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Trotter-Kato Product Formulæ

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To
Galina, Hille, Michiko

Introduction: What Is This Book About? (De quoi parle ce livre?)

1. History and Motivation
Since 1875 it is owing to Sophus Lie we know that for any pair of (noncommutative) finite square matrices $A, B \in \mathcal{M}(\mathbb{R}^d)$ on the d-dimensional Euclidean space $\mathbb{R}^d$, one has for $n \in \mathbb{N}$ the estimates

$$\left\| \left(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n}\right)^n - \mathrm{e}^{-t(A+B)} \right\| \leq O(1/n), \tag{1}$$

$$\left\| \left(\mathrm{e}^{-tA/2n}\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/2n}\right)^n - \mathrm{e}^{-t(A+B)} \right\| \leq O(1/n^2)\,. \tag{2}$$

for $n \to \infty$. Here $\| \cdot \|$ is *any* norm on the matrix space $\mathcal{M}(\mathbb{R}^d)$. Note that the powers $\{\left(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n}\right)^n\}_{n\geq 1}$ are called the *products* (or the product *approximants*), and the powers $\{\left(\mathrm{e}^{-tA/2n}\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/2n}\right)^n\}_{n\geq 1}$ are called the *symmetric* (or *symmetrised*) *products* (or *approximants*) for Lie product formulæ (1) and (2). Extension of the Lie-type product formulæ to infinite-dimensional spaces was a subject of several papers by Yu. L. Daletskiĭ (see, e.g., [Dal60]) in connection with the path-integral representation (*Feynman-Kac* formula) for solutions of evolution equations. Although the first abstract result appeared in 1959 due to a celebrated paper by H. Trotter [Trot59], who extended the *Lie* product formula (1) to the case of strongly continuous contraction semigroups on a *Banach* space $\mathfrak{X}$. There the convergence of the *Lie-Trotter* product formula was proved in the *strong* operator topology on the Banach space of bounded operators $\mathcal{L}(\mathfrak{X})$.

Namely, let A and B be generators of strongly continuous contraction semigroups in a Banach space $\mathfrak{X}$. If the closure C of the operator sum $A + B$ on domain: $\mathrm{dom}\,(A) \cap \mathrm{dom}\,(B)$ is generator of a contraction semigroup, then limit

$$\text{s-}\lim_{n\to\infty} \left(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n}\right)^n = \mathrm{e}^{-tC}\,, \qquad C := \overline{A+B}\,, \tag{3}$$

holds for any $t \geq 0$ in the strong operator topology on $\mathcal{L}(\mathfrak{X})$.

In 1964, E. Nelson pointed out [Nel64] the importance of the *Trotter* product formula (3) for semigroups generated by *Schrödinger* operators, and for the proof of the *Feynman-Kac* formula. It is also to him belongs a simple proof of (3) in a separable Hilbert space $\mathfrak{H}$ where A and B are two non-negative self-adjoint operators such that the operator sum $A + B$ is also self-adjoint [Nel64]. It is striking that much later it was proven in [IT01] and [ITTZ01] that Nelson's conditions ensure convergence of the Trotter product formula (3) in the *operator-norm* topology ($\| \cdot \| - \lim_{n\to\infty}$) on $\mathcal{L}(\mathfrak{H})$, with error bound estimate for the rate of convergence which is (ultimately) *optimal*. This means that it coincides with the rate in (1) for matrices, and it cannot be improved without supplementary conditions, or by a simple symmetrisation of approximants as in (2).

A beautiful and elegant way to treat the Lie-Trotter product formulæ and to prove (3) in the framework of a general theory of strongly continuous contractions on Hilbert and Banach spaces is due to P. Chernoff [Cher68, Cher74]. There he generalised (3) by proposition to study on $\mathfrak{X}$ (or $\mathfrak{H}$) strongly continuous families of contractions $\{F(t)\}_{t\geq 0}$ and the product formula of the type

$$\text{s-}\lim_{n\to\infty} \big(F(t/n)\big)^n = G(t)\,, \quad t \geq 0\,, \tag{4}$$

which is now known as the *Chernoff* product formula. If the closure of the strong right derivative $F'(+0)$ is equal to (minus) generator: $(-C)$ of a contraction semigroup, the arguments in [Cher68] reveal that the strong limit (4) exists and $\{G(t) = e^{-tC}\}_{t\geq 0}$. The *Chernoff theory* and its generalisation for convergence in the operator-norm topology will be developed and systematically employed below in this book.

Further progress in the product formulæ approximations was achieved by T. Kato [Kato78]. In 1978, he obtained the following important results:

(a) Let $A \geq 0$ and $B \geq 0$ be non-negative self-adjoint operators in a separable Hilbert space $\mathfrak{H}$. Denote by $\mathfrak{H}_0$ the subspace

$$\mathfrak{H}_0 := \overline{\text{dom}\,(A^{1/2}) \cap \text{dom}\,(B^{1/2})}\,.$$

Note that it may happen that intersection of operator domains $\text{dom}\,(A) \cap \text{dom}\,(B) = \{0\}$, but the *form-sum* $C := A \dotplus B$ is well-defined and is generator in the subspace $\mathfrak{H}_0$. Under these conditions, the Trotter product formula converges strongly to *degenerate* semigroup $\{e^{-tC}P_0\}_{t>0}$, *locally uniformly* away from zero. That is, one gets (*Kato* product formula):

$$\text{s-}\lim_{n\to\infty} \big(e^{-tA/n}e^{-tB/n}\big)^n = e^{-tC}P_0, \qquad t > 0, \tag{5}$$

uniformly in $t \in [a, b]$, for any $[a, b] \subset \mathbb{R}^+$. Here P_0 denotes the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_0$ and $\mathbb{R}^+ := (0, +\infty)$.

(b) Further, T. Kato [Kato74, Kato78] discovered that the product formula (5) is valid not only for the exponential function e^{-x}, $x \geq 0$, but also for the whole class of Borel measurable functions f and g, which are defined on $\mathbb{R}_0^+ = [0, \infty)$

and satisfy the conditions

$$0 \leq f(x) \leq 1, \qquad f(0) = 1, \qquad f'(+0) = -1, \tag{6}$$

$$0 \leq g(x) \leq 1, \qquad g(0) = 1, \qquad g'(+0) = -1. \tag{7}$$

Kato proved that in this case one obtains instead (5) the product formula

$$\text{s-}\lim_{n\to\infty} (f(tA/n)g(tB/n))^n = \mathrm{e}^{-tC} P_0, \qquad t > 0\,, \tag{8}$$

converging locally uniformly in $\mathbb{R}_0^+$ away from zero. These product formulæ for pairs $f, g \in \mathcal{K}$ are known as the *Trotter-Kato product formulæ* for the set of *generic* Kato functions $\mathcal{K}$ satisfying (6), (7).

The next important step concerning the Trotter product formula is related to the following remarkable result by Dzh. L. Rogava in 1993 [Rog93] about convergence of (3) in the *operator-norm* topology on $\mathcal{L}(\mathfrak{H})$. More precisely:

Let A and B be two bounded from below self-adjoint operators in a separable Hilbert space $\mathfrak{H}$. If dom $(A) \subset$ dom (B) and operator $C = A + B$ is self-adjoint (cf. [Nel64]), then

$$\left\| \left(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n}\right)^n - \mathrm{e}^{-tC} \right\| \leq \frac{c\,\ln(n)}{\sqrt{n}}\,, \qquad c > 0\,, \quad n > 1\,, \tag{9}$$

uniformly in $t \in [0, T]$, $0 < T < \infty$. This *operator-norm* estimate leads to convergence of the Trotter product formula (3) with the error bound estimate (9) for the rate. This convergence is uniform in $t \geq 0$ if the self-adjoint operators A and B are non-negative.

Ever since the discovery by Rogava [Rog93] that the Trotter product formula for strongly continuous semigroups may exhibit convergence in the *operator-norm* topology, a compelling question has emerged regarding the *optimal* operator-norm error bound estimate. This inquiry delves into the error bound estimate for the rate of convergence within the framework of Eqs. (9) and scrutinises its dependency on the pair of generators A and B. The question has become a focal point for both Trotter and Trotter-Kato product formulæ, captivating researches in Hilbert and Banach spaces.

The core focus of this book is to provide an exhaustive account of recent findings pertaining to operator-norm convergent Trotter and Trotter-Kato product formulæ, along with several of their generalisations. The dedication of the book to this exploration underscores its significance in advancing our understanding of these mathematical constructs.

It is noteworthy to mention that, in the realm of Hilbert spaces, Trotter-Kato product formulæ converging in the operator-norm topology have been acknowledged since 1988. However, during this period, there was an absence of estimates regarding the rate of convergence. This was particularly observed for a distinct class of *strongly* continuous at zero semigroups (C_0-semigroups), known as *Gibbs semigroups*, as documented in works such as [Zag88] and [NZ90a, NZ90b]. The established topology of *convergence* was stronger than the operator-norm topology, aligning with the *trace-norm* topology of continuity for Gibbs semigroups *away* from zero; for details, see [Zag19]. This observation extends also to the C_0-semigroups considered in this book, emphasising in particular their *operator-norm* continuity *away* from zero.

The principal thrust of this book is to communicate concrete results regarding the elevation (*uplifting*) of diverse *strongly* convergent Trotter-Kato product formulæ, encompassing those with time dependence, to a state of convergence in the *operator-norm* topology within both Banach and Hilbert spaces. Additionally, the book explores alternative options, including different spaces for semigroups and topologies of convergence of product formulæ, while outlining potential limits to this overarching programme. In essence, the book aims to serve as a comprehensive resource, shedding light on the intricacies of these mathematical concepts and their convergence behaviour in various settings.

2. Overview

Contents of the book is essentially based on original publications by the authors and their co-authors. According to particular subjects, we split the presentation into five parts: Parts I–V, which are accompanying by a few technical Appendices.

Part I contains some standard preliminaries about the C_0-semigroups and their generators. Besides we recall the abstract non-autonomous Cauchy problem (nACP) for linear operator-valued evolution equations. The corresponding product formulæ will be the main subject of Part IV. In the last section of the Part I, we introduce a class of *quasi-sectorial contractions* [CZ01b], [ArZag10], together with suitable (in the operator-norm continuity context) elements of the Chernoff theory including revisions for future purposes.

In Part II, we collect results on the Trotter-Kato product formulæ for *self-adjoint* semigroups. To this aim, the Chernoff theory of approximation is extended from the strong to the operator-norm topology. As a consequence, the Trotter-Kato product formulæ are established in the operator-norm topology first without and then with the error bound estimates. In the last section, the ITTZ Theorem [IT01], [ITTZ01] about the *optimal* operator-norm error bound estimate is proven. Another theorem about the optimal error bound estimate, which is due to the *fractional powers* conditions, [NZ99a], [Tam00], and [INZ04], concludes this part.

Part III is dedicated to extensions of results presented in the Part II to *non-self-adjoint* semigroups and also to certain results about the Trotter product formula in a Banach space. Although we developed non-self-adjoint improvement of the Chernoff theory [CZ01b], [Zag08], [Zag22c], there are no results about optimality of the operator-norm convergence rate for non-self-adjoint semigroups, cf. [CZ99], [CNZ01] and [CNZ02]. These results are partially summarised in [Ca10]. On the other hand, very little results are known about the operator-norm convergent Trotter product formula in Banach spaces [CZ01a], [Zag23]. They are more poor than the corresponding results in Hilbert spaces, and they are still essentially based on the useful properties of *holomorphic* semigroups.

A new result in a Banach space concerns the operator-norm convergence of the Trotter product formula for *contraction* semigroups, [NSZ18a], [Zag23]. It shows in particular that the rate of the operator-norm convergence of the Trotter product formula may be *arbitrary slow*, even if one of generators in formula is bounded; see Example in Sect. 10.3.

In Part IV, we present the results on product formulæ approximations (11.1.2), (11.1.3) of *solution operator* $\{U(t,s)\}_{0\leq s\leq t\leq T}$ (also known as *fundamental solution*, or *propagator*) corresponding to the abstract *non-autonomous* Cauchy problem (nACP) (11.1.1) in Banach, or Hilbert spaces. Note that in Banach space a direct study of product approximants (11.1.2) yields convergence (11.1.3) only in the *strong* operator topology (Sect. 11.1). So, we use a rather *round-about* way to estimate their convergence in the operator-norm topology. This is possible owing to the *Howland-Evans-Neidhardt* theory of nACP ([How74], [Ev76], [Nei79], [Nei81]) based on the *evolution semigroup* method in the Banach space L^p-setting; see (11.3.24). As a consequence, one gets estimates (11.5.68) and (11.5.79) for the the operator-norm rate of convergence.

The Hilbert space L^2-setting (Sect. 12.1) essentially follows the L^p-setting for Banach spaces, but with a better operator-norm estimate of the rate of convergence, Theorem 12.1.19. This improvement is due to the self-adjointness, which yields some number of straightforward estimates which are available in the Hilbert space L^2-setting. In Sect. 12.2, we present the *Ichinose-Tamura* approach to a direct analysis of product approximants for the nACP solution operator in a Hilbert space. It deserves to mention that for the *Lipschitz* condition (12.1.3), this approach yields for convergence rates in (12.1.73) and (12.1.74) the best of the actually known estimate: $O(\ln(n)/n)$, for the operator-norm error bound.

Part V contains two different subjects. The first one (Chap. 13) is a revision in Sect. 13.2 of the well-known strongly convergent Trotter product formula for unitary groups striving to uplift it to convergence to the operator-norm topology. In Sect. 13.2, it is shown that Trotters's conditions are neither sufficient, nor necessary for convergence of the unitary product formula in the operator-norm topology. Our results there on the limit of unitary Trotter product formula in the operator-norm topology are based on a certain *commutator* conditions. Searching for generalisations of Trotters's conditions, we present in Sect. 13.3 the unitary Trotter-Kato product formulæ, which converge only in the operator *locally convex* topology (L^2-topology) of the *Fréchet* space. The fact that we retreated into the corner of convergence in this topology (weaker than the strong operator topology) indicates that we are far from our ultimate aim of *uplifting* this result to the operator-norm topology.

The second subject concerns the *Zeno* product formula. Formally it corresponds to a unitary Trotter product formula when one of the factors is a *degenerate* (semi)*group* $\{P_t\}_{t\in\mathbb{R}}$ of projections: $P_t = P$, for $t \neq 0$, and $P_{t=0} = \mathbb{1}$. First we present the results on convergence of the Zeno product formula in the operator topology of the *Fréchet* space, Sect. 14.1. Then we consider extensions of this result to the *non-exponential* Zeno product formulæ, Sect. 14.2, for the *strong* as well as for the *operator-norm* convergences. In Sect. 14.3, we elucidate and improve the theory developed for the operator topology of the *Fréchet* space. Due to the *Exner-Ichinose* approach, one can show that the exponential Zeno product formula holds in the *strong* operator topology.

3. Guide to the Reader

Part I of this book is a consistent and quite detailed introduction into preliminaries for further reading. For non-specialists, it is indispensable for orientation before entering into the main subject of the book, which concerns a variety of the product formulæ approximations in different spaces and topologies.

The specialists may proceed quickly to Part II and to Parts III–V, which describe original concrete results on the Trotter-Kato product formulæ. The notes to each chapter allow to continue the reading of a more specialised literature. They also contain additional comments that escaped from the main text and that may be also useful for non-specialists.

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The genesis of this book spans more than two decades, tracing its roots back to October 2002, when Hagen Neidhardt visited the Centre de Physique Théorique (CPT-Marseille). During that visit, the seed of an idea was planted, a concept that would mature over the years. We shared this idea with Takashi Ichinose (Kanazawa University), who joined us, providing invaluable support to breathe life into the project. As we embarked on this journey, numerous institutions have played a pivotal rôle in promoting our work.

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The year 2019 brought the sombre news of the passing of our dear friend and co-author, Hagen Neidhardt. His demise on 23 March 2019, at the age of 68, was a profound loss. During his time in the hospital, Hagen expressed the hope that the book project would be completed.

We hope that the present work corresponds to endeavours in accomplishing the requests and ideas that together with Hagen we formulated many years ago. One such request was to dedicate this book to our wives.

Marseille, France — Valentin A. Zagrebnov
Kanazawa, Japan — Takashi Ichinose
December, 2023

Takashi Ichinose, Hagen Neidhardt, Valentin A.Zagrebnov at the Mittag-Leffler Mathematical Institute, July–August 2014

Contents

Part I
Preliminaries

Chapter 1
Semigroups and Their Generators

This chapter is a prelude. It contains a brief account of some basic notions and facts from the theory of strongly continuous semigroups of operators (C_0-semigroups) and their generators. We also introduce here notations and definitions indispensable in the following chapters. Further, we focus here on some special classes of semigroups important for entire of our further presentation. After definition of the strongly continuous exponential function, we consider contraction and quasi-bounded semigroups, emphasising the holomorphic semigroups.

The content of this chapter (except some details like perturbation theory based on the *product fromula*) is standard, and it can be found in many sources (see Notes to Chap. 1).

1.1 The Exponential Function

Let $\mathfrak{X}$ be a Banach space. For any bounded operator $A \in \mathcal{L}(\mathfrak{X})$, one can define for complex $t \in \mathbb{C}$ the exponential function $t \mapsto \mathrm{e}^{-tA} =: U_A(t)$ by the series

$$U_A(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!}(-A)^n, \tag{1.1.1}$$

which is convergent in the operator-norm $\|\cdot\|$ on the Banach space $\mathcal{L}(\mathfrak{X})$ of linear-bounded operators. Therefore, for any $A \in \mathcal{L}(\mathfrak{X})$, the mapping $\mathbb{C} \ni t \mapsto U_A(t) \in$

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$\mathcal{L}(\mathfrak{X})$ is entire $\|\cdot\|$-holomorphic operator-valued function on $\mathbb{C}$. The group property (*group law*)

$$U_A(t_1+t_2) = U_A(t_1)U_A(t_2), \quad t_1, t_2 \in \mathbb{C}, \tag{1.1.2}$$

is a direct consequence of formula (1.1.1) as well as equation

$$\|\cdot\|-\partial_t U_A(t) = (-A)U_A(t) = U_A(t)(-A), \tag{1.1.3}$$

where $\|\cdot\|-\partial_t$ means differentiation in the sense of the operator-norm on $\mathcal{L}(\mathfrak{X})$.

Now, let A be an unbounded operator with a dense in $\mathfrak{X}$ domain $\mathrm{dom}\,(A) \subset \mathfrak{X}$. Then, the existence of the exponential function $U_A(\cdot) : \mathcal{D} \to \mathcal{L}(\mathfrak{X})$ (1.1.3) for domain $\mathcal{D} \subset \mathbb{C}$ is much less obvious. One of the possibilities is to define $U_A(t)$ in a way alternative to (1.1.1), namely, by means of the classical *Euler formula* for exponential.

To this aim, suppose that $A \in \mathcal{C}(\mathfrak{X})$, where $\mathcal{C}(\mathfrak{X})$ denotes the set of *closed* linear operators in Banach space $\mathfrak{X}$, with resolvent $R_A(\zeta) := (A - \zeta\,\mathbb{1})^{-1}$ and *non-empty* resolvent set $\rho(A) := \{\zeta \in \mathbb{C} : R_A(\zeta) \in \mathcal{L}(\mathfrak{X})\}$. If the set $(-\infty, 0) \subset \rho(A)$, then one constructs a sequence of bounded *Euler approximants*:

$$\mathcal{E}_{A,n}(t) := \left(\mathbb{1} + \frac{t}{n}A\right)^{-n}, \quad n = 1, 2, \ldots, \qquad t \geq 0, \tag{1.1.4}$$

with intention to prove the limit (*Euler formula*)

$$U_A(t) = \lim_{n\to\infty}\left(\mathbb{1} + \frac{t}{n}A\right)^{-n}, \tag{1.1.5}$$

for $t \in \mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$, here $\mathbb{R}^+ := (0, +\infty)$, in one of the topologies on the set of bounded operators $\mathcal{L}(\mathfrak{X})$.

The following proposition provides a sufficient condition for that in the *strong operator* topology:

Proposition 1.1.1 *Let $A \in \mathcal{C}(\mathfrak{X})$ be a closed linear operator with dense domain* $\mathrm{dom}\,(A) \subset \mathfrak{X}$ *such that the following two conditions hold* :

(i) $\mathbb{R}^- := (-\infty, 0)$ *belongs to the resolvent set* $\rho(A)$.
(ii) $\|(A + \lambda\mathbb{1})^{-1}\| \leq \lambda^{-1}$ *for* $\lambda > 0$.

Then for $t \geq 0$,

(a) *The limit* (1.1.5) *exists in the strong operator topology to* (*exponential*) *function* $U_A(t)$, *such that* $\|U_A(t)\| \leq 1$, *locally uniformly in* $t \in \mathbb{R}_0^+$.
(b) *The mapping* $t \in \mathbb{R}^+ \mapsto U_A(t)x$ *is continuous for every* $x \in \mathfrak{X}$ (*i.e., strongly continuous*) *with the right limit* $\lim_{t\to+0}\|(U_A(t)\,x - x\| = 0$, *that is,* $U_A(0) = \mathbb{1}$.

(c) dom (A) *is invariant under* $U_A(t)$ *in the sense that* $U_A(t)(\mathrm{dom}\,(A)) \subseteq \mathrm{dom}\,(A)$ *for all* $t \geq 0$.
(d) *The semigroup law* $U_A(t+s) = U_A(t)U_A(s)$, *for all* $t, s \geq 0$.

Proof

(a) Owing to definition (1.1.4) and condition (ii), one infers that Euler approximants are bounded and $\|\mathcal{E}_{A,n}(t)\| \leq 1$ for $t \in \mathbb{R}_0^+$ and natural $n \in \mathbb{N}$. Then for $t > 0$, the operators

$$A\,\mathcal{E}_{A,n+1}(t) = \frac{n}{t}\,(\mathcal{E}_{A,n}(t) - \mathcal{E}_{A,n+1}(t))\,, \quad n \in \mathbb{N}\,, \tag{1.1.6}$$

are bounded. As a consequence, the functions $t \mapsto \mathcal{E}_{A,n}(t)$ are differentiable for $t > 0$ in the operator-norm topology on $\mathcal{L}(\mathfrak{X})$, and by (1.1.6) the operator-norm derivatives

$$\|\cdot\|{-}\partial_t\mathcal{E}_{A,n}(t) = (-A)\,\mathcal{E}_{A,n+1}(t) \in \mathcal{L}(\mathfrak{X})\,, \quad n \in \mathbb{N}\,, \tag{1.1.7}$$

are bounded for $t > 0$.

Now, we elucidate the continuity of functions $t \mapsto \mathcal{E}_{A,n}(t)$, at $t = 0$, where $\mathcal{E}_{A,n}(0) = \mathbb{1}$. To this end, we note that

$$\mathbb{1} - \mathcal{E}_{A,n}(t) = \frac{t}{n}\,A\sum_{k=0}^{n-1} C_{n-1}^{k}\left(\frac{t}{n}\,A\right)^{k}\,\mathcal{E}_{A,n}(t)\,, \quad n \in \mathbb{N}\,. \tag{1.1.8}$$

By iteration of (1.1.6) for $A^k\,\mathcal{E}_{A,n}(t)$ and $1 \leq k \leq n-1$, one concludes that operators

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} C_{n-1}^{k}\left(\frac{t}{n}\,A\right)^{k}\,\mathcal{E}_{A,n}(t)\right\| \leq M_n\,, \quad n \in \mathbb{N}\,, \tag{1.1.9}$$

are bounded for $t \in \mathbb{R}_0^+$. Given that operator A is unbounded, it is not necessarily that operator in the right-hand side of (1.1.8) tends to zero as $t \to +0$ in the operator-norm topology, although it is clear that in virtue of (1.1.9)

$$\|\,(\mathbb{1} - \mathcal{E}_{A,n}(t))\,x\,\| \leq t\,\|A\,x\|\,M_n\,, \quad x \in \mathrm{dom}\,(A)\,, \quad n \in \mathbb{N}\,. \tag{1.1.10}$$

Hence, $\mathbb{R}^+ \ni t \mapsto \mathcal{E}_{A,n}(t)$ is right-continuous at $t = 0$ on dom (A). Then because dom (A) is dense in $\mathfrak{X}$ and for any $n \in \mathbb{N}$ the family of operators $\{\mathbb{1} - \mathcal{E}_{A,n}(t)\}_{t\geq 0}$ is *uniformly* bounded, we can use Proposition A.1.6 (Appendix A.1) to extend the continuity in (1.1.10) from dom (A) to the whole space $\mathfrak{X}$: $\lim_{t\to+0}\|\,(\mathbb{1} - \mathcal{E}_{A,n}(t))\,x\,\| = 0$ for $x \in \mathfrak{X}$. Consequently, the function

$t \mapsto \mathcal{E}_{A,n}(t)$ is continuous at $t = 0$ in the *strong* operator sense

$$\text{s-}\lim_{t\to+0} \mathcal{E}_{A,n}(t) = \mathcal{E}_{A,n}(0) = \mathbb{1}. \tag{1.1.11}$$

To proceed (with the proof of convergence in (1.1.5)), we note that (1.1.7) and (1.1.10) also yield

$$\mathcal{E}_{A,n}(t)\, x - \mathcal{E}_{A,k}(t)\, x = \lim_{\varepsilon\to+0} \int_{\varepsilon}^{t-\varepsilon} d\tau\; \partial_\tau \left[\mathcal{E}_{A,k}(t-\tau)\, \mathcal{E}_{A,n}(\tau)\right] x\,,$$
$$x \in \mathfrak{X}\,,\quad t > 0\,,$$

for $n, k \in \mathbb{N}$ and well-defined (*Bochner*) integral, Proposition 1.6.13. Then using (1.1.7), we obtain the following representation for any $x \in \mathfrak{X}$:

$$\mathcal{E}_{A,n}(t)\, x - \mathcal{E}_{A,k}(t)\, x \;= \tag{1.1.12}$$
$$\lim_{\varepsilon\to+0} \int_{\varepsilon}^{t-\varepsilon} d\tau \left(\frac{\tau}{n} - \frac{t-\tau}{k}\right) A^2\, \mathcal{E}_{A,k+1}(t-\tau)\, \mathcal{E}_{A,n+1}(\tau)\, x\,.$$

If we let $x \in \mathrm{dom}\,(A^2)$, then one may rearrange the integrand in (1.1.12) by commutations in this fashion:

$$\int_{\varepsilon}^{t-\varepsilon} d\tau \left(\frac{\tau}{n} - \frac{t-\tau}{k}\right) \mathcal{E}_{A,k+1}(t-\tau)\, \mathcal{E}_{A,n+1}(\tau)\;\; A^2\, x\,. \tag{1.1.13}$$

Owing to (1.1.7) and (1.1.11), the integrand in (1.1.13) is strongly continuous in $\tau \in [0, t]$. Since in addition $\|\mathcal{E}_{A,n}(t)\| \leq 1$ for $t \in \mathbb{R}_0^+$, we obtain for any $n, k \in \mathbb{N}$ the estimate

$$\|\mathcal{E}_{A,n}(t)x - \mathcal{E}_{A,k}(t)x\| \leq \frac{t^2}{2}\left(\frac{1}{n} + \frac{1}{k}\right) \|A^2 x\|\,,\quad x \in \mathrm{dom}\,(A^2)\,. \tag{1.1.14}$$

This means that $\{\mathcal{E}_{A,n}(t)x\}_{n\geq 1}$ is a *Cauchy* sequence for $x \in \mathrm{dom}\,(A^2)$. Moreover, condition (ii) also implies (see Propositions A.1.6, A.1.8 and Corollary A.1.9(a)) that $\mathrm{dom}\,(A^2)$ is dense in $\mathfrak{X}$ and, in fact, is a *core* for A, Proposition 1.2.10. Then on account of the uniform operator-norm boundedness $\|\mathcal{E}_{A,n}(t) - \mathcal{E}_{A,k}(t)\| \leq 2$, it follows by (1.1.14) and Corollary A.1.9(b) that $\{\mathcal{E}_{A,n}(t)x\}_{n\geq 1}$ is a *Cauchy* sequence of contraction operators on the whole space $\mathfrak{X}$, which is locally uniformly convergent in $t \in \mathbb{R}_0^+$.

As a result, the limit (1.1.5) holds in the strong operator topology

$$U_A(t) := \text{s-}\lim_{n\to\infty} \mathcal{E}_{A,n}(t)\,,\quad t \geq 0\,, \tag{1.1.15}$$

uniformly in $t \in [0, T]$ for any finite interval. This proves (a) and defines the family $\{U_A(t)\}_{t\geq 0}$ of *contractions* $\{\|U_A(t)\| \leq 1\}_{t\geq 0}$, on the Banach space $\mathfrak{X}$.

(b) Since by (1.1.7) and (1.1.9) the mappings $t \mapsto \mathcal{E}_{A,n}(t)$, $n \in \mathbb{N}$, are strongly continuous for $t \in \mathbb{R}_0^+$ and seeing that convergence in (1.1.15) is uniform in t for any finite interval $0 \le t \le T$, the operator-valued function

$$t \mapsto U_A(t) : \mathbb{R}_0^+ \to \mathcal{L}(\mathfrak{X}) \tag{1.1.16}$$

is strongly continuous in $\mathbb{R}^+$ and also at $t = 0$ from the right: $U_A(+0) = \mathbb{1}$. This proves claim (b).

(c) On account of (1.1.8) and (1.1.12), it follows that

$$(\mathcal{E}_{A,n}(t) - \mathbb{1})x = -\int_0^t \mathrm{d}\tau \left(\mathbb{1} + \frac{\tau}{n}A\right)^{-1} \mathcal{E}_{A,n}(\tau)\ Ax, \quad x \in \mathrm{dom}\,(A). \tag{1.1.17}$$

Since

$$\text{s-}\lim_{n\to\infty} \left(\mathbb{1} + \frac{\tau}{n}A\right)^{-1} \mathcal{E}_{A,n}(\tau) = U_A(\tau)$$

uniformly for any finite interval $0 \le \tau \le T$, (1.1.17) yields

$$(U_A(t) - \mathbb{1})x = -\int_0^t \mathrm{d}\tau\ U_A(\tau)\ Ax, \quad x \in \mathrm{dom}\,(A). \tag{1.1.18}$$

On the other hand, due to (1.1.16), the vector-valued family $\{U_A(\tau)(Ax)\}$ is continuous in τ for any $x \in \mathrm{dom}\,(A)$. Therefore, (1.1.18) shows that the vector-valued function $0 \le t \mapsto U_A(t)x$ is differentiable in t for all $x \in \mathrm{dom}\,(A)$

$$\partial_t(U_A(t)x) = U_A(t)(-Ax), \quad t \ge 0,\ x \in \mathrm{dom}\,(A), \tag{1.1.19}$$

where by definition we consider in (1.1.19) at $t = 0$ the right derivative

$$\partial_t\, U_A(+0)\, x := \lim_{t\to+0} \frac{1}{t}(U_A(t) - \mathbb{1})x = -Ax\,, \quad x \in \mathrm{dom}\,(A)\,.$$

The last identity is due to the right continuity (1.1.16) and representation (1.1.18).

Note that in fact (1.1.8) gives more. Indeed, since

$$\left(\mathbb{1} + \frac{t}{n}A\right)^{-1} \mathfrak{X} \subseteq \mathrm{dom}\,(A)\,, \quad t > 0,$$

one has

$$\partial_t \mathcal{E}_{A,n}(t) = -\mathcal{E}_{A,n}(t)\, A \left(\mathbb{1} + \frac{t}{n}A\right)^{-1} = -A\, \mathcal{E}_{A,n}(t) \left(\mathbb{1} + \frac{t}{n}A\right)^{-1} \tag{1.1.20}$$

and

$$\lim_{n\to\infty} A\left(\mathbb{1}+\frac{t}{n}A\right)^{-1}x = \lim_{n\to\infty}\left(\mathbb{1}+\frac{t}{n}A\right)^{-1}Ax = Ax, \quad x\in \mathrm{dom}\,(A)$$

because $A \in \mathcal{C}(\mathfrak{X})$ is a closed operator. Hence, by virtue of (1.1.15) and by closeness of A, we get that for any $x \in \mathrm{dom}\,(A)$

$$\lim_{n\to\infty} \mathcal{E}_{A,n}(t)A\Big(\mathbb{1}+\frac{t}{n}A\Big)^{-1}x = U_A(t)\,A\,x =$$
$$= \lim_{n\to\infty} A\,\mathcal{E}_{A,n}(t)\Big(\mathbb{1}+\frac{t}{n}A\Big)^{-1}x = A\,U_A(t)\,x\,. \tag{1.1.21}$$

For that reason $U_A(t)x \in \mathrm{dom}\,(A)$, that proves (c).

(d) Note that by (1.1.21) the differential equation (1.1.19) takes the form

$$\partial_t(U_A(t)x) = -A\,(U_A(t)x), \quad t>0, \;\; x\in\mathrm{dom}\,(A). \tag{1.1.22}$$

So, by virtue of (b), the *orbit* $\{U_A(t)\,x_0\}_{t\geq 0}$ of vector x_0 under U_A is a solution of the *autonomous* Cauchy problem (ACP)

$$\partial_t\,x(t) = -\,A\,x(t)\,, \qquad t\in\mathbb{R}_0^+, \tag{1.1.23}$$

provided the initial condition $x(t)|_{t=0} = x_0 \in \mathrm{dom}\,(A)$. Consequently, the orbit $\{x(t) := U_A(t)x_0\}_{t\geq 0}$ is differentiable including $t=0$, where the right derivative is $(\partial_t x)(+0) = -\,A\,x(0)$.

Let $x(t)$ be a solution of (1.1.23) with initial condition $x_0 = x(0) \in \mathrm{dom}\,(A)$. This means that the vector-valued function $x(\cdot) : \mathbb{R}^+ \to \mathrm{dom}\,(A)$ is strongly differentiable, which means that

$$\lim_{\delta\to 0}\delta^{-1}(x(t+\delta)-x(t)) = \partial_t x(t)$$

exists, and is such that (1.1.23) holds for $t>0$. By (1.1.19) and (1.1.23), we have

$$\partial_\tau(U_A(t-\tau)x(\tau)) = 0, \quad 0<\tau<t,$$

with the one-side derivative at $\tau = +0$:

$$\lim_{\tau\to+0}\partial_\tau(U_A(t-\tau)x(\tau)) = 0.$$

Thus, for each $t>0$ and arbitrary $0\leq\tau\leq t$, one gets that

$$U_A(t-\tau)\,x(\tau) = U_A(t)\,x(0) = x(t) \in \mathrm{dom}\,(A). \tag{1.1.24}$$

This means that any solution of (1.1.23) has the unique form

$$x(t) = U_A(t)\, x_0, \qquad x_0 = x(0)\;, \tag{1.1.25}$$

that is, $U_A(t-\tau)U_A(\tau)x_0 = U_A(t)x_0$ for any $x_0 \in \mathrm{dom}\,(A)$. Since this holds on the dense set $\mathrm{dom}\,(A)$, we obtain the *functional equation* or *semigroup law*:

$$U_A(s)U_A(\tau) = U_A(s+\tau), \quad s, \tau \geq 0, \tag{1.1.26}$$

on $\mathfrak{X}$. This proves (d) and also motivates for $U_A(t)$, according to the classical Euler formula and the limit (1.1.5), the notation of the *exponential* function

$$U_A(t) := \mathrm{e}^{-tA} \in \mathcal{L}(\mathfrak{X}), \quad t \geq 0, \tag{1.1.27}$$

generated by the operator A. □

We showed above that conditions of Proposition 1.1.1 are *sufficient* for construction of a strongly continuous semigroup of contractions $\{U_A(t)\}_{t\geq 0}$. The converse assertion about *necessity* of these conditions and uniqueness of the construction we shall prove in Proposition 1.2.7 of Sect. 1.2.

1.2 Strongly Continuous and Contraction Semigroups

Proposition 1.1.1 motivates definition of the following fundamental notion for this book: the one-parameter *strongly continuous* semigroup or the C_0-semigroup on a Banach space $\mathfrak{X}$.

Definition 1.2.1 (C_0-semigroups) A family $\{U(t)\}_{t\geq 0}$ of bounded linear operators on $\mathfrak{X}$ is called a strongly continuous one-parameter C_0-semigroup if

(a) $U(0) = \mathbb{1}$
(b) $U(t)U(s) = U(t+s)$, for $t, s \in \mathbb{R}^+_0$ (*functional equation* or the *semigroup law*)
(c) The mapping $t \mapsto U(t)$ is strongly continuous from $\mathbb{R}^+$ into the Banach space $\mathcal{L}(\mathfrak{X})$ of bounded operators and strongly *right-continuous* at $t = 0$:

$$(s_0) \qquad \lim_{t\to+0} \|U(t)\,x - x\| = 0\,, \tag{1.2.1}$$

for every $x \in \mathfrak{X}$, that is, $\text{s-lim}_{t\to+0}\, U(t) = \mathbb{1}$ (*non-degenerate* semigroup) .

Remark 1.2.2 Here and in Sects. 1.4 and 1.5, we show that crucial for the properties of a semigroup is topology of the *right* continuity at $t = 0$. We denote it also as continuity at $t = 0$. On the other hand, it is known that (the *strongest*) topology of continuity of a semigroup away from $t = 0$ may be rather different from that at $t = 0$. Moreover, this topology may even *vary* with $t > 0$ from the *strong* topology to the *operator-norm* and then to the *trace-norm* topology (see examples and comments in Notes to Chap. 1 (Sect. 1.6)).

The strong continuity (1.2.1) makes the C_0-semigroups, in a certain sense, exceptional and the most important among the one-parameter operator semigroups. On account of this remark, we come back to Definition 1.2.1 for scrutinising the continuity condition (s_0) at $t = 0$. In fact, the Proposition 1.2.3, which follows below, says that conditions (a)–(c) are *equivalent* to (a) and (b) and condition (s_0) *only* at the origin $t = 0$. That is, conditions (a) and (b) and (s_0) are necessary and sufficient for the family $\{U(t)\}_{t\geq 0}$ be a C_0-semigroup.

Since (a)–(c) evidently imply (a), (b) and (s_0), we have to prove only the converse statement.

Proposition 1.2.3 *If a family of bounded operators* $\{U(t)\}_{t\geq 0}$ *satisfy* (a), (b) *and* (s_0) *in Definition* 1.2.1, *then*

(1) *There exist constants* $M \geq 1$ *and* $\beta \geq 0$ *such that*

$$\|U(t)\| \leq M\, e^{\beta t} \tag{1.2.2}$$

for all $t \geq 0$.

(2) *The orbit mappings* : $\{t \mapsto U(t)\, x\}_{x\in\mathfrak{X}}$ *are strongly continuous from* $\mathbb{R}_0^+$ *into* $\mathfrak{X}$ *for every* $x \in \mathfrak{X}$. *That is, the one-parameter semigroup* $\{U(t)\}_{t\geq 0}$ *satisfies condition* (c).

Proof

(1) On account of (s_0), one infers that for some $n^* \in \mathbb{N}$

$$M_{n^*} := \sup_{0\,\leq\, t\,\leq\, 1/n^*} \|U(t)\| < \infty\,.$$

Indeed, for otherwise, there would exist a sequence $\{t_n \in [0, 1/n]\}_{n\in\mathbb{N}}$ such that $\|U(t_n)\| > n$. Then it would follow that

$$\limsup_{n\to\infty} \|U(t_n)\, x\| = \infty\,,$$

for some $x \in \mathfrak{X}$, which contradicts to the limit (1.2.1). Note that owing to (a) one also gets that $M_{n^*} \geq 1$.

Now, we put $M := (M_{n^*})^{n^*} \geq 1$. Then by the functional equation (b) and definition of M_{n^*}, we obtain that $\|U(s)\| \leq \|U(s/n^*)\|^{n^*} \leq M$ for $s \in [0, 1]$. For arbitrary $t \in \mathbb{R}_0^+$, there exists $n \in \mathbb{N}$ such that $t = n + s$, where $s \in [0, 1]$. As a consequence of (b) and $M \geq 1$, we deduce (1.2.2) since

$$\|U(t)\| \leq \|U(1)\|^n\ \|U(s)\| \leq M^{n+1} \leq M\, \mathrm{e}^{\beta t}$$

holds for $\beta := \ln(M)$ and each $t \geq 0$.

(2) Let $t_0 > 0$. Then the right continuity at t_0 of the family of bounded operators $\{U(t)\}_{t\geq 0}$ follows from the right continuity (s_0) at zero by virtue of

$$\lim_{\delta\to+0} \|U(t_0+\delta)\,x - U(t_0)\,x\| \leq \|U(t_0)\| \lim_{\delta\to+0} \|U(\delta)\,x - x\| = 0\,, \quad x \in \mathfrak{X}\,.$$

For the proof of the left continuity at t_0, we note that by estimate (1.2.2) and again by condition (s_0)

$$\lim_{\delta\to+0} \|U(t_0-\delta)\,x - U(t_0)\,x\| \leq \lim_{\delta\to+0} \|U(t_0-\delta)\|\,\|x - U(\delta)\,x\| \leq$$
$$\lim_{\delta\to+0} M\,\mathrm{e}^{\beta\,(t_0-\delta)}\,\|x - U(\delta)\,x\| = 0\,, \quad x \in \mathfrak{X}\,.$$

Thus, the strong one-sided continuity (s_0): $\text{s-lim}_{t\to+0} U(t) = U(0)$, together with non-degeneracy (a) : $U(0) = \mathbb{1}$, and the functional equation (b) imply that the orbit mappings: $\{t \mapsto U(t)\,x\}_{x\in\mathfrak{X}}$ are continuous from $\mathbb{R}_0^+$ into $\mathfrak{X}$ for every $x \in \mathfrak{X}$. That is, the one-parameter semigroup $\{U(t)\}_{t\geq 0}$ satisfies the condition (c). □

Remark 1.2.4

(a) A priori, a one-parameter C_0-semigroup $\{U(t)\}_{t\geq 0}$ is *not* continuous for $t > 0$ in the *operator-norm* topology, even though one has Example 1.6.5 of such behaviour in Sect. 1.6. On the other hand, a semigroup, which is operator-norm continuous at $t = +0$, is, in a certain sense, a *trivial* concept (see Sects. 1.4 and 1.10 (Notes to Sect. 1.2)).

(b) Instead, one gets that for a C_0-semigroup $\{U(t)\}_{t\geq 0}$ the family $\{\|U(t)\,x\|\}_{t\geq 0}$ is continuous for each $x \in \mathfrak{X}$. Then function $\mathbb{R}_0^+ \ni t \mapsto \|U(t)\|$ is *lower* semi-continuous and, as a consequence, is measurable. On that account, the family of operator-norms $\{\|U(t)\|\}_{t\geq 0}$ need *not* be continuous.

(c) Moreover, on account of the semigroup law, the measurable function $\mathbb{R}^+ \ni t \mapsto \ln(\|U(t)\|)$ is *subadditive*. As a consequence, we infer that there exists

$$\beta_0 := \inf_{t>0} \frac{1}{t} \ln(\|U(t)\|) = \lim_{t\to\infty} \frac{1}{t} \ln(\|U(t)\|)\,, \tag{1.2.3}$$

which is called the *type* of the C_0-semigroup $\{U(t)\}_{t\geq 0}$.

We note that Proposition 1.1.1 proves the existence of a special class of strongly continuous semigroups (1.1.15) for which $M = 1$ and $\beta = 0$, cf. (1.2.2) in Proposition 1.2.3. This class is generated by operators satisfying the *Hille-Yosida* conditions (i) and (ii) of Proposition 1.1.1.

Definition 1.2.5 An operator-valued function $t \mapsto U(t)$ is called a contraction C_0-semigroup if it is a strongly continuous semigroup and $\|U(t)\| \leq 1$ for $t \in \mathbb{R}_0^+$.

It is important that the *converse* statement of Proposition 1.1.1 is also true. To this aim, one needs to define another basic object.