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Madhumangal Pal
Department of Applied Mathematics
Vidyasagar University
Midnapore, West Bengal, India

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Preface

The presence of uncertainty in the real world has led to the utilization of two primary theories—probability theory and fuzzy mathematics—to address problems associated with uncertainty. Depending on the nature of the problems, these two methods are applied individually or concurrently. Simultaneously, matrices serve as a fundamental tool in mathematics, extensively employed to model problems across various disciplines such as mathematics, physics, statistics, engineering, and more.

The development of the fuzzy matrix in 1977 has proven instrumental in effectively addressing uncertainty. The fuzzy matrix theory, now a robust topic in fuzzy mathematics, holds a distinct mathematics classification number 15B15. The challenges associated with finding eigenvalues and eigenvectors of a fuzzy matrix are notable, particularly in physics and various engineering domains. Topics like the convergence of a fuzzy matrix and the inverse of a fuzzy matrix play a crucial role in uncertain computation.

In the realm of game theory and business, where precise predictions of profit and loss are elusive, representing such uncertainties as triangular fuzzy numbers, interval numbers, trapezoidal fuzzy numbers, etc. becomes essential. Game theoretic problems often involve uncertain payoffs, represented as fuzzy numbers, and can be efficiently modelled using fuzzy matrices. Moreover, solving problems with fuzzy numbers allows for obtaining solutions in a crisp environment by substituting specific values, providing an advantage in real-life problem-solving.

Following Zadeh's development of fuzzy set theory in 1965, researchers extended their focus to various types of fuzzy sets, including interval-valued fuzzy sets and intuitionistic fuzzy sets. Similarly, after the inception of fuzzy matrices in 1977, numerous extensions emerged, such as interval-valued fuzzy matrices, intuitionistic fuzzy matrices, picture fuzzy matrices, neutrosophic fuzzy matrices, spherical fuzzy matrices, m-polar fuzzy matrices, etc.

This book is crafted for postgraduate students and researchers, offering a substantial amount of material to enable instructors to choose topics tailored to their needs. Comprising 12 chapters, each chapter is briefly described below.

Chapter 1 introduces the concept of fuzzy matrix (FM) and highlights its distinctions from crisp matrices, along with its practical applications. Various results, including determinant and adjoint, akin to those in crisp matrices, are discussed. The chapter introduces numerous new operators, each accompanied by a multitude of properties. Fuzzy matrices are defined over diverse fields, such as incline and residuated lattices, expanding their applicability. The importance of both the generalized inverse and conventional inverse in problem-solving is emphasized. The chapter delves into the generalized inverse and regularity of FMs. An essential topic covered is nilpotency, thoroughly defined and studied within the context of fuzzy matrices. Additional topics covered include permanent, norm, and distances between FMs. The chapter explores fuzzy vector space and introduces the concept of distance between Boolean fuzzy matrices. Although the computation of eigenvalues and eigenvectors is a challenging task for FMs, the chapter provides a brief discussion on this crucial topic within the realm of mathematics.

Chapter 2 focuses on interval-valued fuzzy matrices (IVFMs), exploring various types such as symmetric, reflexive, transitive, idempotent, and constant IVFMs. The chapter introduces and defines key concepts like trace, convergence, periodicity, determinant, permanent, and adjoint of IVFMs, providing methods for their evaluation. The properties of similarity relations and invertibility conditions of IVFMs are extensively examined, contributing to a comprehensive understanding of these matrices. The chapter initiates the discussion on different types of ranks, including row rank, column rank, fuzzy rank, and Schein rank of IVFMs. Cross vectors and scalar multiplication of IVFMs are also explored, offering insights into their algebraic properties. While finding eigenvalues for fuzzy matrices is a less-explored area, the chapter provides an outline for finding the eigenvalues and eigenvectors of IVFMs. The proposed results are illustrated with examples, although their establishment for general cases is pending. The chapter also delves into topics such as the g-inverse and regularity of IVFM. Additionally, the Hamacher operator, a t-norm and t-conorm-based operator, is defined on IVFM, contributing to the understanding of operators in this context.

Chapter 3 delves into triangular fuzzy numbers (TFNs) and matrices (TFMs), defining fundamental operations and computational procedures. The chapter begins by introducing elementary operations on triangular fuzzy numbers, drawing parallels with classical matrices. The process of computing the determinant of a triangular fuzzy matrix (TFM) is explained and demonstrated through an example. Various special types of TFMs are introduced, such as pure and fuzzy triangular matrices, symmetric and skew-symmetric matrices, singular and semi-singular matrices, constant matrices, and more. Each type is illustrated with examples to provide a clear understanding. The chapter also presents a method for solving systems of equations with triangular fuzzy numbers as coefficients, contributing to practical problem-solving approaches. Furthermore, the eigenvalues and eigenvectors of TFMs are explored and illustrated, shedding light on their mathematical properties and applications.

Chapter 4 is dedicated to the discussion of Matrices of Interval Numbers (MIN). Moore is credited with introducing this concept for the first time. The chapter outlines various special types of MIN, including symmetric, skew-symmetric, pseudo-skew-symmetric, triangular, constant, etc. Essential operations related to

MIN are defined to provide a comprehensive understanding. The chapter explores the concepts of determinant and adjoint in the context of MIN. Additionally, it introduces readers to the notions of nilpotent MIN, convergence of MIN, regularity, and singularity. A method is presented to solve systems of linear equations that involve interval coefficients, offering practical insights for applications. The chapter also provides a concise review of the comparison between two interval numbers, considering both the optimistic and pessimistic perspectives of decision-makers. Building on these results, the chapter demonstrates the application of MIN in solving matrix game problems and addressing the all-pairs shortest distances problem on a graph.

Chapter 5 provides an in-depth exploration of intuitionistic fuzzy matrices (IFM). Various properties of IFM are meticulously examined, complemented by illustrative examples for better understanding. The chapter introduces unique properties of intuitionistic fuzzy determinants and adjoints, showcasing results that deviate from those observed in crisp matrices. Additionally, a range of new operators is defined, yielding novel and significant results. The chapter expands the scope of IFMs by incorporating new types, such as concentration and dilation, and incorporates linguistic terms like "very," "more or less," "highly," "very very," among others, to enhance the study of IFMs. Discussions on convergence, similarity, symmetry, and other characteristics are presented with examples for clarity. The chapter also delves into specific types of IFMs, including circulant, generalized, and nilpotent IFMs. Furthermore, intuitionistic fuzzy eigenvalues and eigenvectors are thoroughly investigated, along with insights into the group inverse and generalized inverse of IFMs, complete with their respective properties. The solvability of systems involving intuitionistic fuzzy linear equations is a focal point. Distances between IFMs are defined, encompassing various types, each accompanied by a comprehensive examination of their properties.

In Chap. 6, attention is directed towards the exploration of the Interval-Valued Intuitionistic Fuzzy Matrix (IVIFM), an uncertain matrix that has received limited attention in research endeavours. The chapter delves into various aspects of IVIFM, encompassing topics such as determinant, adjoint, cofactor, and constant IVIFM. Furthermore, the partition of IVIFMs is thoroughly examined, including insights into block IVIFMs. The discussion extends to the definition and study of different types of distances between IVIFMs, accompanied by a presentation of a few properties. An application showcasing the utilization of distances on IVIFMs is also provided, demonstrating the practical implications of these measures. The chapter concludes by highlighting the ample opportunities available for further advancements and enhancements in this intriguing and relatively unexplored area.

Chapter 7 provides a comprehensive exploration of another significant extension of the fuzzy matrix known as the bipolar fuzzy matrix (BFM). In bipolar fuzzy sets, each element is characterized by two membership values: a positive membership value and a negative membership value. The chapter commences by introducing bipolar algebra and bipolar fuzzy relations. Subsequently, the bipolar fuzzy matrix is defined, accompanied by the construction of its geometric diagram. The chapter delves into various aspects, including results on the transitive closure and power convergence of bipolar fuzzy matrices. Leveraging bipolar fuzzy algebra, the chapter establishes the concept of a bipolar fuzzy vector space, unravelling numerous intriguing properties such as subspace, basis, and dimension. Similar to fuzzy matrices, BFMs exhibit three distinct ranks: row rank, column rank, and fuzzy rank. The chapter elucidates the relationship among these ranks, noting that, unlike general fuzzy matrices, they are not necessarily equal for BFMs. Furthermore, the chapter develops several properties and employs the cross vector to investigate such ranks. Scalar multiplication for a BFM is introduced, showcasing compelling properties. The discussion extends to the study of eigenvalues and eigenvectors for BFMs, addressing aspects like idempotence, diagonal dominance, and spectral radius.

Chapter 8 delves into the picture fuzzy matrix (PFM), a significant extension of the fuzzy matrix (FM) and particularly the intuitionistic fuzzy matrix (IFM). The matrix incorporates the concept of "neutral," in addition to membership and non-membership values. Notably, Dogra and Pal introduced the PFM for the first time in 2020, with only a handful of papers published on PFMs since then. The chapter introduces two variants: restricted PFM and special restricted PFM, shedding light on two types of cuts for the special restricted square PFM and presenting noteworthy results for these cuts. The discussion also encompasses determinants and adjoints of square PFMs, revealing intriguing findings. This exploration is considered a generalization of IFM. The chapter concludes with an application of PFMs.

Chapter 9 introduces Pythagorean fuzzy matrices (PyFMs), spherical fuzzy matrices (SFMs), and T-spherical fuzzy matrices (TSFMs). PyFM is an extension of IFMs (intuitionistic fuzzy matrices), while SFM and TSFM are extensions of picture fuzzy matrices. In PyFM, the sum of squares of membership and non-membership values is constrained to be less than or equal to 1, introducing a novel type of uncertain matrix. SFMs impose a similar constraint, where the sum of squares of all components of each element must be less than or equal to 1. TSFMs extend this concept further, limiting the sum of the qth power (a given integer) of all components of each element to be less than or equal to 1. The chapter explores various aspects of these matrices, including new operators, determinants, adjoints, convergence, nilpotency, etc., supported by relevant examples. It defines the similarity between two PyFMs and provides necessary examples to illustrate this concept. However, it acknowledges that there is ample room for further research and development in these matrix types, indicating the existence of unexplored opportunities and potential advancements in this area.

Chapter 10 begins by discussing neutrosophic fuzzy sets (NFS), single-valued neutrosophic fuzzy sets, non-standard intervals, and related concepts in its initial section. Subsequently, the chapter delves into the definition of neutrosophic fuzzy matrices (NFMs) and explores the convergence of these matrices. In NFMs, each element is characterized by three types of membership values. The chapter further introduces fuzzy neutrosophic matrices (FNMs), highlighting their fundamental properties, order relations, determinants, adjoint, and more. It is emphasized that NFMs and FNMs differ, with each element in both matrices comprising three components, and their values range between 0 and 1, inclusive. Another category, namely neutrosophic matrices (RNMs), is presented, where each element contains two parts—one certain and another indeterminate—and these values can be any real numbers. The chapter extensively covers topics such as inverse, eigenvalues, and eigenvectors, and provides methods for solving systems of linear algebraic equations and simple non-linear equations, supported by illustrative examples. Additionally, the chapter introduces an extension of these matrices, termed refined neutrosophic matrices, where each element consists of three components: the first being certain, while the other two represent indeterminacy. Despite limited exploration, these matrices open avenues for further research.

Chapter 11 introduces a novel category of fuzzy matrix known as the fuzzy matrix with uncertain rows and columns (FMURC). In contrast to conventional fuzzy matrices that presume certainty in both rows and columns, this innovative matrix acknowledges uncertainty in all rows and columns. The application of this concept extends to fuzzy matrices (FMs), interval-valued fuzzy matrices (IVFMs), and intuitionistic fuzzy matrices (IFMs). Pal introduced these matrices in 2015, incorporating a new addition method and introducing several innovative concepts. The chapter defines various null and identity matrices, along with density and balanced matrices, elucidating many properties associated with these matrix types. These matrices find significant utility in representing fuzzy graphs, especially in situations where uncertainty characterizes all the vertices and edges.

Chapter 12 introduces the concept of the m-polar fuzzy matrix (mPFM), where each element is represented as an m-dimensional vector. Each component of the vector signifies a membership value associated with a specific attribute among the m attributes. The components are independent, ranging between 0 and 1. The chapter covers fundamental arithmetic operations and operators applicable to this matrix. Various topics such as determinant, convergence, regularity, density, transitive closure, permanent, and aggregation operation are explored and elucidated through examples. The chapter also defines several distinct types of matrices, detailing their respective properties. Additionally, the m-polar fuzzy matrix with uncertain columns and rows (mPFMUCRs) is introduced, incorporating new terminologies and operators, while the density of these matrices is subject to examination.

Each chapter concludes by presenting a set of open problems that warrant further investigation.

To the best of our knowledge, there is currently no existing book on fuzzy matrix theory and its applications. We hope this book will prove valuable to postgraduate students, research scholars, and experts, facilitating a deeper understanding of fuzzy matrix theory. The content of the book offers an interconnected presentation of fundamental ideas, concepts, and results.

The completion of this book was made possible through the assistance of various research articles and books, for which we express our heartfelt gratitude. The moral and loving support, as well as continuous encouragement, received from my students, family members, and relatives played an indispensable role in the realization of this project.

We encourage readers to share their comments, criticisms, suggestions, corrections, etc., as their valuable input can contribute to enhancing the content in future editions. Your feedback is welcomed, and we look forward to incorporating the experience gained from the first edition into subsequent editions of the book.

Midnapore, India Madhumangal Pal

Contents

Chapter 1
Fuzzy Matrices

Abbreviations

FS	Fuzzy set
FM	Fuzzy matrix
$\mathscr{M}_{mn}^{FM}$	Set of all FMs of order $m \times n$
$\mathscr{M}_{n}^{FM}$	Set of all FMs of order $n \times n$
FA	Fuzzy algebra
FR	Fuzzy relation
FV	Fuzzy vector
FD	Fuzzy determinant
FPM	Fuzzy permutation matrix
Per	Permanent

1.1 Introduction

Like classical (crisp) matrix theory, fuzzy matrix (FM) is also a very useful tool for modelling many uncertain problems that arise in sciences, engineering, social sciences, and many other areas. In crisp matrices, the elements are either real numbers or complex numbers or sometimes vectors, but in FMs, the elements are membership values. In the Boolean matrix, the elements are either 0 or 1 and the two basic operations addition and multiplication are max and min, i.e. $a+b = \max\{a, b\}$ and $a.b = \min\{a, b\}$. Here, 0 and 1 represent two states of a system, such as on and off of an electrical network, etc. Whereas in FM the elements are any real number in the closed interval [0, 1], so it is a multi-state logic, i.e. it is used to represent infinite many situations. The addition and multiplication rules are the same as a Boolean matrix. Fuzzy matrices are used to model problems of many fields, e.g. fuzzy relations, fuzzy relational equations, pattern classification, knowledge-based systems, etc.

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Many works have been done on FMs after the development by Thomason in 1977 [107] and it has a separate AMS Subject Classification, 15B15. Thomason [107] studied the power convergence of FMs. Kim and Roush [43] investigated the canonical form of an idempotent matrix. Hashimoto [26] and Kolodziejczyk [47] introduced the concept of the canonical form of a transitive matrix a strongly transitive matrix. Kim [44] defined and presented several properties of the adjoint of a square fuzzy matrix. Unlike the crisp determinant, the fuzzy determinant is defined for FM by Kim et al. [45] and more investigation is made on it by Ragab and Emam [80, 81]. The controllable fuzzy matrices are studied by Xin [108, 109]. Hemasinha et al. [28] investigated iterates of fuzzy circulates matrices. Ragab and Emam [81] presented some properties of the min-max composition of fuzzy matrices.

Pal et al. did many works on fuzzy and related matrices, such as new operators on fuzzy matrices [96], intuitionistic fuzzy matrices [67], generalized intuitionistic fuzzy matrices [9], similarity relations, invertibility and eigenvalues of intuitionistic fuzzy matrix [61] and for bipolar fuzzy matrix [62], inverse of intuitionistic fuzzy matrices [75, 76], triangular fuzzy matrices [98], circulant triangular fuzzy number matrices [8], complex nilpotent matrices [15], norm [71], interval-valued fuzzy matrices [97], rank of interval-valued fuzzy matrices [63], picture fuzzy matrix [17], bipolar fuzzy matrices [72]. New types of fuzzy matrices are introduced whose rows and columns are uncertain, see for the fuzzy matrices [70], for intuitionistic fuzzy matrices [73], for interval-valued fuzzy matrices [69].

Some of the above results are extended to generalized FMs, i.e. matrices over a special type of semiring. The transitivity of matrices over path algebra (i.e. additively idempotent semiring) is discussed by Hashimoto [27]. The determinant theory, powers and nilpotent conditions of matrices over a distributive lattice are investigated by Zhang [110, 111] and Tan [103]. Also, some works have been done over incline, which is a special type of semiring, a particular case of path algebra but extended than distributive lattice. The transitive closure, convergence of powers and adjoint of generalized FMs over incline are discussed in [19].

Fuzzy logic finds extensive applications across diverse fields such as science, technology, medical science, and social sciences. Notably, it is employed in fuzzy graph theory [1–7, 22, 29, 30, 53–56, 59, 64–66, 68, 77, 78, 88–93, 99, 101], fuzzy topological indices [31–37], fuzzy intersection graph [82–85], fuzzy algebra [14–16, 40, 79, 94, 95], supply chain management [87], fuzzy inventory control [57, 58], fuzzy decision making [38, 39, 41], and numerous other domains.

In this chapter, FMs are investigated. The determinant/permanent of an FM, its generalized inverse, eigenvalue, rank, new operators on FMs, and distance between two FMs are presented. The FMs over the incline and residuated lattice are very new concepts; these are also presented here. The solution of a fuzzy system of equations and nilpotent FMs is discussed. The fuzzy vector spaces and regular fuzzy matrices are defined and presented with several properties.

1.2 Definitions and Preliminaries

First of all the FM is defined over semiring. Let us define the semiring.

Definition 1.1 A **semiring** is a set R with two binary operations $+, \times$, from $R \times R$ to R, which satisfy

(i) $x + y = y + x$.
(ii) $x + (y + z) = (x + y) + z$.
(iii) $x(yz) = (xy)z$.
(iv) $x(y + z) = xy + xz$.
(v) $(y + z)x = yx + zx$.

A semiring is called **commutative** if $xy = yx$ holds for all x, y. The symbols 0 and 1 are denoted an additive and a multiplicative identity, respectively. If 0, 1 exist they are unique and $1.0 = 0.1 = 0$.

Example 1.1 The Boolean algebra $\{0, 1\}$, the fuzzy algebra $[0, 1]$ under the operations $x + y = \sup\{x, y\}$, $x.y = \inf\{x, y\}$ are all commutative semirings.

Definition 1.2 A **permutation matrix** is a square binary matrix that has exactly one entry 1 in each row and each column and 0s elsewhere.

Example 1.2 An example of permutation matrix of order 3×3 is given by $\mathscr{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Let $\mathbb{I}$ be the set of all real numbers between 0 and 1, i.e. $\mathbb{I} = \{x : 0 \leq x \leq 1\}$. The following operations are defined for all $x, y, \alpha \in \mathbb{I}$.

$$\text{(i)} \quad x + y \ \ (\text{or } x \vee y) = \max\{x, y\} \tag{1.1}$$

$$\text{(ii)} \quad x.y \quad \text{or } xy \quad (\text{or } x \wedge y) = \min\{x, y\} \quad \text{sometimes it is denoted by } x.y \tag{1.2}$$

$$\text{(iii)} \quad x@y = \frac{x + y}{2}, \quad + \text{ is ordinary addition} \tag{1.3}$$

$$\text{(iv)} \quad x\$y = \sqrt{x.y}, \quad . \text{ is ordinary multiplication} \tag{1.4}$$

$$\text{(v)} \quad x \ominus y = \begin{cases} x, & \text{if } x > y \\ 0, & \text{if } x \leq y \end{cases} \tag{1.5}$$

$$\text{(vi)} \quad x^{(\alpha)}(\text{upper } \alpha\text{-cut}) = \begin{cases} 1, & \text{if } x \geq \alpha \\ 0, & \text{if } x < \alpha \end{cases} \tag{1.6}$$

$$\text{(vii)} \quad x_{(\alpha)}(\text{lower } \alpha\text{-cut}) = \begin{cases} x, & \text{if } x \geq \alpha \\ 0, & \text{if } x < \alpha \end{cases} \tag{1.7}$$

$$\text{(viii)} \quad x^c \quad (\text{complement}) \quad = 1 - x. \tag{1.8}$$

It may be noted that the values of $x \vee y,\ x \wedge y,\ x \ominus y,\ x \oplus y,\ x \odot y,\ x^{(\alpha)},\ x_{(\alpha)}$ and x^c belong to the set $\mathbb{I}$.

An algebraic structure $(R, +, \cdot)$ is called a semiring if $(R, +)$ is an Abelian monoid (with identity 0), $(R, \cdot)$ is a monoid (with identity 1), $\cdot$ distributes over + from both sides, $a0 = 0a = 0$ for all $a \in R$ and $0 \neq 1$. In this chapter the semiring is denoted by R.

Before going to define FMs formally we consider an example. Let us consider a group of four people p_1, p_2, p_3, p_4 and three characteristics, viz. smart, tall, and intelligent. The people p_1, p_2, p_3, p_4 are crisp, they are fixed four people. But the characteristics smart, tall, intelligent cannot be measured in a crisp way, these quantities are surely fuzzy quantities. We cannot measure precisely the smartness of a pupil, and in this case we insert a gradation to measure the smartness among the people. And this gradation is called the membership value of the smartness of the people.

Similarly, the characteristics tall, intelligent are also fuzzy quantities and these are measured by fuzzy membership values.

Here, we may construct two sets such as $X = \{p_1, p_2, p_3, p_4\}$ and $Y = \{$smart, tall, intelligent$\}$.

The fuzzy relation between two sets X and Y is a fuzzy set in the Cartesian product $X \times Y$ characterized by a membership function μ_ρ,

$$\mu_\rho : X \times Y \to [0, 1].$$

We denote $\mu_\rho(p_j, smart)\ by\ \mu_{sj};\ \ j = 1, 2, 3, 4$, which represent the membership value of smartness of the people p_j. Similarly, the membership value of tallness and intelligence of the people $p_j (j = 1, 2, 3, 4)$ are denoted, respectively, as, $\mu_\rho(p_j, tall)\ by\ \mu_{tj}$ and $\mu_\rho(p_j, intelligence)\ by\ \mu_{ij}$.

The fuzzy matrix representing the relation ρ between the sets X and Y is given by

$$R = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} \text{smart} & \text{tall} & \text{intelligent} \end{array} \\ \begin{pmatrix} \mu_{s_1} & \mu_{t_1} & \mu_{i_1} \\ \mu_{s_2} & \mu_{t_2} & \mu_{i_2} \\ \mu_{s_3} & \mu_{t_3} & \mu_{i_3} \\ \mu_{s_4} & \mu_{t_4} & \mu_{i_4} \end{pmatrix} \end{array}.$$

Not that all the elements of the matrix are real numbers on [0, 1]. Now, the main question is ‘What are new in fuzzy matrix ?’ Following are the answer.

(i) Fuzzy matrix is the extension of classical matrix.
(ii) In fuzzy matrix many new operators are defined.
(iii) The elements are taken from different fields.
(iv) Several verities of fuzzy matrices are available, like fuzzy sets.

Let us consider the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

For crisp matrix, a_{ij} is either a real or a complex number, only the basic operators are addition, subtraction, multiplication, and division. No geometric representation is possible as the elements may be very large and very small also.

Whereas for FMs

(a) The elements may be membership values, non-membership values, with certain restrictions.
(b) The elements may be fuzzy numbers.
(c) Several new operators are available for different types of FMs.

Now, we formally defined the FM below.

Definition 1.3 (Fuzzy Matrix) A FM $\mathscr{P}$ of order $m \times n$ is defined as $\mathscr{P} = (p_{ij})_{m\times n}$ where p_{ij} is the membership value of the ijth element of $\mathscr{P}$.

A special type of fuzzy matrix called Boolean fuzzy matrix is defined below.

Definition 1.4 (Boolean Fuzzy Matrix) A FM $\mathscr{Q} = (q_{ij})_{m\times n}$ is said to be a Boolean fuzzy matrix of order $m \times n$ if all the elements of $\mathscr{Q}$ are either 0 or 1.

The set of all FMs of order $m \times n$ is denoted by $\mathscr{M}_{mn}^{FM}$ and that of order $n \times n$ is denoted by $\mathscr{M}_{n}^{FM}$.

In FMs as well as Boolean matrices, the addition and multiplication operators are defined as

$$a + b = a \vee b = \max\{a, b\}, \quad a.b = a \wedge b = \min\{a, b\}.$$

These arithmetic operators differentiate fuzzy matrices from crisp matrices.

Some basic operators on FMs are defined below.

Let $\mathscr{P} = (p_{ij}), \mathscr{Q} = (q_{ij}) \in \mathscr{M}_{mn}^{FM}$.

$$\text{(i)} \quad \mathscr{P} + \mathscr{Q} \ \ (or \, \mathscr{P} \vee \mathscr{Q}) = (p_{ij} \vee q_{ij}) \tag{1.9}$$

$$\text{(ii)} \quad \mathscr{P}.\mathscr{Q} \ \ (or \, \mathscr{P} \wedge \mathscr{Q}) = (p_{ij} \wedge q_{ij}) \tag{1.10}$$

$$\text{(iii)} \quad \mathscr{P} \ominus \mathscr{Q} = (p_{ij} \ominus q_{ij}) \tag{1.11}$$

$$\text{(vi)} \quad \mathscr{P} @ \mathscr{Q} = \left(\frac{p_{ij} + q_{ij}}{2} \right) \tag{1.12}$$

$$\text{(vii)} \quad \mathscr{P} \$ \mathscr{Q} = (\sqrt{p_{ij}.q_{ij}}) \tag{1.13}$$

$$\text{(viii)} \quad \mathscr{P}^{(\alpha)} = (p_{ij}^{(\alpha)}) \quad \text{(upper } \ \alpha - \text{cut fuzzy matrix)} \tag{1.14}$$

$$\text{(ix)} \quad \mathscr{P}_{(\alpha)} = (p_{ij(\alpha)}) \quad \text{(lower } \ \alpha - \text{cut fuzzy matrix)} \tag{1.15}$$

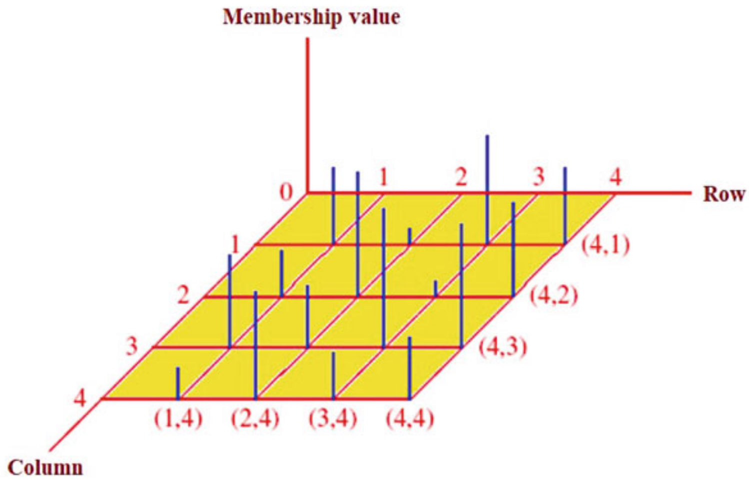

Fig. 1.1 Geometrical representation of the FM $\mathscr{P}^{\dagger}$

$$\text{(x)} \quad \mathscr{P}^T = (p_{ji}) \quad \text{(the transpose of } \mathscr{P}\text{)} \tag{1.16}$$

$$\text{(xi)} \quad \mathscr{P}^c = (1 - p_{ij}) \quad \text{(the complement of } \mathscr{P}\text{)} \tag{1.17}$$

$$\text{(xii)} \quad A \le B \quad \text{if and only if} \quad a_{ij} \le b_{ij} \quad \text{for all} \quad i, j. \tag{1.18}$$

(xiii) For any two fuzzy matrices $\mathscr{P}$ and $\mathscr{Q}$,

$$\min(\mathscr{P}, \mathscr{Q}) = \begin{cases} \mathscr{P}, \text{ if } \mathscr{P} \le \mathscr{Q} \\ \mathscr{Q}, \text{ if } \mathscr{Q} \le \mathscr{P}. \end{cases} \tag{1.19}$$

Since the elements of the FMs are bounded and bounded within the closed interval [0, 1], so every FM can be visualized as a three dimensional diagram. Whereas this is not possible for classical matrix without any proper scaling. To illustrate this fact, let us consider the FM $\mathscr{P}$ as follows:

$$\mathscr{P}^{\dagger} = \begin{pmatrix} 0.5 & 0.1 & 0.7 & 0.5 \\ 0.3 & 0.8 & 0.1 & 0.6 \\ 0.6 & 0.4 & 0.9 & 0.8 \\ 0.2 & 0.7 & 0.3 & 0.4 \end{pmatrix}.$$

The geometrical representation of the FM $\mathscr{P}^{\dagger}$ is shown in Fig. 1.1.

In the following we define some special types of matrices. Let $R = (r_{ij})$ be an $n \times n$ fuzzy matrix. Then,

(i) $\mathscr{P}$ is **reflexive** if and only if $p_{ii} = 1$ for all $i = 1, 2, \ldots, n$.
(ii) $\mathscr{P}$ is **irreflexive** if and only if $p_{ii} = 0$ for all $i = 1, 2, \ldots, n$.

(iii) $\mathscr{P}$ is **nearly irreflexive** if and only if $p_{ii} \leq p_{ij}$ for all $i, j = 1, 2, \ldots, n$.
(iv) $\mathscr{P}$ is **symmetric** if and only if $\mathscr{P}^T = \mathscr{P}$.
(v) $\mathscr{P}$ is **constant** if and only if $p_{ij} = p_{kj}$ for all $i, j, k = 1, 2, \ldots, n$.
(vi) $\mathscr{P}$ is **identity** if and only if $p_{ii} = 1$ and $p_{ij} = 0$ $(i \neq j)$ for all i, j. The identity matrix of order $n \times n$ is denoted by I_n.
(vii) $\mathscr{P}$ is **weakly reflexive** if $p_{ii} \geq p_{ij}$ for all i, j.
(viii) $\mathscr{P}$ is **diagonal** if $p_{ii} \geq 0$ and $p_{ij} = 0$, $(i \neq j)$ for all i, j.

If all the entries of a FM are 0 (respectively, 1) then we denote it by $\mathscr{O}$ (respectively, $\mathscr{U}$). Throughout the paper we assume that $\mathscr{P} = (p_{ij})$, $\mathscr{Q} = (q_{ij})$, $\mathscr{R} = (r_{ij})$ and $\mathscr{S} = (s_{ij})$.

1.3 Fuzzy Determinant and Adjoint

Like crisp determinant, fuzzy determinant is also a very useful tool for modelling or solving many problems. In this section, determinant and adjoint of FMs are discussed. The work presented in this section are taken from [45, 80, 110].

Definition 1.5 ([45]) The **determinant** of an $n \times n$ FM $\mathscr{P}$ is denoted by $det(\mathscr{P})$ and is defined as

$$det(\mathscr{P}) = \sum_{\rho \in S_n} p_{1\rho(1)} p_{2\rho(2)} \cdots p_{n\rho(n)}, \tag{1.20}$$

where S_n denotes the symmetric group of all permutations over the symbols $\{1, 2, \ldots, n\}$.

Similar to crisp matrices, the following results are also valid for FMs.

Property 1.1 Suppose $\mathscr{P} \in \mathscr{M}_n^{FM}$.

(i) If $\mathscr{P}$ contains a zero row (or column), then $det(\mathscr{P}) = 0$.
(ii) If $\mathscr{P}$ is triangular, then $det(\mathscr{P})$ is the product of the diagonal elements.
(iii) If a FM $\mathscr{Q}$ obtained from $\mathscr{P}$ by multiplying the jth row of $\mathscr{P}$ (or jth column) by a scalar $\kappa \in (0, 1]$, then $\kappa \; det(\mathscr{P}) = det(\mathscr{Q})$.

For crisp matrices A, B, $det(AB) = det(A)det(B)$, but this result is not true for FMs.

Theorem 1.1 ([44]) *If $\mathscr{P}, \mathscr{Q} \in \mathscr{M}_n^{FM}$, then $det(\mathscr{P}\mathscr{Q}) \geq det(\mathscr{P})det(\mathscr{Q})$.*

Example 1.3 Let $\mathscr{P}$ and $\mathscr{Q}$ be two FMs, where

$$\mathscr{P} = \begin{pmatrix} 0.2 & 0.1 & 0.8 \\ 0.4 & 0.9 & 0.7 \\ 0.4 & 0.9 & 0.1 \end{pmatrix} \text{ and } \mathscr{Q} = \begin{pmatrix} 0.8 & 0.3 & 0.4 \\ 0.2 & 0.0 & 0.2 \\ 0.8 & 0.0 & 0.0 \end{pmatrix}.$$

The $det(\mathscr{P})$ is calculated as follows:

$$\begin{aligned} det(\mathscr{P}) &= 0.2\begin{vmatrix} 0.9 & 0.7 \\ 0.9 & 0.1 \end{vmatrix} + 0.1\begin{vmatrix} 0.4 & 0.7 \\ 0.4 & 0.1 \end{vmatrix} + 0.8\begin{vmatrix} 0.4 & 0.9 \\ 0.4 & 0.9 \end{vmatrix} \\ &= 0.2[0.1 + 0.7] + 0.1[0.1 + 0.4] + 0.8[0.4 + 0.4] \\ &= (0.2)(0.7) + (0.1)(0.4) + (0.8)(0.4) = 0.2 + 0.1 + 0.4 \\ &= 0.4. \end{aligned}$$

$det(\mathscr{Q})$ is

$$\begin{aligned} det(\mathscr{Q}) &= 0.8\begin{vmatrix} 0.0 & 0.2 \\ 0.0 & 0.0 \end{vmatrix} + 0.3\begin{vmatrix} 0.2 & 0.2 \\ 0.8 & 0.0 \end{vmatrix} + 0.4\begin{vmatrix} 0.2 & 0.0 \\ 0.8 & 0.0 \end{vmatrix} \\ &= 0.8[0.0 + 0.0] + 0.3[0.0 + 0.2] + 0.4[0.0 + 0.0] \\ &= (0.8)(0.0) + (0.3)(0.2) + (0.4)(0.0) = 0.0 + 0.2 + 0.0 \\ &= 0.2. \end{aligned}$$

Now, $\mathscr{P}\mathscr{Q} = \begin{pmatrix} 0.2 & 0.1 & 0.8 \\ 0.4 & 0.9 & 0.7 \\ 0.4 & 0.9 & 0.1 \end{pmatrix} \begin{pmatrix} 0.8 & 0.3 & 0.4 \\ 0.2 & 0.0 & 0.2 \\ 0.8 & 0.0 & 0.0 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 & 0.2 \\ 0.7 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.4 \end{pmatrix}$ and $det(AB) = 0.3$.

In this case, $det(\mathscr{P}\mathscr{Q}) = 0.3$, $det(\mathscr{P})\ det(\mathscr{Q}) = 0.4 \wedge 0.2 = 0.2$.

The adjoint of FMs can be defined as in crisp matrices.

Definition 1.6 ([45]) The **adjoint of a FM** Adjoint of FM $\mathscr{P}$ is denoted by $adj(\mathscr{P})$ and its ijth elements is defined by $det(\mathscr{P}_{ji})$ where $det(\mathscr{P}_{ji})$ is the determinant of order $(n-1) \times (n-1)$ obtained from the FM $\mathscr{P}$ by deleting row j and column i.

Alternately, the determinant $det(\mathscr{P}_{ji})$ can be obtained from $det(\mathscr{P})$ by setting p_{ji} to 1 and for all other rows the jth elements $p_{jk}, k \neq i$ is set to 0.

Note 1.1 The ijth element q_{ij} of $adj(\mathscr{P})$ can be rewritten as

$$q_{ij} = \sum_{\rho \in S_{n_j n_i}} \prod_{t \in n_j} p_{t\rho(t)},$$

where $n_j = \{1, 2, \ldots, n\} \setminus \{j\}$ and $S_{n_j n_i}$ is the set of all permutations of the set n_j over the set n_i.

This representation is used to prove several properties related to $adj(\mathscr{P})$.

Property 1.2 Let $\mathscr{P}, \mathscr{Q} \in \mathscr{M}_n^{FM}$. Then

(i) $\mathscr{P} \leq \mathscr{Q}$ implies $adj(\mathscr{P}) \leq adj(\mathscr{Q})$.

(ii) $adj(\mathscr{P}) + adj(\mathscr{Q}) \leq adj(\mathscr{P} + \mathscr{Q})$.
(iii) $adj(\mathscr{P}^T) = (adj\mathscr{P})^T$.

Proof

(i) Let $A = adj(\mathscr{P})$ and $B = adj(\mathscr{Q})$. Then the ijth elements of A and B can be written as

$$a_{ij} = \sum_{\rho \in S_{n_j n_i}} \prod_{t \in n_j} p_{t\rho(t)} \text{ and } b_{ij} = \sum_{\rho \in S_{n_j n_i}} \prod_{t \in n_j} q_{t\rho(t)}.$$

By definition, it is easy to observed that $a_{ij} \leq b_{ij}$, because $p_{t\rho(t)} \leq q_{t\rho(t)}$ for all $t \neq j, \rho(t) \neq i$.

Hence, $adj(\mathscr{P}) \leq adj(\mathscr{Q})$.

(ii) By definition of $+$ operator, $\mathscr{P}, \mathscr{Q} \leq \mathscr{P} + \mathscr{Q}$. So, $adj(\mathscr{P})$, $adj(\mathscr{Q}) \leq adj(\mathscr{P} + \mathscr{Q})$, and hence $adj(\mathscr{P}) + adj(\mathscr{Q}) \leq adj(\mathscr{P} + \mathscr{Q})$.
(iii) Straightforward.

□

Property 1.3 Let $\mathscr{P} \in \mathscr{M}_n^{FM}$. Then

(i) $\mathscr{P} adj(\mathscr{P}) \geq det(\mathscr{P}) I_n$.
(ii) $(adj(\mathscr{P}))\mathscr{P} \geq det(\mathscr{P}) I_n$.
(iii) If $\mathscr{P}$ has a zero row, $adj(\mathscr{P})\mathscr{P} = \mathscr{O}$.

Property 1.4 Let $\mathscr{P}, \mathscr{Q}, \mathscr{R} \in \mathscr{M}_n^{FM}$. Then $\mathscr{P}\mathscr{R} \leq \mathscr{Q}\mathscr{R}$.

The following result holds for FM, but not for crisp matrices.

Property 1.5 Let $\mathscr{P} \in \mathscr{M}_n^{FM}$. Then $det(\mathscr{P}) = det(adj(\mathscr{P}))$.

Property 1.6 ([107]) Let $\mathscr{P} \in \mathscr{M}_n^{FM}$ be a reflexive FM. Then there exits an integer $\lambda \leq n - 1$ such that $adj(\mathscr{P}) = \mathscr{P}^\lambda$, where $\mathscr{P}^\lambda$ is idempotent.

Property 1.7 Let $\mathscr{P} \in \mathscr{M}_n^{FM}$. Then

(i) $adj(\mathscr{P}^2) = (adj\mathscr{P})^2 = adj\mathscr{P}$.
(ii) If $\mathscr{P}$ is idempotent, then $adj(\mathscr{P}) = \mathscr{P}$.
(iii) $adj(\mathscr{P})$ is reflexive.
(iv) $adj(adj(\mathscr{P})) = adj(\mathscr{P})$.
(v) $adj(\mathscr{P}) \geq \mathscr{P}$.
(vi) $\mathscr{P}(adj(\mathscr{P})) = (adj(\mathscr{P}))\mathscr{P} = adj(\mathscr{P})$.

Proof

(i) Since $\mathscr{P}$ is reflexive, so $\mathscr{P}^2$ is also reflexive. Thus, $adj(\mathscr{P}^2) = (\mathscr{P}^2)^\lambda = (\mathscr{P}^\lambda)^2 = (adj(\mathscr{P}))^2$. Again, $\mathscr{P}^\lambda$ is idempotent, $(adj(\mathscr{P}))^2 = adj(\mathscr{P})$.
(ii) Since $\mathscr{P}$ is idempotent, $\mathscr{P}^\lambda = \mathscr{P}$. Also, $adj(\mathscr{P}) = \mathscr{P}^\lambda$ for some $\lambda \leq n - 1$. Thus, $adj(\mathscr{P}) = \mathscr{P}$.

Other proofs are left to the reader.

□

Definition 1.7 A FM $\mathscr{P} \in \mathscr{M}_n^{FM}$ is said to be circular if and only if $(\mathscr{P}^2)^T \leq \mathscr{P}$, i.e. $p_{jk}p_{ki} \leq p_{ij}$ for all $k = 1, 2, \ldots, n$.

Theorem 1.2 *Let* $\mathscr{P} \in \mathscr{M}_n^{FM}$.

(i) If $\mathscr{P}$ *is symmetric, then* $adj(\mathscr{P})$ *is symmetric.*
(ii) If $\mathscr{P}$ *is transitive, then* $adj(\mathscr{P})$ *is transitive.*
(iii) If $\mathscr{P}$ *is circular, then* $adj(\mathscr{P})$ *is circular.*

Example 1.4 Let $\mathscr{P} = \begin{pmatrix} 0.5 & 0.6 & 0.5 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.8 & 0.6 \end{pmatrix}$

Then $\mathscr{P}^2 = \begin{pmatrix} 0.5 & 0.6 & 0.5 \\ 0.4 & 0.8 & 0.4 \\ 0.6 & 0.7 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.6 & 0.5 \\ 0.4 & 0.8 & 0.4 \\ 0.6 & 0.7 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.6 & 0.5 \\ 0.4 & 0.8 & 0.4 \\ 0.5 & 0.7 & 0.5 \end{pmatrix} \leq \mathscr{P}$.

This shows that $\mathscr{P}$ is transitive.

Now, $adj(\mathscr{P}) = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{pmatrix}$

$(adj(\mathscr{P}))^2 = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.5 \end{pmatrix} \leq (adj(\mathscr{P})$.

Thus, $adj(\mathscr{P})$ is transitive.

This verifies (ii) of the above theorem.

Property 1.8 For any FM $\mathscr{P} \in \mathscr{M}_n^{FM}$, then $\mathscr{P}adj(\mathscr{P})$ is transitive.

Proof Let $A = \mathscr{P}adj(\mathscr{P})$. Then the ijth element a_{ij} of A is

$$a_{ij} = \sum_{k=1}^{n} p_{ik}det(\mathscr{P}_{jk}) = p_{i\lambda}\det(\mathscr{P}_{j\lambda}).$$

(Assume that $p_{i\lambda}\det(\mathscr{P}_{j\lambda})$ is maximum among all other terms.)

Now,

$$\begin{aligned} a_{ij}^{(2)} &= \sum_{k=1}^{n} p_{is}p_{sj} = \sum_{s=1}^{n} \left[\left(\sum_{l=1}^{n} p_{il}det(\mathscr{P}_{sl}) \right) \left(\sum_{t=1}^{n} p_{st}det(\mathscr{P}_{jt}) \right) \right] \\ &= \sum_{s=1}^{n} p_{iu}det(\mathscr{P}_{su})p_{sv}det(\mathscr{P}_{jv}) \leq p_{iu}det(\mathscr{P}_{jv}) \\ &\leq p_{i\lambda}\det(\mathscr{P}_{j\lambda}), \end{aligned}$$

for some $u, v \in \{1, 2, \ldots, n\}$.

Hence, $(\mathscr{P}\ adj(\mathscr{P}))^2 \leq \mathscr{P}\ adj(\mathscr{P})$. □

1.4 Operators Based on t-Norm and t-Conorm

The triangular norm (t-norm) and conorm are very important functions in studying fuzzy mathematics.

Definition 1.8 A **triangular norm** (or t-norm) is a binary operation T defined on the unit interval [0, 1], i.e. $T : [0, 1] \times [0, 1] \to [0, 1]$, which satisfies the following four axioms:

(i) $T(1, a) = a$, (boundary condition).
(ii) $T(a, b) = T(b, a)$, (commutativity).
(iii) $T(a, T(b, c)) = T(T(a, b), c)$, (associativity).
(iv) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, (monotonicity)

for all $a, b, c, d \in [0, 1]$.

For a t-norm, there is a **triangular conorm** (dual of T) is the function $S : [0, 1] \times [0, 1] \to [0, 1]$ defined by $S(a, b) = 1 - T(1 - a, 1 - b)$.
Several t-norms and t-conorm are defined in literature, some of the commonly used norms are defined below.

(i) **(Minimum T_M and maximum S_M)**

$$T_M(a, b) = \min\{a, b\},\ S_M(a, b) = \max\{a, b\}.$$

(ii) **(Product T_P and probabilistic sum S_P)**

$$T_P(a, b) = a.b,\ S_P(a, b) = a + b - a.b.$$

(iii) **(Einstein product T_E and Einstein sum S_E)**

$$T_E(a, b) = \frac{a.b}{1 + (1 - a)(1 - b)},\ S_E(a, b) = \frac{a + b}{1 + a.b}.$$

(iv) **(Drastic product T_D and Drastic sum S_D)**

$$T_D(a, b) = \begin{cases} 0, & \text{if } (a, b) \in [0, 1)^2 \\ \min\{a, b\} & \text{otherwise} \end{cases} \quad S_D(a, b) = \begin{cases} 1, & \text{if } (a, b) \in (0, 1]^2 \\ \max\{a, b\} & \text{otherwise} \end{cases}.$$

(v) **(Hamacher t-norm and t-conorm T_H and S_H)**

$$T_H(a, b) = \frac{a.b}{\lambda + (1-\lambda)(a + b - a.b)},\ S_H(a, b) = \frac{a + b - a.b - (1-\lambda)a.b}{1 - (1-\lambda)a.b},$$

$\lambda \geq 0$.

(vi) **(Dombi t-norm and t-conorm T_{Dom} and S_{Dom})**

$$T_{Dom}(a,b) = \frac{1}{1+\left\{\left(\frac{1-a}{a}\right)^k + \left(\frac{1-b}{b}\right)^k\right\}^{1/k}},$$

$$S_{Dom}(a,b) = 1 - \frac{1}{1+\left\{\left(\frac{a}{1-a}\right)^k + \left(\frac{b}{1-b}\right)^k\right\}^{1/k}}, \quad k \geq 1.$$

Hamacher norm is more generalized t-norm and t-conorm. When $\lambda = 1$, then the Hamacher t-norm and t-conorm reduce to $T_P(a,b) = a.b$, $S_P(a,b) = a+b-a.b$; when $\lambda = 2$, then Hamacher t-norm and t-conorm reduce to Einstein t-norm T_E and t-conorm S_E, respectively.

Many operators on FMs can be defined from the t-norms and t-conorms. Here, two cases are discussed.

1.4.1 Operators Based on T_P and S_P

Here, the product and probabilistic sum are consider as the t-norm and t-conorm and based on these norms two new operators are defined for FMs. Lot of properties are also presented. The main contribution of this section is taking from [96]. Based on these t-norm and s-norm, the following two operators are defined.

$$\text{(i)} \quad x \oplus_p y = x + y - x.y \quad \text{and} \tag{1.21}$$

$$\text{(ii)} \quad x \odot_p y = x.y, \tag{1.22}$$

where the operations '+', '−', and '.' are ordinary addition, subtraction, and multiplication, respectively.

These two operators are utilize for the FMs as defined below.

$$\text{(i)} \quad \mathscr{P} \oplus_p \mathscr{Q} = (p_{ij} + q_{ij} - p_{ij}.q_{ij}) \tag{1.23}$$

$$\text{(ii)} \quad \mathscr{P} \odot_p \mathscr{Q} = (p_{ij}.q_{ij}). \tag{1.24}$$

The operators $\oplus_p$ and $\oplus_p$ are repeatedly used to defined a new type of power of FMs.

$$\text{(iii)} \quad \mathscr{P}^{[k+1]} = \mathscr{P}^{[k]} \odot_p \mathscr{P}, \mathscr{P}^{[1]} = \mathscr{P}, k = 1, 2, \ldots \tag{1.25}$$

$$\text{(iv)} \quad [k+1]\mathscr{P} = [k]\mathscr{P} \oplus_p \mathscr{P}, [1]\mathscr{P} = \mathscr{P}, k = 1, 2, \ldots \tag{1.26}$$

It is very interesting that the matrices $\mathscr{P}^{[k]}$ and $[k]\mathscr{P}$ converge to null matrix $\mathscr{O}$ and the matrix $\mathscr{U}$, respectively.

Lemma 1.1 *Let* $\mathscr{P} = (p_{ij}) \in \mathscr{M}_n^{FM}$.

(i) If $p_{ij} < 1$ *for all* i, j, *then* $\lim_{k\to\infty} \mathscr{P}^{[k]} = \mathscr{O}$.
(ii) If $p_{ij} > 0$ *for all* i, j, *then* $\lim_{k\to\infty} [k]\mathscr{P} = \mathscr{U}$.

Proof For FMs $\mathscr{P} = (p_{ij})$ and $\mathscr{Q} = (q_{ij})$, $\mathscr{P} \odot_p \mathscr{Q} = (p_{ij}.q_{ij})$.
Therefore, $\mathscr{P} \odot_p \mathscr{P} = \mathscr{P}^{[2]} = (p_{ij}^2)$, $\mathscr{P}^{[3]} = \mathscr{P}^{[2]} \odot_p \mathscr{P} = (p_{ij}^3)$.
In general, $\mathscr{P}^{[k]} = (p_{ij}^k)$ for any positive integer k.
Hence, $\lim_{k\to\infty} \mathscr{P}^{[k]} = \mathscr{O}$.
Again, $[2]\mathscr{P} = \mathscr{P} \oplus_p \mathscr{P} = (2p_{ij} - p_{ij}^2) = (1 - (1 - p_{ij})^2)$. Also, $1 - p_{ij} \le 1$. Thus, $[k]\mathscr{P} = (1 - (1 - p_{ij})^k)$ for positive integer k, and hence $\lim_{k\to\infty} [k]\mathscr{P} = \mathscr{U}$. □

If $0 \le a, b \le 1$, $a.b \le a$ and $a.b \le b$, then $a.b \le \min(a, b)$.
Thus, $\mathscr{P} \odot_p \mathscr{Q} \le \min\{\mathscr{P}, \mathscr{Q}\}$.

Property 1.9 Let $\mathscr{P}$ and $\mathscr{Q}$ be two fuzzy matrices.

(i) $\mathscr{P} \oplus_p \mathscr{Q} \ge \mathscr{P} \odot_p \mathscr{Q}$.
(ii) If $\mathscr{P}$ and $\mathscr{Q}$ are symmetric, then $\mathscr{P} \oplus_p \mathscr{Q}$ and $\mathscr{P} \odot_p \mathscr{Q}$ are symmetric.
(iii) If $\mathscr{P}$ and $\mathscr{Q}$ are nearly irreflexive, then $\mathscr{P} \oplus_p \mathscr{Q}$ and $\mathscr{P} \odot_p \mathscr{Q}$ are nearly irreflexive.

The operator $\oplus_p$ is expanding while the operator $\odot_p$ is contracting. That is, if the operator $\oplus_p$ is used repeatedly on a FM say $\mathscr{P}$, then it converses to $\mathscr{U}$. In case of the operator $\odot$ the FM converges to $\mathscr{O}$.

Property 1.10 For any fuzzy matrix $\mathscr{P}$,

(i) $\mathscr{P} \oplus_p \mathscr{P} \ge \mathscr{P}$.
(ii) $\mathscr{P} \odot_p \mathscr{P} \le \mathscr{P}$.

The following results are obvious.
Both the operators $\oplus_p$ and $\odot_p$ are commutative as well as associative. But, unfortunately, the operators $\oplus_p$ and $\odot_p$ do not obey the De Morgan's laws over transpose; it follows the following rules.

Property 1.11 Let $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$ be three FMs. Then

(i) $(\mathscr{P} \oplus_p \mathscr{Q})^T = \mathscr{P}^T \oplus_p \mathscr{Q}^T$.
(ii) $(\mathscr{P} \odot_p \mathscr{Q})^T = \mathscr{P}^T \odot_p \mathscr{Q}^T$.
(iii) If $\mathscr{P} \le \mathscr{Q}$ then $\mathscr{P} \oplus_p \mathscr{R} \le \mathscr{Q} \oplus_p \mathscr{R}$ and $\mathscr{P} \odot_p \mathscr{R} \le \mathscr{Q} \odot_p \mathscr{R}$.

Property 1.12 For any $\mathscr{P} \in \mathscr{M}_n^{FM}$,

(i) $I_n \oplus_p (\mathscr{P} \oplus_p \mathscr{P}^T)$ is reflexive and symmetric.
(ii) $\mathscr{P} \ominus I_n$ is irreflexive.
(iii) $\mathscr{P} \oplus_p \mathscr{P}^T$ is nearly irreflexive and symmetric.
(iv) $I_n \oplus_p (\mathscr{P} \oplus_p \mathscr{P}^T) = I_n \vee (\mathscr{P} \oplus_p \mathscr{P}^T)$.

Proof

(iii) Let $\mathscr{R} = \mathscr{P} \oplus_p \mathscr{P}^T$, i.e. $r_{ij} = p_{ij} + p_{ji} - p_{ij}.p_{ji} = r_{ji}$.
Therefore, $\mathscr{R}$ is symmetric.
Again, $r_{ii} = 2p_{ii} - p_{ii}^2$. Since $\mathscr{P}$ is nearly irreflexive, $p_{ii} \leq p_{ij}$.
Therefore, $1 - p_{ii} \geq 1 - p_{ij}$.
Now, $r_{ij} - r_{ii} = \{1 - (1 - p_{ij}).(1 - p_{ji})\} - \{1 - (1 - p_{ii}).(1 - p_{ii})\}$
$= (1 - p_{ii}).(1 - p_{ii}) - (1 - p_{ij}).(1 - p_{ji}) \geq 0$.
Therefore, $\mathscr{P} \oplus_p \mathscr{P}^T$ is nearly irreflexive and symmetric.
Other proofs are trivial.

□

1.4.2 Operators Based on T_H and S_H

Like previous operators, one can defined addition and multiplication operators for FMs based on the Hamacher t-norm and t-conorm.

When $\lambda = 0$, the Hamacher t-norm and t-conorm become

$$T_H(a, b) = \frac{a.b}{a + b - a.b}, \quad S_H(a, b) = \frac{a + b - 2a.b}{1 - a.b}.$$

These t-norm and t-conorm are used to defined multiplication and addition operators. The Hamacher sum ($\oplus_H$) and product ($\odot_H$) between two FMs $\mathscr{P} = (p_{ij})$ and $\mathscr{Q} = (q_{ij})$ are defined as

$$\mathscr{P} \oplus_H \mathscr{Q} = \left(\frac{p_{ij} + q_{ij} - 2p_{ij}q_{ij}}{1 - p_{ij}q_{ij}}\right), \text{ and } \mathscr{P} \odot_H \mathscr{Q} = \left(\frac{p_{ij}q_{ij}}{p_{ij} + q_{ij} - p_{ij}q_{ij}}\right).$$

It can be shown by direct calculation that

$$\frac{ab}{a + b - ab} \leq \frac{a + b - 2ab}{1 - ab} \quad \text{for any } a, b \in (0, 1]. \tag{1.27}$$

Some properties are presented below.

Property 1.13 For any FMs $\mathscr{P}, \mathscr{Q}$,

(i) $\mathscr{P} \odot_H \mathscr{Q} \leq \mathscr{P} \oplus_H \mathscr{Q}$.
(ii) $\mathscr{P} \oplus_H \mathscr{P} \geq \mathscr{P}$.
(iii) $\mathscr{P} \odot_H \mathscr{P} \leq \mathscr{P}$.

Proof

(i) Follows from definition of $\odot_H$ and $\oplus_H$.
(ii) The ijth element of $\mathscr{P} \oplus_H \mathscr{P}$ is $\frac{2p_{ij} - 2p_{ij}^2}{1 - p_{ij}^2} = \frac{2p_{ij}}{1 + p_{ij}} \geq p_{ij}$, for all i, j.
(since $p_{ij}^2 - p_{ij} \leq 0$, or $p_{ij}^2 + p_{ij} - 2p_{ij} \leq 0$, i.e. $p_{ij}^2 + p_{ij} \leq 2p_{ij}$)