{rd}}\mathrm{DP}_-)$*-third-order counter-clockwise center.*

- *For* $a_{j_1j_1 0} > 0$ *and* $a_{j_2j_2 0} < 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1 1}) = \underbrace{(\mathrm{DP}_-, 3^{\mathrm{rd}}\mathrm{DP}_+)}_{\text{third-order CW center}} . \tag{1.25}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1 1})$ *is a* $(\mathrm{DP}_-,3^{\mathrm{rd}}\mathrm{DP}_+)$*-third-order clockwise center.*

- *For* $a_{j_1j_1 0} < 0$ *and* $a_{j_2j_2 0} < 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1 1}) = \underbrace{(\mathrm{UP}_-, 3^{\mathrm{rd}}\mathrm{UP}_-)}_{\text{third-order negative saddle}} . \tag{1.26}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1 1})$ *is a* $(\mathrm{UP}_-,3^{\mathrm{rd}}\mathrm{UP}_-)$*-third-order negative saddle.*

(iii) For $\Delta_{j_1j_2} < 0$ *and* $\Delta_{j_2j_1} > 0$*, the standard form is*

$$\begin{aligned} \dot{x}_{j_2} &= a_{j_2j_2 0}(x_{j_1} - a_{j_2j_1 1})(x_{j_1} - a_{j_2j_1 2})(x_{j_1} - a_{j_2j_1 3}), \\ \dot{x}_{j_1} &= a_{j_1j_1 0}(x_{j_2} - a_{j_1j_2 1})\left[(x_{j_1} - a_{j_1j_1})^2 + b_{j_1j_1}\right] \end{aligned} \tag{1.27}$$

where

$$\begin{aligned} & a_{j_1j_1} = -\frac{1}{2}B_{j_1j_1}, b_{j_1j_1} = \frac{1}{4}(-\Delta_{j_1j_1}), \\ & b_{j_2j_1 2}, b_{j_2j_1 3} = -\frac{1}{2}(B_{j_2j_1} \pm \sqrt{\Delta_{j_2j_1}}), \\ & \{a_{j_2j_1 1}, a_{j_2j_1 2}, a_{j_2j_1 3}\} = \mathrm{sort}\{b_{j_2j_1 1}, b_{j_2j_1 2}, b_{j_2j_1 2}\}, \\ & a_{j_2j_1 s_1} < a_{j_2j_1 s_2}, s_1, s_2 \in \{1,2,3\}, s_1 < s_2. \end{aligned} \tag{1.28}$$

The first integral manifold is

$$
\begin{aligned}
&\frac{1}{2}\left[(x_{j_1}-a_{j_2j_1s_1})^2-(x_{j_10}-a_{j_2j_1s_1})^2\right]+(2a_{j_1j_1}-a_{j_2j_1s_2}-a_{j_2j_1s_3})(x_{j_1}-x_{j_10})\\
&+\frac{1}{2}\left\{(2a_{j_1j_1}-a_{j_2j_1s_2}-a_{j_2j_1s_3})(a_{j_1j_1}-a_{j_2j_1s_1})\right.\\
&\left.+\left[(a_{j_1j_1}-a_{j_2j_1s_2})(a_{j_1j_1}-a_{j_2j_1s_3})-b_{j_1j_1}\right]\right\}\ln\frac{|\,(x_{j_1}-a_{j_1j_1})^2+b_{j_1j_1}\,|}{|\,(x_{j_10}-a_{j_1j_1})^2+b_{j_1j_1}\,|}\\
&+\left\{(a_{j_1j_1}-a_{j_2j_1s_1})\left[(a_{j_1j_1}-a_{j_2j_1s_2})(a_{j_1j_1}-a_{j_2j_1s_3})-b_{j_1j_1}\right]\right.\\
&\left.-b_{j_1j_1}(2a_{j_1j_1}-a_{j_2j_1s_2}-a_{j_2j_1s_3})\right\}\frac{1}{\sqrt{b_{j_1j_1}}}(\arctan\frac{x_{j_1}-a_{j_1j_1}}{\sqrt{b_{j_1j_1}}}-\arctan\frac{x_{j_10}-a_{j_1j_1}}{\sqrt{b_{j_1j_1}}})\\
&=\frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2}-a_{j_1j_21})^2-(x_{j_20}-a_{j_1j_21})^2\right].
\end{aligned}
\tag{1.29}
$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_1s_1})$ $(s_1=1,2,3)$ *has the following properties:*

- *For* $a_{j_1j_10}>0$ *and* $a_{j_2j_20}\prod_{s_2=1,s_2\neq s_1}^{3}(a_{j_2j_1s_1}-a_{j_2j_1s_2})>0$,

$$(a_{j_1j_21},a_{j_2j_1s_1})=\underbrace{(\mathrm{UP}_+,\mathrm{UP}_+)}_{\text{positive saddle}}. \tag{1.30}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_1s_1})$ *is a* $(\mathrm{UP}_+,\mathrm{UP}_+)$-*positive saddle.*
- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}\prod_{s_2=1,s_2\neq s_1}^{3}(a_{j_2j_1s_1}-a_{j_2j_1s_2})>0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_+,\mathrm{DP}_-)}_{\text{CCW center}}. \tag{1.31}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_1s_1})$ *is a* $(\mathrm{DP}_+,\mathrm{DP}_-)$-*counter-clockwise center.*
- *For* $a_{j_1j_10}>0$ *and* $a_{j_2j_20}\prod_{s_2=1,s_2\neq s_1}^{3}(a_{j_2j_1s_1}-a_{j_2j_1s_2})<0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_-,\mathrm{DP}_+)}_{\text{CW center}}. \tag{1.32}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_1s_1})$ *is a* $(\mathrm{DP}_-,\mathrm{DP}_+)$-*clockwise center.*
- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}\prod_{s_2=1,s_2\neq s_1}^{3}(a_{j_2j_1s_1}-a_{j_2j_1s_2})<0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{UP}_-,\mathrm{UP}_-)}_{\text{negative saddle}}. \tag{1.33}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_1s_1})$ *is a* $(\mathrm{UP}_-,\mathrm{UP}_-)$-*negative saddle.*

(iv) For $\Delta_{j_1j_2}=0$ *and* $\Delta_{j_2j_1}<0$*, the standard form is*

$$\begin{aligned}\dot{x}_{j_2}&=a_{j_2j_20}(x_{j_1}-a_{j_2j_11})\left[(x_{j_1}-a_{j_2j_1})^2+b_{j_2j_1}\right],\\ \dot{x}_{j_1}&=a_{j_1j_10}(x_{j_2}-a_{j_1j_21})(x_{j_1}-a_{j_1j_11})^2\end{aligned}\tag{1.34}$$

where

$$\begin{aligned}a_{j_1j_11}&=-\frac{1}{2}B_{j_1j_1},a_{j_2j_11}=b_{j_2j_11},\\ a_{j_2j_1}&=-\frac{1}{2}B_{j_2j_1},b_{j_2j_1}=\frac{1}{4}(-\Delta_{j_2j_1}).\end{aligned}\tag{1.35}$$

(iv₁) The first integral manifold for $a_{j_2j_11}\neq a_{j_1j_11}$ *is*

$$\begin{aligned}&\frac{1}{2}\left[(x_{j_1}-a_{j_2j_11})^2-(x_{j_10}-a_{j_2j_11})^2\right]\\ &+2(a_{j_1j_11}-a_{j_2j_1})(x_{j_1}-x_{j_10})+\left\{2(a_{j_1j_11}-a_{j_2j_1})(a_{j_1j_11}-a_{j_2j_11})\right.\\ &\left.+\left[(a_{j_1j_11}-a_{j_2j_1})^2+b_{j_2j_1}\right]\right\}\ln\frac{|x_{j_1}-a_{j_1j_11}|}{|x_{j_10}-a_{j_1j_11}|}\\ &-(a_{j_1j_11}-a_{j_2j_11})\left[(a_{j_1j_11}-a_{j_2j_1})^2+b_{j_2j_1}\right]\left(\frac{1}{x_{j_1}-a_{j_1j_11}}-\frac{1}{x_{j_10}-a_{j_1j_11}}\right)\\ &=\frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2}-a_{j_1j_21})^2-(x_{j_20}-a_{j_1j_21})^2\right].\end{aligned}\tag{1.36}$$

(iv₁ₐ) The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *has the following properties:*

- *For* $a_{j_1j_10}(a_{j_2j_11}-a_{j_1j_11})^2>0$ *and* $a_{j_2j_20}>0$*,*

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{UP}_+,\mathrm{UP}_+)}_{\text{positive saddle}}.\tag{1.37}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{UP}_+,\mathrm{UP}_+)$*-positive saddle.*
- *For* $a_{j_1j_10}(a_{j_2j_11}-a_{j_1j_11})^2<0$ *and* $a_{j_2j_20}>0$*,*

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_+,\mathrm{DP}_-)}_{\text{CCW center}}.\tag{1.38}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{DP}_+,\mathrm{DP}_-)$*-counter-clockwise center.*
- *For* $a_{j_1j_10}(a_{j_2j_11}-a_{j_1j_11})^2>0$ *and* $a_{j_2j_20}<0$*,*

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_-,\mathrm{DP}_+)}_{\text{CW center}}.\tag{1.39}$$

  The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{DP}_-,\mathrm{DP}_+)$*-clockwise center.*

- *For* $a_{j_1j_10}(a_{j_2j_11} - a_{j_1j_11})^2 < 0$ *and* $a_{j_2j_20} < 0$,

$$(a_{j_1j_21}, a_{j_2j_11}) = \underbrace{(\mathrm{UP}_-, \mathrm{UP}_-)}_{\text{negative saddle}}. \tag{1.40}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_11})$ *is a* (UP$_-$,UP$_-$)*-negative saddle.*

(iv$_{1b}$) The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *has the following properties:*

- *For* $a_{j_1j_10} > 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_11}) > 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\mathrm{UP:UP, pF})}_{\text{hyperbolic-secant-to-hyperbolic flow } (+)}. \tag{1.41}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (UP:UP,pF)*-positive hyperbolic-secant-to-hyperbolic flow.*

- *For* $a_{j_1j_10} < 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_11}) > 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\mathrm{DP:DP, pF})}_{\text{hyperbolic-to-hyperbolic-secant flow } (+)}. \tag{1.42}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (DP:DP,pF)*-positive hyperbolic-to-hyperbolic-secant flow.*

- *For* $a_{j_1j_10} > 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_11}) < 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\mathrm{DP:DP, nF})}_{\text{hyperbolic-to-hyperbolic-secant flow } (-)}. \tag{1.43}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (DP:DP,nF)*-negative hyperbolic-secant-to-hyperbolic flow.*

- *For* $a_{j_1j_10} < 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_11}) < 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\mathrm{UP:UP, nF})}_{\text{hyperbolic-secant-to-hyperbolic flow } (-)}. \tag{1.44}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (UP:UP,nF)*-negative hyperbolic-secant to hyperbolic flow.*

(iv$_2$) The first integral manifold for $a_{j_2j_11} = a_{j_1j_11}$ *is*

$$\begin{aligned}
&\frac{1}{2}\left[(x_{j_1} - a_{j_1j_11})^2 - (x_{j_10} - a_{j_1j_11})^2\right] \\
&+2(a_{j_1j_11} - a_{j_2j_1})(x_{j_1} - x_{j_10}) + \left[(a_{j_1j_11} - a_{j_2j_1})^2 + b_{j_2j_1}\right] \ln \frac{|\, x_{j_1} - a_{j_1j_11} \,|}{|\, x_{j_10} - a_{j_1j_11} \,|} \\
&= \frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2} - a_{j_1j_21})^2 - (x_{j_20} - a_{j_1j_21})^2\right].
\end{aligned} \tag{1.45}$$

(iv$_{2a}$) The infinite-equilibrium of $x^*_{j_1} = a_{j_2 j_1 1} = a_{j_1 j_1 1}$ with $\bar{x}_{j_2} \neq a_{j_1 j_2 1}$ has the following properties:

- *For $a_{j_1 j_1 0}(\bar{x}_{j_2} - a_{j_1 j_2 1}) > 0$ and $a_{j_2 j_2 0} > 0$,*

$$(a_{j_1 j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{US}, \mathrm{DU})}_{\text{down-up upper-saddle}}. \tag{1.46}$$

  The infinite-equilibrium of $x^*_{j_1} = a_{j_2 j_1 1} = a_{j_1 j_1 1}$ is a (US,DU)-*down-up asymptotic upper-saddle.*

- *For $a_{j_1 j_1 0}(\bar{x}_{j_2} - a_{j_1 j_2 1}) < 0$ and $a_{j_2 j_2 0} > 0$,*

$$(a_{j_1 j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{LS}, \mathrm{UD})}_{\text{up-down lower-saddle}}. \tag{1.47}$$

  The infinite-equilibrium of $x^*_{j_1} = a_{j_2 j_1 1} = a_{j_1 j_1 1}$ is an (LS,UD)-*up-down asymptotic lower-saddle.*

- *For $a_{j_1 j_1 0}(\bar{x}_{j_2} - a_{j_1 j_2 1}) > 0$ and $a_{j_2 j_2 0} < 0$,*

$$(a_{j_1 j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{US}, \mathrm{UD})}_{\text{up-down upper-saddle}}. \tag{1.48}$$

  The infinite-equilibrium of $x^*_{j_1} = a_{j_2 j_1 1} = a_{j_1 j_1 1}$ is a (US,UD)-*up-down asymptotic upper-saddle.*

- *For $a_{j_1 j_1 0}(\bar{x}_{j_2} - a_{j_1 j_2 1}) < 0$ and $a_{j_2 j_2 0} < 0$,*

$$(a_{j_1 j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{LS}, \mathrm{DU})}_{\text{down-up lower-saddle}}. \tag{1.49}$$

  The infinite-equilibrium of $x^*_{j_1} = a_{j_2 j_1 1} = a_{j_1 j_1 1}$ is an (LS,DU)-*down-up asymptotic lower-saddle.*

(iv$_{2b}$) The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 1})$ with $a_{j_1 j_1 1} = a_{j_2 j_1 1}$ has the following properties:

- *For $a_{j_1 j_1 0} > 0$ and $a_{j_2 j_2 0} > 0$,*

$$(a_{j_1 j_2 1}, a_{j_2 j_1 1}) = \underbrace{({}_{\mathrm{UD}}\mathrm{LS} : {}_{\mathrm{DU}}\mathrm{US}, \mathrm{DP}_- : \mathrm{UP}_+)}_{\text{hyperbolic lower-to-upper saddle}}. \tag{1.50}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1 j_2 1}, a_{j_2 j_1 1})$ for $a_{j_1 j_1 1} = a_{j_2 j_1 1}$ is a (${}_{\mathrm{UD}}$LS: ${}_{\mathrm{DU}}$US, DP$_-$:UP$_+$)-hyperbolic lower-to-upper saddle.

- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}>0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{({}_{\rm DU}{\rm US}:{}_{\rm UD}{\rm LS},{\rm UP}_-:{\rm DP}_+)}_{\text{hyperbolic-secant upper-to-lower saddle}}. \tag{1.51}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *for* $a_{j_1j_11}=a_{j_2j_11}$ *is a* (${}_{\rm DU}$US: ${}_{\rm UD}$LS, UP$_-$:DP$_+$)-*hyperbolic-secant upper-to-lower saddle.*

- *For* $a_{j_1j_10}>0$ *and* $a_{j_2j_20}<0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{({}_{\rm DU}{\rm LS}:{}_{\rm UD}{\rm US},{\rm UP}_+:{\rm DP}_-)}_{\text{hyperbolic-secant lower-to-upper saddle}}. \tag{1.52}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *for* $a_{j_1j_11}=a_{j_2j_11}$ *is a* (${}_{\rm DU}$LS: ${}_{\rm UD}$US, UP$_+$:DP$_-$)-*hyperbolic-secant lower-to-upper saddle.*

- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}<0$,

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{({}_{\rm UD}{\rm US}:{}_{\rm DU}{\rm LS},{\rm DP}_+:{\rm UP}_-)}_{\text{hyperbolic upper-to-lower saddle}}. \tag{1.53}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *for* $a_{j_1j_11}=a_{j_2j_11}$ *is a* (${}_{\rm UD}$US: ${}_{\rm DU}$LS, DP$_+$:UP$_-$)-hyperbolic upper-to-lower saddle.

(v) For $\Delta_{j_1j_2}=0$ *and* $\Delta_{j_2j_1}=0$, *the standard form is*

$$\begin{aligned}\dot{x}_{j_2}&=a_{j_2j_20}(x_{j_1}-a_{j_2j_1s_1})(x_{j_1}-a_{j_2j_1s_2})^2,\\ \dot{x}_{j_1}&=a_{j_1j_10}(x_{j_2}-a_{j_1j_21})(x_{j_1}-a_{j_1j_11})^2\end{aligned} \tag{1.54}$$

where

$$a_{j_1j_11}=-\frac{1}{2}B_{j_1j_1},a_{j_2j_1s_1}=b_{j_2j_11},a_{j_2j_1s_2}=-\frac{1}{2}B_{j_2j_1}. \tag{1.55}$$

(v_1) The first integral manifold for $a_{j_1j_11}\neq a_{j_2j_1s_1}\neq a_{j_2j_1s_2}$ *is*

$$\begin{aligned}&\frac{1}{2}\left[(x_{j_1}-a_{j_2j_1s_1})^2-(x_{j_10}-a_{j_2j_1s_1})^2\right]\\&+2(a_{j_1j_11}-a_{j_2j_1s_2})(x_{j_1}-x_{j_10})\\&+\left[(a_{j_1j_11}-a_{j_2j_1s_2})^2+2(a_{j_1j_11}-a_{j_2j_1s_1})(a_{j_1j_11}-a_{j_2j_1s_2})\right]\ln\frac{|\,x_{j_1}-a_{j_1j_11}\,|}{|\,x_{j_10}-a_{j_1j_11}\,|}\\&-(a_{j_1j_11}-a_{j_2j_1s_1})(a_{j_1j_11}-a_{j_2j_1s_2})^2\left(\frac{1}{x_{j_1}-a_{j_1j_11}}-\frac{1}{x_{j_10}-a_{j_1j_11}}\right)\\&=\frac{1}{2}\frac{a_{j_1j_10}}{a_{j_2j_20}}\left[(x_{j_2}-a_{j_1j_21})^2-(x_{j_20}-a_{j_1j_21})^2\right].\end{aligned} \tag{1.56}$$

(v_{1a}) The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_1})$ *has the following properties:*

- *For* $a_{j_1j_1 0}(a_{j_2j_1s_1} - a_{j_1j_1 1})^2 > 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 > 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1s_1}) = \underbrace{(\mathrm{UP}_+, \mathrm{UP}_+)}_{\text{positive saddle}}. \tag{1.57}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_1})$ *is a* (UP_+,UP_+)*-positive saddle.*
- *For* $a_{j_1j_1 0}(a_{j_2j_1s_1} - a_{j_1j_1 1})^2 < 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 > 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1s_1}) = \underbrace{(\mathrm{DP}_+, \mathrm{DP}_-)}_{\text{CCW center}}. \tag{1.58}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_1})$ *is a* (DP_+,DP_-)*-counter-clockwise center.*
- *For* $a_{j_1j_1 0}(a_{j_2j_1s_1} - a_{j_1j_1 1})^2 > 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 < 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1s_1}) = \underbrace{(\mathrm{DP}_-, \mathrm{DP}_+)}_{\text{CW center}}. \tag{1.59}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_1})$ *is a* (DP_-,DP_+)*-clockwise center.*
- *For* $a_{j_1j_1 0}(a_{j_2j_1s_1} - a_{j_1j_1 1})^2 < 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 < 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1s_1}) = \underbrace{(\mathrm{UP}_-, \mathrm{UP}_-)}_{\text{negative saddle}}. \tag{1.60}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_1})$ *is a* (UP_-,UP_-)*-negative saddle.*

(v_{1b}) The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_2})$ *with* $a_{j_1j_1 1} \neq a_{j_2j_1s_2}$ *has the following properties:*

- *For* $a_{j_1j_1 0}(a_{j_2j_1s_2} - a_{j_1j_1 1})^2 > 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_2} - a_{j_2j_1s_1}) > 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1s_2}) = \underbrace{(\mathrm{UP}, \mathrm{US})}_{\text{up-parabola upper-saddle}}. \tag{1.61}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_2})$ *is a* (UP,US)*-up-parabola upper-saddle.*
- *For* $a_{j_1j_1 0}(a_{j_2j_1s_2} - a_{j_1j_1 1})^2 < 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_2} - a_{j_2j_1s_1}) > 0$,

$$(a_{j_1j_2 1}, a_{j_2j_1s_2}) = \underbrace{(\mathrm{DP}, \mathrm{US})}_{\text{down-parabola upper-saddle}}. \tag{1.62}$$

  The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_2 1}, a_{j_2j_1s_2})$ *is a* (DP,US)*-down-parabola upper-saddle.*

- *For* $a_{j_1j_10}(a_{j_2j_1s_2} - a_{j_1j_11})^2 > 0$ *and* $a_{j_2j_20}(a_{j_2j_1s_2} - a_{j_2j_1s_1}) < 0$,

$$(a_{j_1j_21}, a_{j_2j_1s_2}) = \underbrace{(\text{DP}, \text{LS})}_{\text{down-parabola lower-saddle}}. \tag{1.63}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_1s_2})$ *is a* (DP,LS)-*down-parabola lower-saddle.*

- *For* $a_{j_1j_10}(a_{j_2j_1s_2} - a_{j_1j_11})^2 < 0$ *and* $a_{j_2j_20}(a_{j_2j_1s_2} - a_{j_2j_1s_1}) < 0$,

$$(a_{j_1j_21}, a_{j_2j_1s_2}) = \underbrace{(\text{UP}, \text{LS})}_{\text{up-parabola lower-saddle}}. \tag{1.64}$$

The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_1s_2})$ *is a* (UP,LS)-*up-parabola lower-saddle.*

– *The parabola-saddles are the appearing bifurcations of saddle and center.*

(v_{1c}) The equilibrium of $(x^*_{j_2}, x^*_{j_1}) = (a_{j_1j_21}, a_{j_2j_11})$ *has the following properties:*

- *For* $a_{j_1j_10} > 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_1s_1})(a_{j_1j_11} - a_{j_2j_1s_2})^2 > 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\text{UP}:\text{UP}, \text{pF})}_{\text{hyperbolic-secant-to-hyperbolic flow }(+)}. \tag{1.65}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (UP:UP, pF)-*positive hyperbolic-secant-to-hyperbolic flow.*

- *For* $a_{j_1j_10} < 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_1s_1})(a_{j_1j_11} - a_{j_2j_1s_2})^2 > 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\text{DP}:\text{DP}, \text{pF})}_{\text{hyperbolic-to-hyperbolic-secant flow }(+)}. \tag{1.66}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (DP:DP, pF)-*positive hyperbolic-to-hyperbolic-secant flow.*

- *For* $a_{j_1j_10} > 0$ *and* $a_{j_2j_20}(a_{j_1j_11} - a_{j_2j_1s_1})(a_{j_1j_11} - a_{j_2j_1s_2})^2 < 0$,

$$(a_{j_1j_11}, a_{j_1j_21}) = \underbrace{(\text{DP}:\text{DP}, \text{nF})}_{\text{hyperbolic-to-hyperbolic-secant flow }(-)}. \tag{1.67}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_11}, a_{j_1j_21})$ *is a* (DP:DP, nF)-*negative hyperbolic-secant-to-hyperbolic flow.*

- *For* $a_{j_1j_1 0}<0$ *and* $a_{j_2j_2 0}(a_{j_1j_1 1}-a_{j_2j_1 s_1})(a_{j_1j_1 1}-a_{j_2j_1 s_2})^2<0$,

$$(a_{j_1j_1 1},a_{j_1j_2 1})=\underbrace{(\mathrm{UP:UP,nF})}_{\text{hyperbolic-secant-to-hyperbolic flow }(-)}. \tag{1.68}$$

The equilibrium of $(x_{j_1}^*,x_{j_2}^*)=(a_{j_1j_1 1},a_{j_1j_2 1})$ *is a* (UP:UP, nF)-*negative hyperbolic-secant-to-hyperbolic flow.*

(v_2) For $a_{j_2j_1 s_1}=a_{j_2j_1 s_2}=a_{j_2j_1 1}$, *the standard form is*

$$\begin{aligned}\dot{x}_{j_2}&=a_{j_2j_2 0}(x_{j_1}-a_{j_2j_1 1})^3,\\ \dot{x}_{j_1}&=a_{j_1j_1 0}(x_{j_2}-a_{j_1j_2 1})(x_{j_1}-a_{j_1j_1 1})^2.\end{aligned} \tag{1.69}$$

The first integral manifold for $a_{j_2j_1 1}\neq a_{j_1j_1 1}$ *is*

$$\begin{aligned}&\frac{1}{2}\left[(x_{j_1}-a_{j_1j_1 1})^2-(x_{j_1 0}-a_{j_1j_1 1})^2\right]+\frac{3}{2}(a_{j_1j_1 1}-a_{j_2j_1 1})(x_{j_1}-x_{j_1 0})\\ &+3(a_{j_1j_1 1}-a_{j_2j_1 1})^2\ln\frac{|x_{j_1}-a_{j_1j_1 1}|}{|x_{j_1 0}-a_{j_1j_1 1}|}-(a_{j_1j_1 1}-a_{j_2j_1 1})^3\left(\frac{1}{x_{j_1}-a_{j_1j_1 1}}-\frac{1}{x_{j_1 0}-a_{j_1j_1 1}}\right)\\ &=\frac{1}{2}\frac{a_{j_1j_1 0}}{a_{j_2j_2 0}}\left[(x_{j_2}-a_{j_1j_2 1})^2-(x_{j_2 0}-a_{j_1j_2 1})^2\right].\end{aligned} \tag{1.70}$$

(v_{2a}) The equilibrium of $(x_{j_2}^*,x_{j_1}^*)=(a_{j_1j_2 1},a_{j_2j_1 1})$ *has the following properties:*

- *For* $a_{j_1j_1 0}(a_{j_2j_1 1}-a_{j_1j_1 1})^2>0$ *and* $a_{j_2j_2 0}>0$,

$$(a_{j_1j_2 1},a_{j_2j_1 1})=\underbrace{(\mathrm{UP}_+,3^{\mathrm{rd}}\mathrm{UP}_+)}_{\text{third-order positive saddle}}. \tag{1.71}$$

The equilibrium of $(x_{j_2}^*,x_{j_1}^*)=(a_{j_1j_2 1},a_{j_2j_1 1})$ *is a* (UP$_+$,3$^{\mathrm{rd}}$UP$_+$)-*third-order positive saddle.*

- *For* $a_{j_1j_1 0}(a_{j_2j_1 1}-a_{j_1j_1 1})^2<0$ *and* $a_{j_2j_2 0}>0$,

$$(a_{j_1j_2 1},a_{j_2j_1 1})=\underbrace{(\mathrm{DP}_+,3^{\mathrm{rd}}\mathrm{DP}_-)}_{\text{third-order CCW center}}. \tag{1.72}$$

The equilibrium of $(x_{j_2}^*,x_{j_1}^*)=(a_{j_1j_2 1},a_{j_2j_1 1})$ *is a* (DP$_+$,3$^{\mathrm{rd}}$DP$_-$)-*third-order counter-clockwise center.*

- *For* $a_{j_1j_10}(a_{j_2j_11}-a_{j_1j_11})^2>0$ *and* $a_{j_2j_20}<0$*,*

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{DP}_-,3^{\mathrm{rd}}\mathrm{DP}_+)}_{\text{third-order CW center}}. \tag{1.73}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{DP}_-,3^{\mathrm{rd}}\mathrm{DP}_+)$*-third-order clockwise center.*

- *For* $a_{j_1j_10}(a_{j_2j_11}-a_{j_1j_11})^2<0$ *and* $a_{j_2j_20}<0$*,*

$$(a_{j_1j_21},a_{j_2j_11})=\underbrace{(\mathrm{UP}_-,3^{\mathrm{rd}}\mathrm{UP}_-)}_{\text{third-order negative saddle}}. \tag{1.74}$$

The equilibrium of $(x^*_{j_2},x^*_{j_1})=(a_{j_1j_21},a_{j_2j_11})$ *is a* $(\mathrm{UP}_-,3^{\mathrm{rd}}\mathrm{UP}_-)$*-third-order negative saddle.*

– *The third-order saddles are the appearing and switching bifurcations of saddle, center, and saddle.*
– *The third-order centers are the appearing and switching bifurcations of center, saddle, and center.*

(v_{2b}*) The equilibrium of* $(x^*_{j_1},x^*_{j_2})=(a_{j_1j_11},a_{j_1j_21})$ *with* $a_{j_1j_11}\neq a_{j_1j_2s_1},a_{j_1j_2s_2}$ *has the following properties:*

- *For* $a_{j_1j_10}>0$ *and* $a_{j_2j_20}(a_{j_1j_11}-a_{j_2j_11})^3>0$*,*

$$(a_{j_1j_11},a_{j_1j_21})=\underbrace{(\mathrm{UP{:}UP,pF})}_{\text{hyperbolic-secant-to-hyperbolic flow }(+)}. \tag{1.75}$$

The equilibrium of $(x^*_{j_1},x^*_{j_2})=(a_{j_1j_11},a_{j_1j_21})$ *is a* (UP:UP, pF)*-positive hyperbolic-secant-to-hyperbolic flow.*

- *For* $a_{j_1j_10}<0$ *and* $a_{j_2j_20}(a_{j_1j_11}-a_{j_2j_11})^3>0$*,*

$$(a_{j_1j_11},a_{j_1j_21})=\underbrace{(\mathrm{DP{:}DP,pF})}_{\text{hyperbolic-to-hyperbolic-secant flow }(+)}. \tag{1.76}$$

The equilibrium of $(x^*_{j_1},x^*_{j_2})=(a_{j_1j_11},a_{j_1j_21})$ *is a* (DP:DP,pF)*-positive hyperbolic-to-hyperbolic-secant flow.*

- *For* $a_{j_1j_10}>0$ *and* $a_{j_2j_20}(a_{j_1j_11}-a_{j_2j_11})^3<0$*,*

$$(a_{j_1j_11},a_{j_1j_21})=\underbrace{(\mathrm{DP{:}DP,nF})}_{\text{hyperbolic-to-hyperbolic-secant flow }(-)}. \tag{1.77}$$

The equilibrium of $(x^*_{j_1},x^*_{j_2})=(a_{j_1j_11},a_{j_1j_21})$ *is a* (DP:DP, nF)*-negative hyperbolic-secant-to-hyperbolic flow.*

- *For* $a_{j_1j_1 0} < 0$ *and* $a_{j_2j_2 0}(a_{j_1j_1 1} - a_{j_2j_1 1})^3 < 0$,

$$(a_{j_1j_1 1}, a_{j_1j_2 1}) = \underbrace{(\mathrm{UP:UP, nF})}_{\text{hyperbolic-secant-to-hyperbolic flow } (-)}. \tag{1.78}$$

The equilibrium of $(x^*_{j_1}, x^*_{j_2}) = (a_{j_1j_1 1}, a_{j_1j_2 1})$ *is a* (UP:UP,nF)-*negative hyperbolic-secant-to-hyperbolic flow.*

(v_3) The first integral manifold for $a_{j_2j_1s_1} = a_{j_1j_1 1}$ *is*

$$\frac{1}{2}\left[(x_{j_1} - a_{j_1j_1 1})^2 - (x_{j_1 0} - a_{j_1j_1 1})^2\right] = \frac{1}{2}\frac{a_{j_1j_1 0}}{a_{j_2j_2 0}}\left[(x_{j_2} - a_{j_1j_2 1})^2 - (x_{j_2 0} - a_{j_1j_2 1})^2\right]. \tag{1.79}$$

(v_{3a}) The infinite-equilibrium of $x^*_{j_1} = a_{j_2j_1s_1} = a_{j_1j_1 1}$ *with* $\bar{x}_{j_2} \neq a_{j_1j_2 1}$ *has the following properties:*

- *For* $a_{j_1j_1 0}(\bar{x}_{j_2} - a_{j_1j_2 1}) > 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 > 0$,

$$(a_{j_1j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{US, DU})}_{\text{down-up upper-saddle}}. \tag{1.80}$$

The infinite-equilibrium of $x^*_{j_1} = a_{j_2j_1s_1} = a_{j_1j_1 1}$ *is a* (US,DU)-*down-up asymptotic upper-saddle.*

- *For* $a_{j_1j_1 0}(\bar{x}_{j_2} - a_{j_1j_2 1}) < 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 > 0$,

$$(a_{j_1j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{LS, UD})}_{\text{up-down lower-saddle}}. \tag{1.81}$$

The infinite-equilibrium of $x^*_{j_1} = a_{j_2j_1s_1} = a_{j_1j_1 1}$ *is an* (LS,UD)-*up-down asymptotic lower-saddle.*

- *For* $a_{j_1j_1 0}(\bar{x}_{j_2} - a_{j_1j_2 1}) > 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 < 0$,

$$(a_{j_1j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{US, UD})}_{\text{up-down upper-saddle}}. \tag{1.82}$$

The infinite-equilibrium of $x^*_{j_1} = a_{j_2j_1s_1} = a_{j_1j_1 1}$ *is a* (US,UD)-*up-down asymptotic upper-saddle.*

- *For* $a_{j_1j_1 0}(\bar{x}_{j_2} - a_{j_1j_2 1}) < 0$ *and* $a_{j_2j_2 0}(a_{j_2j_1s_1} - a_{j_2j_1s_2})^2 < 0$,

$$(a_{j_1j_1 1}, \bar{x}_{j_2}) = \underbrace{(\mathrm{LS, DU})}_{\text{down-up lower-saddle}}. \tag{1.83}$$

The infinite-equilibrium of $x^*_{j_1} = a_{j_2j_1s_1} = a_{j_1j_1 1}$ *is an* (LS,DU)-*down-up asymptotic lower-saddle.*