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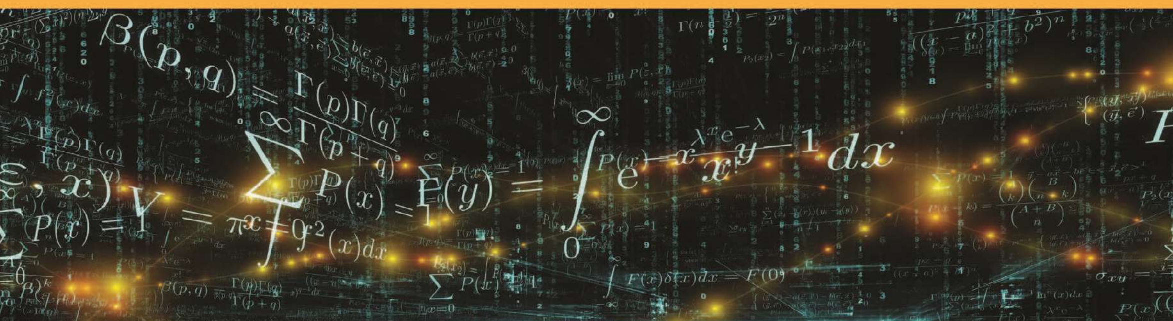
Traditional Functional-Discrete Methods for the Problems of Mathematical Physics

New Aspects

Volodymyr Makarov
Nataliya Mayko



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Preface

New Aspects of the Traditional Functional-Discrete Methods for the Problems of Mathematical Physics

This book is based on the authors' latest research focusing on obtaining weighted accuracy estimates of numerical methods for solving boundary value and initial value problems. The idea of such estimates is based on Volodymyr Makarov's observation that due to the Dirichlet boundary condition for a differential equation in a canonical domain (e.g. on an interval or in a rectangle), the accuracy of the approximate solution in the mesh nodes near the boundary of the domain is higher compared to the accuracy in the mesh nodes away from the boundary. The study commenced about 30 years ago with the finite-difference scheme for the two-dimensional elliptic equation with the generalized solution from Sobolev spaces and later continued for other types of problems: quasilinear stationary and non-stationary equations with boundary conditions, boundary value problems for equations with fractional derivatives, the Cauchy problem and boundary value problems for abstract differential equations in Hilbert and Banach spaces, etc. For brevity, to name the influence that boundary and initial conditions have on the accuracy of the approximate solution, we choose to use the wording *boundary effect* or *initial effect*. Thus, we obtain a priori accuracy weighted estimates, taking into account the boundary and initial effects. These effects are quantitatively described by means of a suitable weight function, which characterizes the distance of a point to the boundary of the domain.

To our best knowledge, there are very few publications addressing these issues. It is our hope that the present book will meet this need and thus help to inspire new generations of students, researchers and practitioners. We also sincerely hope that our approach, methods and techniques developed in the book will contribute not only to the theory of the numerical analysis but also to its applications, since awareness of the boundary and initial effects makes it possible to use a greater mesh step near the boundary of the domain. Since the finite-difference approximations and the mesh schemes proposed and studied in this book are traditional and not exotic, they can be used for solving a wide range of problems in physics, engineering, chemistry, biology, finance, etc.

The target audience of our book is graduate and postgraduate students, specialists in numerical analysis, computational and applied mathematics, and engineers. As in books like ours, the analytical and numerical components are closely intertwined, we expect the potential reader to have fluency in both univariate and multivariate analysis, familiarity with ordinary and partial differential equations, basic knowledge of functional analysis, advanced knowledge of numerical analysis, and be at ease with modern scientific computing. These mathematical prerequisites will make the text much easier to understand.

We are deeply grateful to Professor Nikolaos Limnios and Professor Dmytro Koroliuk for their suggestion to submit the manuscript, to Professor Ivan Gavrilyuk for many fruitful discussions, and to Professor Vyacheslav Ryabichev for his valuable software advice and constant professional assistance. We also express our gratitude to the team at ISTE Group for useful recommendations and careful preparation of our book for publication. We are immensely thankful to our families for everyday understanding, support and encouragement.

Volodymyr MAKAROV,
Institute of Mathematics of the National Academy of Sciences of Ukraine,
Nataliya MAYKO,
Taras Shevchenko National University of Kyiv,
Kyiv, Ukraine
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Introduction

It is well known that the vast majority of boundary value and initial value problems cannot be solved exactly and require the use of appropriate approximate methods. An important characteristic of any approximate method is its accuracy. To estimate the accuracy, we traditionally use a certain discretization parameter: a mesh step, the number of terms of the partial sum of the series, etc.

However, for both theoretical and practical reasons, it is also important to take into account the influence of other factors, for example, the so-called boundary and initial effects. Precisely, the boundary effect means that due to the Dirichlet boundary condition for a differential equation in the canonical domain, the accuracy of the approximate solution near the boundary of the domain is higher compared to the accuracy further from the boundary. A similar situation is observed for non-stationary equations near those mesh nodes where the initial condition is set.

For the quantitative characteristics of the boundary or initial effect, we can take an a priori error estimate (in a certain mesh norm) with a certain weight function, which characterizes the distance of a point inside the domain to the boundary of the domain. The idea of such estimates was first announced by Volodymyr Makarov in Makarov (1987) for an elliptic equation in case of generalized solutions from Sobolev spaces and developed further in publications for quasilinear stationary and non-stationary equations. Since the concept was quite new, there were (and still are) very few publications on this subject. In some respects, the same issue is studied in the works of Galba (1985) and Molchanov and Galba (1990). However, they assume only the classical smoothness of solutions and do not consider time-dependent problems.

In this book, we develop our previous studies and present some new results on the impact of initial and boundary conditions on the accuracy of the following methods: the finite-difference method for elliptic and parabolic equations, the discrete method for solving equations with fractional derivatives, and the Cayley transform method for abstract differential equations in Hilbert and Banach spaces. Regardless of the type of problem or method, our main focus is always on obtaining weighted estimates with a proper weight function.

For a better understanding of the reasoning and easier navigation through the computation, some information is assumed to be known to the reader from classical mathematics courses, while the rest is provided directly in the text. Some of the formulas may seem a bit long and cumbersome, but this is partly because we are trying to be as detailed as possible and help the reader follow the calculations with ease.

This book consists of five chapters. Chapters 1 and 2 are devoted to the study of the accuracy of finite-difference schemes for stationary and non-stationary equations respectively, taking into account the influence of boundary and initial conditions (in the sense of Makarov as mentioned above).

The finite-difference method is historically one of the first and most recognized numerical methods for solving problems of mathematical physics, mainly due to its universality and convenience in practical implementation. In recent decades, it has gained considerable popularity due to growing interest in the study of nonlinear processes in various fields of physics, chemistry, seismology, ecology, etc. Mathematical models of such phenomena involve nonlinear partial differential equations. For example, in aerodynamics and hydrodynamics, the one-dimensional quasilinear Burgers parabolic equation arises as an adequate mathematical model of turbulence. A special case of the Burgers equation is the quasilinear transport equation (the Hopf equation), which is the simplest equation describing discontinuous flows or flows with shock waves. In biology, ecology, physiology, combustion theory, crystallization theory, plasma physics, etc., the Fisher–Kolmogorov–Petrovsky–Piskunov equation (the Fisher–KPP equation) plays an important role as the simplest semi-linear parabolic equation. The propagation of shallow water waves that weakly and nonlinearly interact, ion acoustic waves in plasma, acoustic waves on crystal lattices, etc. are often modeled by the Korteweg–de Vries equation (the KdV equation). Many publications are devoted to finite-difference schemes for solving problems for elliptic and parabolic equations with dynamic conjugation conditions at the contact boundary (which is associated with the presence of concentrated heat capacities in a heat-conducting medium) and/or dynamic boundary conditions (which model heat conduction in a solid body in contact with fluid, as well as processes in semiconductor devices). In the mathematical modeling of some processes in ecology, physics and technology, when it is impossible to set the exact values of the desired solution at the boundary of a domain, problems with non-local boundary conditions usually arise.

These and many other examples demonstrate that the finite-difference method is actively developing and is widely used to solve current scientific and technical problems. At the same time, there are very few publications dedicated to the study of the initial and boundary effects in the above sense, and our book is a certain step towards filling this gap. One of the first such works is the announcement (Makarov 1989) that deals with the problem

$$Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + q(x)u(x) = f(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \Gamma = \partial\Omega,$$

where

$$\nu \sum_{i=1}^{2} \xi_i^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \le \mu \sum_{i=1}^{2} \xi_i^2 \quad \forall x = (x_1, x_2) \in \Omega,\ \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$
$$\nu > 0,\ a_{ij}(x) = a_{ji}(x),\ q(x) \in C(\bar{\Omega}),\ q(x) \ge 0,$$

and $\Omega = \{(x_1, x_2) : 0 < x_\alpha < 1, \alpha = 1, 2\}$ is a unit square. The problem is discretized by the finite-difference scheme

$$Ay \equiv -\frac{1}{2} \sum_{i,j=1}^{2} \left[\left(a_{ij}^{-0.5i} y_{\bar{x}_j} \right)_{x_i} + \left(a_{ij}^{+0.5i} y_{x_j} \right)_{\bar{x}_i} \right] + qy = \varphi(x),\ x \in \omega, \qquad [I.1]$$
$$y(x) = 0,\ x \in \gamma,$$

where $\omega = \omega_1 \times \omega_2$, $\omega_\alpha = \{x_\alpha = i_\alpha h_\alpha : i_\alpha = 1, 2, \ldots, N_\alpha - 1, h_\alpha = 1/N_\alpha\}$, $\alpha = 1, 2$; γ is a boundary of the mesh ω. Some traditional notations for finite-difference schemes from Samarskii (2001) are used here, for example: $a_{11}^{\pm 0.5_1} = a_{11}(x_1 \pm 0.5h_1, x_2)$,

$$y_{x_1} = \frac{y(x_1 + h_1, x_2) - y(x_1, x_2)}{h_1},\ y_{\bar{x}_1} = \frac{y(x_1, x_2) - y(x_1 - h_1, x_2)}{h_1}, \text{ etc.}$$

The main result was presented in the following statement.

THEOREM.– *Let* $\varphi(x), f(x), a_{ij}(x) \in W_2^3(\Omega)$ *and* $u(x) \in W_2^4(\Omega)$. *Then, there exists* $h_0 > 0$ *such that for all* $h \in (0, h_0]$ *the accuracy of the finite-difference scheme [I.1] is characterized by the weighted estimate*

$$\left\| \rho^{-1/2}(x)\left[y(x) - u(x)\right] \right\|_{C(\omega)} \le Mh^2 \, \| u \|_{W_2^4(\Omega)},$$

with the weight function $\rho(x) = \min\left\{x_1 x_2, x_1(1-x_2), (1-x_1)x_2, (1-x_1)(1-x_2)\right\}$.

This idea is further developed in the present book for other types of boundary conditions for elliptic and parabolic equations. It is worth mentioning that the important stages in obtaining such weighted estimates are the evaluation of discrete Green's functions and the analysis of approximation errors. Each time, when it is necessary to estimate discrete Green's functions, we apply the following proposition, which is formulated and proved in Samarskii et al. (1987, p. 54).

MAIN LEMMA.– *Let the following assumptions be fulfilled: 1)* $A: H \to H$ *is a self-adjoint operator acting in a Hilbert space H; 2)* $B: H^* \to H$ *is a linear operator; 3) the inverse operator* A^{-1} *exists; 4)* $\| B^* v \|_* \le \gamma \| Av \|$ *for all* $v \in H$, *where* $B^*: H \to H^*$ *is the adjoint operator of* B, $(y, v)_*$ *and* $\| v \|_* = \sqrt{(v, v)_*}$ *are an inner product and an associate norm in* H^* *respectively. Then,* $\| A^{-1} B v \| \le \gamma \| v \|_*$ *for all* $v \in H^*$.

Similarly, when it comes to estimating an approximation error for a generalized solution from Sobolev spaces, we refer to the Bramble–Hilbert lemma (e.g. Samarskii et al. (1987, p. 29)). We recall it here for convenience.

LEMMA (BRAMBLE–HILBERT).– *Let* $\Omega \subset \mathbb{R}^n$ *be an open convex bounded set of the diameter* $d > 0$, *let* $l(u)$ *be a bounded linear functional in the space* $W_2^m(\Omega)$ *with* $0 < m = \bar{m} + \lambda$, *where* $\bar{m}$ *is a positive non-negative number and* $0 < \lambda \le 1$, *namely:*

$$| l(u) | \le M \left\{ \sum_{j=0}^{\bar{m}} d^{2j} \, | u |_{W_2^j(\Omega)}^2 + d^{2m} \, | u |_{W_2^m(\Omega)}^2 \right\}^{1/2},$$

and let $l(u)$ *turn into zero on polynomials of degree* $\bar{m}$ *of variables* $x_1, x_2, \ldots, x_n$. *Then, there exists a positive constant* $\bar{M}$, *which is dependent on* Ω *and independent of* $u(x)$, *such that the following inequality holds true*:

$$|l(u)| \leq M\bar{M}d^m |u|_{W_2^m(\Omega)} \quad \forall u \in W_2^m(\Omega).$$

The study of the boundary and initial effects is also of great interest for new classes of problems, for example, related to the application of fractional integro-differentiation. In Chapter 3, we address the accuracy of the mesh methods for solving boundary value problems for differential equations with fractional derivatives.

For almost 300 years (from 1695 until recently) this branch of classical analysis was no more than an abstract mathematical theory. However, over the past several decades, fractional analysis has found wide applications in the construction of adequate mathematical models of many natural and social phenomena, as evidenced by a considerable number of publications (e.g. Kilbas et al. (2006); Sabatier et al. (2007); Nakagawa et al. (2010), to mention a few). Due to the ability to model hereditary phenomena with long memory, fractional analysis is widely used in viscoelasticity problems, models of anomalous diffusion (in particular, subdiffusion), control theory, electrodynamics and nonlinear hydroacoustics, for multidimensional signal processing in radiophysics, etc. However, exact solutions of such problems can be found only in a few (mostly linear) cases. The integral nature of the fractional derivative (in contrast to the classical derivative, which is local in nature) complicates the construction, analysis and implementation of approximate methods. For example, one of such problems is a considerable increase in costs associated with large data storage due to systems of linear equations with large, densely filled matrices. This requires adaptation of known and development of new approaches in the field of fractional numerical analysis, which is actively developing and constantly updated (Li and Zeng 2012; Jin et al. 2017; Jovanović et al. 2019).

Throughout this chapter, we use exclusively the *left Riemann–Liouville derivative of order* $\alpha > 0$ for a function $f(x)$:

$${}_{a}^{RL}D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \quad n = \lfloor \alpha \rfloor + 1.$$

It is quite natural that discretization of the fractional derivative is an important step in the construction of an effective approximate method for solving any fractional differential equation. The most widely used approximations of fractional derivatives can be roughly divided into two groups. The first group includes convolution-type quadrature formulas, while the second group includes the so-called L1 and L2 schemes. Each group has its advantages and disadvantages, which is discussed in detail by Jin et al. (2019). For instance, quadrature formulas are flexible, convenient for analysis, have good stability properties, but are applicable mainly to uniform meshes. The strengths of the L1 and L2 methods are flexibility, easy implementation and the possibility of generalization for non-uniform meshes, while the disadvantages are sophisticated analysis and the first order of accuracy (in the case of direct application without proper correction).

However, in our book, we take another approach. Whether it is a one-dimensional equation with constant or variable coefficients, the Dirichlet boundary value problem for Poisson's equation with the fractional derivative for one of the two variables in a unit square, or the two-dimensional Goursat problem, we reduce each of them to an integral equation of the second kind. Then, we study the kernel and apply the fixed-point iteration to show that a solution of the problem belongs to a particular Sobolev class. After that we propose a mesh scheme and study its convergence in some discrete norm with a weight function, taking into account the boundary condition.

Chapters 4 and 5 are devoted to the Cayley transform method first proposed in Arov and Gavrilyuk (1993) and Arov et al. (1995). This method is designed for the constructive representation of exact and approximate solutions of abstract differential equations in Hilbert and Banach spaces. One of the advantages of this method is the automatic dependence of its accuracy on the smoothness of the input data. This means that the Cayley transform method belongs to the methods *without saturation of accuracy* according to Babenko (2002), and is therefore optimal in a certain sense. The construction of such methods is a topical issue of numerical analysis.

The importance of the Cayley transform method is also explained by the observation that mathematical models of many processes studied in science and technology can be written in the form of differential equations in Banach and Hilbert spaces, namely in the form of the Cauchy problem for the first-order differential equation:

$$\begin{aligned} u'(t)+Au(t)=f(t), \quad x\in(0,T], \\ u(0)=u_0, \end{aligned} \qquad [I.2]$$

the Cauchy problem for the second-order differential equation:

$$\begin{aligned} u''(t)+Au(t)=f(t), \quad x\in(0,T], \\ u(0)=u_0,\ u'(0)=u_1, \end{aligned} \qquad [I.3]$$

and the boundary value problem for the second-order differential equation:

$$\begin{gathered} u''(x) - Au(x) = -f(x), \quad x \in (0,1), \\ u(0) = u_0, \quad u(1) = u_1. \end{gathered} \qquad [I.4]$$

Here, A is a closed linear operator with the dense domain $D(A)$ in a Banach space E (or a self-adjoint positive definite operator with the dense domain $D(A)$ in a Hilbert space *H*), u_0 and u_1 are given vectors from E (or from *H*), $f(\cdot)$ and $u(\cdot)$ are respectively a given function and an unknown solution with values in E (or in *H*).

For example, in the case of a Hilbert space $H = L_2(0,1)$ and the operator $Au(x) = -u''(x),\ D(A) = H^2(0,1) \cap \overset{\circ}{H}{}^1(0,1)$, the Cauchy problems [I.2] and [I.3] turn into the initial–boundary value problems for a parabolic and hyperbolic equations respectively:

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,1),\ t \in (0,T], \\ u(0,t) = 0,\ u(1,t) = 0,\ t \in [0,T], \\ u(x,0) = u_0(x),\ x \in [0,1], \end{gathered}$$

$$\begin{gathered} u_{tt}(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,1),\ t \in (0,T], \\ u(0,t) = 0,\ u(1,t) = 0,\ t \in [0,T], \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in [0,1]. \end{gathered}$$

Similarly, in the case of a Hilbert space $H = L_2(0,1)$ and the operator

$$Au(y) = -u''(y),\ D(A) = H^2(0,1) \cap \overset{\circ}{H}{}^1(0,1),$$

the boundary value problem [I.4] takes the form of the Dirichlet boundary value problem for Poisson's equation:

$$\begin{gathered} u_{xx} + u_{yy} = -f(x,y), \quad (x,y) \in \Omega = (0,1)^2, \\ u(x,y) = 0, \quad (x,y) \in \partial\Omega. \end{gathered}$$

For convenience, we briefly recall the results of the pioneering publication (Arov and Gavrilyuk 1993). In a Hilbert space H and for a bounded operator A, it studies the Cauchy problem

$$\begin{aligned} x'(t) + Ax(t) = 0, \quad t > 0, \\ x(0) = x_0, \end{aligned} \qquad [I.5]$$

and proves that the solution $x(t)$ can be represented by the series

$$x(t) = e^{-\gamma t} \sum_{p=0}^{\infty} (-1)^p L_p^{(0)}(2\gamma t) \left[y_{\gamma,p} + y_{\gamma,p+1} \right], \qquad [I.6]$$

where $\gamma > 0$ is an arbitrary number, $L_p^{(\alpha)}(t) = \sum_{k=0}^{p} \binom{p+\alpha}{p-k} \frac{(-t)^k}{k!}$ are the Laguerre polynomials and the sequence $(y_{\gamma,p})$ satisfies the recurrence relation

$$y_{\gamma,p+1} = T_\gamma y_{\gamma,p} = T_\gamma^{p+1} y_{\gamma,0}, \; p = 0,1,\ldots, \quad y_{\gamma,0} = x_0,$$

and therefore $y_{\gamma,p}$ can be effectively found from the recurrent sequence of the operator equations (with the same operator and different right-hand sides):

$$(\gamma I + A) y_{\gamma,p+1} = (\gamma I - A) y_{\gamma,p}, \; p = 0,1,\ldots, \quad y_{\gamma,0} = x_0 .$$

The partial sum of series [I.6] is then taken as an approximate solution of problem [I.5]:

$$x_N(t) = e^{-\gamma t} \sum_{p=0}^{N} (-1)^p L_p^{(0)}(2\gamma t) \left[y_{\gamma,p} + y_{\gamma,p+1} \right].$$

The accuracy of this approximation is characterized by the estimate

$$\sup_{t \ge 0} \| x(t) - x_N(t) \| \le \frac{q_\gamma^{N+1}}{1 - q_\gamma} \| x_0 \| \quad (0 < q_\gamma < 1),$$

which indicates that the proposed Cayley transform method is exponentially convergent.

Other approaches to the construction of approximate solutions of operator differential equations are used in Gorodnii (1998) and Kashpirovskii and Mytnik (1998).

The results obtained in Arov and Gavrilyuk (1993) and Arov et al. (1995) were then extended to other abstract problems in Hilbert and Banach spaces and subsequently summarized in a monograph (Gavrilyuk and Makarov 2004). Our book continues this tradition and develops the Cayley transform method even further – now taking into account the influence of boundary and initial conditions. Therefore, the proposed technique of obtaining weighted estimates with a proper weight function meets both challenges – taking into account the boundary and initial effects and also the smoothness of input data (e.g. coefficients and the right-hand side of the equation, initial vectors, etc.).

With this brief introduction, we sincerely hope that the reader will share our interest in the issues discussed above and embark on a journey of new research and discovery.

1

Elliptic Equations in Canonical Domains with the Dirichlet Condition on the Boundary or its Part

1.1. A standard finite-difference scheme for Poisson's equation with mixed boundary conditions

We consider here the following boundary value problem:

$$
\begin{aligned}
&-\Delta u = f(x), \quad x \in D, \\
&-\frac{\partial u}{\partial x_1} + \sigma u(x) = 0, \quad x \in \Gamma_{-1}, \\
&u(x) = 0, \quad x \in \Gamma \setminus \Gamma_{-1},
\end{aligned}
\qquad [1.1]
$$

where $x = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator in a Cartesian coordinate system, $D = \{x = (x_1, x_2) : 0 < x_\alpha < l_\alpha, \alpha = 1, 2\}$ is a rectangle, $\Gamma = \partial D$ is a boundary of D, $\Gamma_{-1} = \{x = (0, x_2) : 0 < x_2 < l_2\}$ is the left side of D, $f(x)$ is a given function, $\sigma = \text{const} \geq 0$.

1.1.1. *Discretization of the BVP*

To construct and study the discrete analogue of problem [1.1], we use the traditional notation of the theory of finite-difference schemes (e.g. Samarskii (2001)). We introduce the following sets of nodes:

$$\omega_\alpha = \{i_\alpha h_\alpha,\ i_\alpha = 1, \ldots, N_\alpha - 1,\ h_\alpha = l_\alpha / N_\alpha\},\ N_\alpha \geq 2 \text{ is an integer number,}$$

$$\omega_\alpha^- = \omega_\alpha \cup \{0\}, \quad \omega_\alpha^+ = \omega_\alpha \cup \{1\}, \quad \bar{\omega}_\alpha = \omega_\alpha \cup \{0\} \cup \{1\},$$

$$\omega = \omega_1 \times \omega_2, \quad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2, \quad \gamma = \bar{\omega} \setminus \omega; \qquad [1.2]$$

$$\gamma_{-\alpha} = \{x_\alpha = 0,\ x_{3-\alpha} \in \omega_{3-\alpha}\}, \quad \gamma_{+\alpha} = \{x_\alpha = l_\alpha,\ x_{3-\alpha} \in \omega_{3-\alpha}\}, \quad \alpha = 1, 2.$$

We also use the operators of the exact finite-difference schemes:

$$T_2 v(x_1, x_2) = \frac{1}{h_2^2} \int_{x_2 - h_2}^{x_2 + h_2} \left(h_2 - |x_2 - \xi_2|\right) v(x_1, \xi_2)\, d\xi_2, \quad x \in \omega \cup \gamma_{-1},$$

$$T_1 v(x_1, x_2) = \begin{cases} \dfrac{1}{h_1^2} \displaystyle\int_{x_1 - h_1}^{x_1 + h_1} \left(h_1 - |x_1 - \xi_1|\right) v(\xi_1, x_2)\, d\xi_1, & x \in \omega, \\ \dfrac{2}{h_1^2} \displaystyle\int_0^{h_1} \left(h_1 - \xi_1\right) v(\xi_1, x_2)\, d\xi_1, & x \in \gamma_{-1}, \end{cases}$$

Using the relations

$$T_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} = u_{\bar{x}_\alpha x_\alpha}, \quad x \in \omega,$$

$$T_1 1 = 1,\ T_1 x_1 = \frac{h_1}{3},\ T_1 x_2 = x_2,\ T_1 \frac{\partial^2 u}{\partial x_1^2} = \frac{2}{h_1}\left(u_{x_1} - \frac{\partial u}{\partial x_1}\right), \quad x \in \gamma_{-1},$$

we approximate problem [1.1] by the finite-difference scheme

$$\begin{aligned} -\Lambda y(y) &= \varphi(x), \quad x \in \omega \cup \gamma_{-1}, \\ y(x) &= 0, \quad x \in \gamma \setminus \gamma_{-1}, \end{aligned} \qquad [1.3]$$

where $\varphi(x) = T_1 T_2 f(x),\ \Lambda = \Lambda_1 + \Lambda_2$,

$$\Lambda_1 y(x) = \begin{cases} y_{\bar{x}_1 x_1}, & x \in \omega, \\ \dfrac{2}{h_1}(y_{x_1} - \sigma y), & x \in \gamma_{-1}, \end{cases} \qquad \Lambda_2 y(x) = \begin{cases} y_{\bar{x}_2 x_2}, & x \in \omega, \\ \left(1 + \dfrac{h_1 \sigma}{3}\right) y_{\bar{x}_2 x_2}, & x \in \gamma_{-1}. \end{cases}$$

For the error $z(x) = y(x) - u(x)$, we have the problem

$$\begin{aligned} -\Lambda z(x) &= \psi(x), \quad x \in \omega \cup \gamma_{-1}, \\ z(x) &= 0, \quad x \in \gamma \setminus \gamma_{-1}, \end{aligned} \qquad [1.4]$$

where $\psi(x)$ is the approximation error:

$$\psi(x) = T_1 T_2 f(x) + \Lambda u(x) = -\Lambda_1 \eta_1(x) - \eta_{2\bar{x}_2 x_2}(x), \qquad [1.5]$$

$$\eta_1(x) = (T_2 u)(x) - u(x), \ x \in \omega \cup \gamma_{-1}, \quad \eta_2(x) = \begin{cases} (T_1 u)(x) - u(x), & x \in \omega, \\ (T_1 u)(x) - u(x) - \dfrac{h_1}{3} \sigma u(x), & x \in \gamma_{-1}. \end{cases}$$

1.1.2. *Properties of the finite-difference operators*

We denote by H a set of mesh functions defined on $\bar{\omega}$ and equal to zero on $\gamma \setminus \gamma_{-1}$. The inner product and the associate norm in H are defined by the formulas

$$(y, v) = \sum_{x \in \omega} h_1 h_2 y(x) v(x) + \frac{h_1}{2} \sum_{x \in \gamma_{-1}} h_2 y(x) v(x),$$

$$\| v \| = \| v \|_{L_2(\omega)} = \sqrt{(v, v)} = \left(\sum_{x \in \omega} h_1 h_2 v^2(x) + \frac{h_1}{2} \sum_{x \in \gamma_{-1}} h_2 v^2(x) \right)^{1/2}.$$

We introduce the difference operators

$$A_\alpha, A : H \to H, \quad A_\alpha = -\Lambda_\alpha, \ \alpha = 1, 2, \quad A = A_1 + A_2 = -\Lambda.$$

(If necessary, a function defined on ω is set equal to zero for $x \in \gamma \setminus \gamma_{-1}$ and equal to arbitrary values for $x \in \gamma_{-1}$.)

Then, the finite-difference scheme [1.3] can be written as the operator equation

$$Ay = \varphi, \quad y, \varphi \in H, \qquad [1.6]$$

and similarly problem [1.4] can be written as the operator equation

$$Az = \psi, \quad z, \psi \in H.$$

LEMMA 1.1.– *The difference operator A is symmetric and positive definite.*

PROOF.– The difference operators A_1 and A_2 are symmetric and positive definite in H. Indeed, for A_1, we have

$$(A_1 y, v) = \sum_{x\in\omega} h_1 h_2 (-y_{\bar{x}_1 x_1}) v + \frac{h_1}{2} \sum_{x\in\gamma_{-1}} h_2 \frac{-2}{h_1}(y_{x_1} - \sigma y) v =$$

$$= \sum_{x\in\omega_1^+\times\omega_2} h_1 h_2 y_{\bar{x}_1} v_{\bar{x}_1} + \sigma \sum_{x\in\gamma_{-1}} h_2 y v$$

and therefore

$$(A_1 y, y) = \sum_{x\in\omega_1^+\times\omega_2} h_1 h_2 y_{\bar{x}_1}^2 + \sigma \sum_{x\in\gamma_{-1}} h_2 y^2 \ge \sum_{x_2\in\omega_2} h_2 \sum_{x_1\in\omega_1^+} h_1 y_{\bar{x}_1}^2 \ge$$

$$\ge \sum_{x_2\in\omega_2} h_2 \frac{2}{l_1^2}\left(\sum_{x_1\in\omega_1} h_1 y^2(x) + \frac{h_1}{2} y(0, x_2) \right) = \frac{2}{l_1^2} \| y \|^2 .$$

And similarly for A_2, we obtain

$$(A_2 y, v) = \sum_{x\in\omega} h_1 h_2 (-y_{\bar{x}_2 x_2}) v + \frac{h_1}{2} \sum_{x\in\gamma_{-1}} h_2 \left(1 + \frac{h_1\sigma}{3}\right)(-y_{\bar{x}_2 x_2}) v =$$

$$= \sum_{x\in\omega_1\times\omega_2^+} h_1 h_2 y_{\bar{x}_2} v_{\bar{x}_2} + \left(1 + \frac{h_1\sigma}{3}\right)\frac{h_1}{2} \sum_{\substack{x_2\in\omega_2^+ \\ (x_1=0)}} h_2 y_{\bar{x}_2} v_{\bar{x}_2}$$

which gives

$$(A_2 y, y) = \sum_{x\in\omega_1\times\omega_2^+} h_1 h_2 y_{\bar{x}_2}^2 + \left(1 + \frac{h_1\sigma}{3}\right)\frac{h_1}{2} \sum_{\substack{x_2\in\omega_2^+ \\ (x_1=0)}} h_2 y_{\bar{x}_2}^2 \ge$$

$$\ge \sum_{x_1\in\omega_1} h_1 \sum_{x_2\in\omega_2^+} h_2 y_{\bar{x}_2}^2 + \frac{h_1}{2} \sum_{\substack{x_2\in\omega_2^+ \\ (x_1=0)}} h_2 y_{\bar{x}_2}^2 \ge$$

$$\geq \sum_{x_1\in\omega_1} h_1 \frac{8}{l_2^2} \sum_{x_2\in\omega_2} h_2 y^2(x) + \frac{h_1}{2}\frac{8}{l_2^2} \sum_{x_2\in\omega_2} h_2 y^2(0,x_2) = \frac{8}{l_2^2} \| y \|^2 .$$

Then, the difference operator $A = A_1 + A_2$ is also symmetric and positive definite. The lemma is proved.

It follows from Lemma 1.1 that there exist the inverse operator A^{-1} and thus the discrete problem [1.6] is uniquely solvable. We can now prove the following proposition.

LEMMA 1.2.– *It holds*

$$\| A^{-1}B_k v \| \leq \sqrt{\frac{6}{11}} \| v \|_k \text{ for all } v\in H_k \quad (k=1,2), \qquad [1.7]$$

where the operator B_k, the space H_k and the norm $||v||_k$ are defined further in the text.

PROOF.– Applying summation by parts and the ε-inequality for $\varepsilon = 1/(4\sigma)$, we have

$$\| Ay \|^2 = (Ay, Ay) = (A_1 y + A_2 y, A_1 y + A_2 y) = \| A_1 y \|^2 + \| A_2 y \|^2 + 2(A_1 y, A_2 y) \geq$$

$$\geq 2(A_1 y, A_2 y) = 2\sum_{x\in\omega} h_1 h_2 y_{\bar{x}_1 x_1} y_{\bar{x}_2 x_2} + 2\frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2 \frac{2}{h_1}(y_{x_1} - \sigma y)\left(1 + \frac{h_1\sigma}{3}\right) y_{\bar{x}_2 x_2} =$$

$$= 2\sum_{x\in\omega_1^- \times \omega_2^-} h_1 h_2 y_{x_1 x_2}^2 - 2\frac{h_1\sigma}{3}\sum_{\substack{x_2\in\omega_2^- \\ (x_1=0)}} h_2 y_{x_1 x_2} y_{x_2} + 2\sigma\left(1 + \frac{h_1\sigma}{3}\right)\sum_{\substack{x_2\in\omega_2^- \\ (x_1=0)}} h_2 y_{x_2}^2 \geq$$

$$\geq 2\sum_{x\in\omega_1^- \times \omega_2^-} h_1 h_2 y_{x_1 x_2}^2 - 2\frac{h_1\sigma}{3}\left(\frac{1}{4\sigma}\sum_{\substack{x_2\in\omega_2^- \\ (x_1=0)}} h_2 y_{x_1 x_2}^2 + \sigma\sum_{\substack{x_2\in\omega_2^- \\ (x_1=0)}} h_2 y_{x_2}^2\right) +$$

$$+ 2\sigma\left(1 + \frac{h_1\sigma}{3}\right)\sum_{\substack{x_2\in\omega_2^- \\ (x_1=0)}} h_2 y_{x_2}^2 =$$

$$= 2 \sum_{x \in \omega_1^- \times \omega_2^-} h_1 h_2 y_{x_1 x_2}^2 - 2 \frac{h_1 \sigma}{3} \frac{1}{4\sigma} \sum_{\substack{x_2 \in \omega_2^- \\ (x_1 = 0)}} h_2 y_{x_1 x_2}^2 + 2\sigma \sum_{\substack{x_2 \in \omega_2^- \\ (x_1 = 0)}} h_2 y_{x_2}^2 \geq$$

$$\geq 2 \sum_{x \in \omega_1^- \times \omega_2^-} h_1 h_2 y_{x_1 x_2}^2 - \frac{h_1}{6} \sum_{\substack{x_2 \in \omega_2^- \\ (x_1 = 0)}} h_2 y_{x_1 x_2}^2 \geq \left(2 - \frac{1}{6}\right) \sum_{x \in \omega_1^- \times \omega_2^-} h_1 h_2 y_{x_1 x_2}^2 = \frac{11}{6} \| B_1^* y \|_1^2 ,$$

where

$$B_1^* : H \to H_1, \ B_1^* y = -y_{x_1 x_2}, x \in \omega_1^- \times \omega_2^-,$$

is a difference operator acting from H into the space H_1 of mesh functions defined on $\omega_1^- \times \omega_2^-$ with the inner product and the associate norm

$$(y, v)_1 = \sum_{x \in \omega_1^- \times \omega_2^-} h_1 h_2 y(x) v(x), \quad \| y \|_1 = \sqrt{(y, y)_1} = \left(\sum_{x \in \omega_1^- \times \omega_2^-} h_1 h_2 y^2(x) \right)^{1/2}.$$

Applying summation by parts, we have

$$(B_1^* y, w)_1 = - \sum_{x \in \omega_1^- \times \omega_2^-} h_1 h_2 y_{x_1 x_2} w = - \sum_{x \in \omega} h_1 h_2 y w_{\bar{x}_1 \bar{x}_2} - \frac{h_1}{2} \sum_{x \in \gamma_{-1}} h_2 y \frac{2}{h_1} w_{\bar{x}_2} = (y, B_1 w),$$

where $B_1 : H_1 \to H$ is the adjoint operator of $B_1^* : H \to H_1$,

$$B_1 w(x) = - \begin{cases} w_{\bar{x}_1 \bar{x}_2}, & x \in \omega, \\ \dfrac{2}{h_1} w_{\bar{x}_2}, & x \in \gamma_{-1}. \end{cases}$$

Similarly, we have

$$\| Ay \|^2 = \| A_1 y \|^2 + \| A_2 y \|^2 + 2(A_1 y, A_2 y) \geq 2(A_1 y, A_2 y) =$$

$$= 2 \sum_{x \in \omega} h_1 h_2 y_{\bar{x}_1 x_1} y_{\bar{x}_2 x_2} + 2 \frac{h_1}{2} \sum_{x \in \gamma_{-1}} h_2 \frac{2}{h_1} (y_{x_1} - \sigma y) \left(1 + \frac{h_1 \sigma}{3}\right) y_{\bar{x}_2 x_2} =$$

$$= 2\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y_{x_1\bar{x}_2}^2 - 2\frac{h_1\sigma}{3}\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{x_1\bar{x}_2}y_{\bar{x}_2} + 2\sigma\left(1+\frac{h_1\sigma}{3}\right)\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{\bar{x}_2}^2 \geq$$

$$\geq 2\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y_{x_1\bar{x}_2}^2 - 2\frac{h_1\sigma}{3}\left(\frac{1}{4\sigma}\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{x_1\bar{x}_2}^2 + \sigma\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{\bar{x}_2}^2\right)+$$

$$+ 2\sigma\left(1+\frac{h_1\sigma}{3}\right)\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{\bar{x}_2}^2 =$$

$$= 2\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y_{x_1\bar{x}_2}^2 - 2\frac{h_1\sigma}{3}\frac{1}{4\sigma}\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{x_1\bar{x}_2}^2 + 2\sigma\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{\bar{x}_2}^2 \geq$$

$$\geq 2\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y_{x_1\bar{x}_2}^2 - \frac{h_1}{6}\sum_{\substack{x_2\in\omega_2^+\\(x_1=0)}} h_2y_{x_1\bar{x}_2}^2 \geq \left(2-\frac{1}{6}\right)\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y_{x_1\bar{x}_2}^2 = \frac{11}{6}\| B_2^*y\|_2^2 ,$$

where

$$B_2^*: H\to H_2,\ \ B_2^*y = -y_{x_1\bar{x}_2},\ x\in\omega_1^-\times\omega_2^+,$$

is a difference operator acting from H into the space H_2 of mesh functions defined on $\omega_1^-\times\omega_2^+$ with the inner product and the associate norm

$$(y,v)_2 = \sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y(x)v(x),\quad \| y\|_2 = \sqrt{(y,y)_2} = \left(\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y^2(x)\right)^{1/2}.$$

We find

$$(B_2^*y,w)_2 = -\sum_{x\in\omega_1^-\times\omega_2^+} h_1h_2y_{x_1\bar{x}_2}w = -\sum_{x\in\omega} h_1h_2yw_{\bar{x}_1x_2} - \frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2y\frac{2}{h_1}w_{x_2} = (y,B_2w),$$

where $B_2: H_2\to H$ is the adjoint operator of $B_2^*: H\to H_2$,

$$B_2 w(x) = -\begin{cases} w_{\bar{x}_1 x_2}, & x \in \omega, \\ \dfrac{2}{h_1} w_{x_2}, & x \in \gamma_{-1}. \end{cases}$$

Applying the main lemma from Samarskii et al. (1987, p. 54) to the operators $A, B_1, B_2,$ we obtain estimate [1.7] and thus complete the proof.

1.1.3. *Discrete Green's function*

We denote by $G(x,\xi)$ Green's function of the finite-difference problem [1.4]:

$$-G_{\bar{\xi}_1\xi_1}(x,\xi) - G_{\bar{\xi}_2\xi_2}(x,\xi) = \frac{\delta(x_1,\xi_1)\,\delta(x_2,\xi_2)}{h_1 h_2}, \quad \xi \in \omega, \tag{1.8}$$

$$-\frac{2}{h_1}\left(G_{\xi_1}(x,\xi) - \sigma G(x,\xi)\right) - \left(1 + \frac{h_1\sigma}{3}\right) G_{\bar{\xi}_2\xi_2}(x,\xi) = \frac{2}{h_1}\frac{\delta(x_1,\xi_1)\,\delta(x_2,\xi_2)}{h_2}, \quad \xi \in \gamma_{-1},$$

$$G(x,\xi) = 0, \quad \xi \in \gamma \setminus \gamma_{-1},$$

where $\delta(m,n)$ is the Kronecker delta symbol and $\xi = (\xi_1, \xi_2)$.

LEMMA 1.3.– *For the error $z(x)$, the following estimate holds true:*

$$|z(x)| \le \sqrt{\frac{6}{11}}\,\rho(x)\,\|\psi\|, \quad x \in \omega \cup \gamma_{-1},$$

where $\rho(x) = \min\left\{\sqrt{(l_1 - x_1)(l_2 - x_2)},\ \sqrt{(l_1 - x_1)x_2}\right\}$.

PROOF.– Using the Heaviside step function

$$H(s) = \begin{cases} 1, & s \ge 0, \\ 0, & s < 0, \end{cases}$$