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Mohammad Enayati · Jean-Pierre Gazeau · Hamed Pejhan · Anzhong Wang

The de Sitter (dS) Group and Its Representations

An Introduction to Elementary Systems and Modeling the Dark Energy Universe

Second Edition

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Mohammad Enayati
Department of Physics
Razi University
Kermanshah, Iran

Jean-Pierre Gazeau
Laboratoire Astroparticule et Cosmologie, CNRS
Université Paris Cité
Paris, France

Hamed Pejhan
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia, Bulgaria

Anzhong Wang
Department of Physics, GCAP-CASPER
Baylor University
Waco, TX, USA

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Preface

Upon conducting a brief literature search, one can discover an extensive collection of books, articles, and other resources that offer readers pedagogical or advanced materials essential for comprehending different facets of a consistent quantum field theory (QFT) approach to elementary systems in curved spacetimes. These resources cover a wide range of topics, including the examination of QFT in de Sitter (dS) spacetime, which is particularly relevant to the current study. [We would like to point out in passing that dS spacetime is one of the components of the ΛCDM standard model, where Λ stands for the currently observed acceleration of the Universe expansion (positive "cosmological constant") and CDM for "Cold Dark Matter". Additionally, dS spacetime serves as a fundamental example of curved spacetimes on which QFT can be, to a certain extent, developed with a strong mathematical foundation.] Nevertheless, it is noteworthy that a significant aspect in the formulation of elementary systems, namely, consistency with Wigner's approach, is largely disregarded by the majority of these references, as we further explain now.

First, it is important to recognize that both the field theoretical formulation and the phenomenological treatment of an elementary system, particularly when it comes to interpretation, rely on fundamental concepts such as energy, momentum, mass, and spin. These concepts owe their existence to principles of invariance, specifically the principle of invariance under the Poincaré group, which represents the relativity $\sim$ kinematical group of flat Minkowski spacetime. [Let us recall that, as highlighted by Wigner, quantum elementary systems are associated with (projective) unitary irreducible representations (UIRs) of the relativity group or one of its coverings. More specifically, in the framework of Einstein-Poincaré relativity, Wigner demonstrated that the rest mass m and spin s of an elementary system act as the two invariants that characterize the corresponding UIR of the Poincaré group.]

However, in curved spacetime, any interpretation based on the relativity group of flat Minkowski spacetime becomes physically irrelevant. Additionally, in curved spacetimes in general (excluding dS and anti-dS (AdS) spacetimes), there are no nontrivial groups of motion, leading to the absence of a direct or unique extension of the aforementioned physical concepts (?!).

Consequently, while it is feasible to extend the essential differential equations, such as the Klein-Gordon and Dirac equations, which characterize elementary systems in flat Minkowski spacetime to formulations that exhibit general covariance in curved spacetimes, these mathematical frameworks ultimately cannot be connected to physical elementary systems in the sense given above.

The main motivation behind this manuscript is precisely to address this overlooked aspect. We aim to thoroughly examine the construction of ("free") elementary systems in the global structure of dS spacetime, in accordance with the Wigner framework, as associated with UIRs of the dS (relativity) group. The manuscript delves into the conceptual challenges that arise in formulating such systems and presents a mathematically rigorous exploration of the known results. Several key areas receive particular attention, including: the "smooth" transition from classical to quantum theory; physical content under vanishing curvature, from the point of view of a local ("tangent") Minkowskian observer; and thermal interpretation (at the quantum level), in the sense of the Gibbons-Hawking temperature.

We examine three decompositions of the dS group that hold significance in describing dS spacetime and the classical phase spaces of elementary systems residing within it. We explore the construction of (projective) dS UIRs derived from these group decompositions. The (projective) Hilbert spaces that carry these UIRs, in some restricted sense, serve as identification for the quantum ("one-particle") state spaces of elementary systems in dS spacetime. By employing a well-established Fock procedure based on the Wightman-Gärding axioms and analyticity requirements in the complexified pseudo-Riemannian manifold, we develop a coherent formulation of QFT for elementary systems in dS spacetime. This QFT formulation for dS closely mirrors the corresponding formulation in Minkowski spacetime, albeit with the traditional spectral condition replaced by a specific geometric Kubo-Martin-Schwinger (KMS) condition that represents a precise thermal manifestation of the associated "vacuum" states.

We conclude our study by revisiting a coherent and univocal definition of mass within the framework of dS relativity. This definition, articulated in terms of invariant parameters that define the properties of dS UIRs, furnishes a precise interpretation for terms like "massive" and "massless" fields in dS relativity, akin to their analogs in Minkowski spacetime derived through group contraction methodologies.

Going through the above process, this manuscript seeks to address a wide readership, specifically targeting theoretical and mathematical physicists with an interest in QFT in curved spacetime, cosmology, and quantization. Its comprehensive and pedagogical resources are designed to facilitate the understanding of various mathematical aspects pertaining to the dS group. These aspects encompass the dS group's Lie manifold, Lie algebra, and (co-)adjoint orbits, with the latter assuming significance as potential classical elementary systems within the framework of dS spacetime. Furthermore, the manuscript explores their quantum counterparts, namely the dS UIRs, shedding light on quantum elementary systems within the dS spacetime context.

Kermanshah, Iran — Mohammad Enayati
Paris, France — Jean-Pierre Gazeau
Sofia, Bulgaria — Hamed Pejhan
Waco, USA — Anzhong Wang

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Contents

Acronyms

AdS	Anti-de Sitter
AdS_4	1 + 3-dimensional anti-de Sitter
CMB	Cosmic microwave background
dS	de Sitter
dS_2	1 + 1-dimensional de Sitter
dS_4	1 + 3-dimensional de Sitter
GNS	Gel'fand-Naimark-Segal
KMS	Kubo-Martin-Schwinger
QCD	Quantum chromodynamics
QED	Quantum electrodynamics
QFT	Quantum field theory
QGP	Quark-gluon plasma
UIR	Unitary irreducible representation

Part I

de Sitterian Elementary Systems: A Brief Introduction

1 Introduction: Wigner's Elementary Systems in de Sitter (dS) Spacetime

This chapter offers readers an insight into the content of this book, including its motivations and methodology. It serves as a window that allows readers to familiarize themselves with the book's structure, providing a reading guide and outlining the main conventions used throughout its pages.

1.1 Wigner's Vision

Quantum elementary systems are associated with (projective) unitary irreducible representations (UIRs) of the relativity group (or one of its coverings). This seminal point of view was first put forward in the context of Einstein-Poincaré relativity by Wigner in his famous paper in 1939 [1] (see also Ref. [2]), where the rest mass m and the spin s of an (Einsteinian) elementary system are shown to be the two invariants that characterize the associated UIR of the Poincaré group (the group of motions of flat Minkowski spacetime). He was followed by Inönü [3], Lévy-Leblond [4], and Voisin [5] who applied the Wigner ideas to Galilean systems, and by Gürsey [6] and Fronsdal [7, 8] who extended them to de Sitter (dS) and anti-dS (AdS) systems, respectively.

This book follows in the footsteps of Wigner and offers a thorough exploration of constructing (free) elementary systems within the global structure of dS spacetime. The level of exposition is carefully calibrated to accommodate both experts and beginners, ensuring that this study presents something intriguing and valuable for readers from all backgrounds.

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1.2 Motivations

In the context of modern theories of elementary systems (both field theory and the phenomenological treatment), the formulation of a physical theory, and the interpretation in particular, rests upon the notions of energy, momentum, mass, and spin, whose existence literally stems from the principle of invariance under the Poincaré group [1, 2]. Physicists, however, are well aware that modern theories of elementary systems cannot in the end be based on the Poincaré group. What is needed is a theory of elementary systems, or at least a consistent framework, that respects the full general covariance of Einstein's view of spacetime as a Riemannian manifold. However, once we move beyond the familiar flat Minkowski spacetime, a significant challenge arises in extending physical models for elementary systems. This challenge stems from the absence of nontrivial groups of motion in more general Riemannian spaces, rendering the literal or unique extension of the aforementioned physical concepts nonexistent or exceedingly difficult to establish.

Here, we put aside the suggestion that the important differential equations (Klein-Gordon and Dirac) may easily be generalized to forms that possess general covariance. In the above sense, this suggestion is almost totally irrelevant. Frankly speaking, the modern theories of elementary systems are not primarily studies in differential equations [7, 9].

There is, of course, a specific class of Riemannian spaces in which the road to generalizations is well marked, in the sense given by Fronsdal in 1965 [7]: "*A physical theory that treats spacetime as Minkowskian flat must be obtainable as a well-defined limit of a more general physical theory, for which the assumption of flatness is not essential.*" Poincaré relativity indeed can be considered as the idealistic null-curvature limit of two possible curved-spacetime relativities of maximal symmetry. Technically, a four-dimensional Riemannian space may admit a continuous group of symmetry, preserving the metric $g_{\mu\nu}$, with up to ten essential parameters. The maximum number (which is the same number as flat Minkowski spacetime) is merely realized for a space of constant curvature $1/R$ (R being the radius of curvature, $0 < R < \infty$).

Those spacetimes, which meet flat Minkowski spacetime as the curvature goes to zero ($R \to \infty$), are the ordinary dS and AdS spacetimes, the maximally symmetric solutions to the vacuum Einstein's equations with, respectively, positive and negative cosmological constant Λ ($R = \sqrt{3/|\Lambda|}$) [10]. The former, dS spacetime, admits $\mathrm{SO}_0(1,4)$ (or its universal covering $\mathrm{Sp}(2,2)$) as the group of motions. It is essentially finite in extension [11]; considering any point p and any timelike direction in that point, the geodesics through p, perpendicular to the chosen timelike direction, are finite. AdS spacetime, on the other side, is infinite in extension; analogous geodesics possess infinite lengths and are completely spacelike. The AdS group of motions is $\mathrm{SO}_0(2,3)$ (or its double covering $\mathrm{Sp}(4,\mathbb{R})$, or even its universal covering $\widetilde{\mathrm{SO}_0(2,3)}$).

Interestingly, as Minkowski spacetime is the limit $R \to \infty$ of the ordinary dS and AdS spacetimes, the Poincaré group can be obtained as a contraction of either $\mathrm{SO}_0(1,4)$ or $\mathrm{SO}_0(2,3)$ (or any of their coverings); UIRs of the dS and AdS groups, analogous to their shared Poincaré contraction limit, are characterized by two invariant parameters of the spin and energy scales (note that, in the AdS case, the latter should be read as the rest energy). These remarkable attributes enable the extension of Wigner's elementary system definition to dS and AdS relativities, to a certain extent.[1]

In this study, our primary focus lies on the dS scenario. Beyond the aforementioned conceptual concerns, this choice is spurred by the pivotal significance of the dS metric within inflationary cosmological scenarios, which postulate that our Universe underwent a dS phase during its initial stages [18]. Additionally, it is driven by the aspiration to formulate potential frameworks for late-time cosmology, given that recent data suggests the necessity of a small positive cosmological constant or dark energy [19].

1.3 Content at a Glance

As highlighted earlier, the primary focus of this manuscript revolves around the inclusion of a more expansive relativity group that surpasses the limitations of the Poincaré group to some extent, aiming to provide a more comprehensive description of elementary systems. To lay a solid foundation for our discussions, it is essential here to briefly explore the concept and significance of relativity groups within a broader context.

From a technical perspective, to study a physical system P, one needs a *frame*, that is, a correspondence between P and a mathematical structure M describing the set of states of P measured with respect to this frame. In this context, the relativity group G is the group of frame transformations. Then, the rule "physical laws are independent of the frame" turns into "the structure of M is invariant under G". This structure is a symplectic manifold (called phase space) at the classical level and a (projective) Hilbert space at the quantum level. This system is called an elementary system [1, 2] when one does not deal with internal variables. Therefore, the different states, that appear, are merely due to a change of frames and nothing else. This implies that the action of the group G on M is transitive, i.e., is a co-adjoint representation or a (projective) UIR at the classical and quantum levels, respectively.

In this context, we explore elementary systems by considering the dS group $\mathrm{Sp}(2,2)$.[2] We start from scratch to be able to present the foundations step-by-step in a mathematically rigorous way. We employ three types of decomposition of the $\mathrm{Sp}(2,2)$ group. The first one, called space-time-Lorentz decomposition, is nonstandard and yields a global, but nonunique, decomposition of the group, while the other two are well known in semi-simple group theory [21, 22] and, respectively, called Cartan and Iwasawa decompositions. These group decom-

[1] We note in passing that the (A)dS group-theoretical structures serve a wider variety of practical applications in modern physics than what we have mentioned above. The study of the Hydrogen atom, for instance, well illustrates several aspects of the application of such structures in quantum mechanics (see, for instance, Refs. [12–17]).

[2] To gain a comprehensive understanding of the various mathematical aspects related to the AdS counterpart of this construction, readers are referred to Ref. [20].

positions provide the basic mathematical ingredients of our discussion, namely, (related) families of group cosets $\mathrm{Sp}(2,2)/\underline{S}$, where $\underline{S}$ stands for (closed) subgroups of $\mathrm{Sp}(2,2)$ yielded by these decompositions. As a matter of fact, each phase space of dS elementary systems, more accurately, each transitive manifold under the action of the $\mathrm{Sp}(2,2)$ co-adjoint representations (say $\mathrm{Sp}(2,2)$ co-adjoint orbit), being a symplectic manifold and carrying a natural $\mathrm{Sp}(2,2)$-invariant (Liouville) measure, is a homogeneous space homeomorphic to an even-dimensional group coset $\mathrm{Sp}(2,2)/\underline{S}$, where $\underline{S}$ plays the role of stabilizer subgroup of some orbit point [23, 24].

The $\mathrm{Sp}(2,2)$ co-adjoint orbits, naively speaking, the group cosets $\mathrm{Sp}(2,2)/\underline{S}$, also possess very rich analytic structures, which *underlie*[3] (projective) Hilbert spaces that carry UIRs of the $\mathrm{Sp}(2,2)$ group. According to the physical point of view adopted in this manuscript, this remarkable feature in a well-established process allows for a "smooth" transition from classical to the quantum formulation of dS elementary systems; the phase spaces of dS elementary systems quantize into (projective) dS UIRs. These UIRs consist of three distinguished series, respectively, called principal, complementary, and *discrete (plus the so-called degenerate UIRs)*[4] series [34–38]. The UIRs corresponding to the principal series contract to the massive UIRs of the Poincaré group, in such a way that exhaust the whole set of the latter [39, 40]. Hence, they are called dS massive representations. The situation for the dS massless cases, however, is more subtle; the dS group has no UIR analogous to the so-called massless infinite-spin UIRs of the Poincaré group. Massless representations of the dS group then are naturally distinguished as those with a unique extension to the UIRs of the conformal group $\mathrm{SO}_0(2,4)$, while that extension is equivalent to the conformal extension of the Poincaré massless UIRs (of course, this correspondence exhausts the whole set of the Poincaré massless UIRs) [41–43]. It follows that the dS massless scalar case coincides with a specific UIR of the complementary series, while the dS massless higher-spin cases correspond to the UIRs lying at the lower end of the discrete series. All other dS representations either have nonphysical Poincaré contraction limit or do not have Poincaré contraction limit at all.[5]

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[3] In order to address this matter, one could adopt a comprehensive program for the quantization of functions or distributions, which would involve the incorporation of the complete set of covariant (integral, geometric, deformation, ...) quantization techniques as outlined in Refs. [25–32], for instance.

[4] Note that the term "degenerate" applies to elements within the set of dS UIRs that do not belong to the three series mentioned above. However, in this study, we specifically examine these elements as individual members, referred to as scalar UIRs, within the discrete series. The genuine discrete series comprises square-integrable UIRs, as detailed, for example, in Ref. [33]. Consequently, these scalar UIRs, denoted in this book as $\Pi_{p,0}$, should indeed be characterized as "degenerate" in this particular sense.

[5] It is important to emphasize that the preceding statement does not suggest in any way that the remaining portion of the dS UIRs without a counterpart in Minkowski spacetime is physically insignificant. Quite the opposite, it is entirely valid to investigate all dS UIRs within a coherent framework, encompassing both the mathematical aspects (group representation) and the physical aspects (field quantization).

Upon identifying dS massive and massless elementary systems through the lens of group representation theory, the avenue opens for addressing the corresponding covariant quantum field theories (QFTs), following the trajectory outlined by Wightman and Gärding in their seminal paper [44]. The very problem, that naturally arises here, is the absence of a true spectral condition, which plagues QFT in dS spacetime [45, 46]. Actually, no matter what machinery of QFT is employed to quantize a field in dS spacetime, while it is rather straightforward to formalize the requirements of locality (microcausality) and covariance, it is impossible to formulate any condition on the spectrum of the "energy" operator (even worse, it is impossible to define such a global object at all). Owing to this inherent ambiguity, each distinct dS field model gives rise to a multitude of inequivalent QFTs, a phenomenon that manifests as the nonuniqueness of the vacuum state. Often, each of these QFTs finds relevance in specific time coordinate selections, thereby inducing corresponding frequency splitting. Consequently, to establish a coherent QFT interpretation of dS elementary systems, beyond the dS group representation theory (as posited by Wigner) and the Wightman and Gärding axioms, an additional criterion is required to substitute for the conventional spectral condition.

In Refs. [47, 48],[6] Bros et al. have proposed that by appropriately adapting certain well-known concepts of complex Minkowski spacetime to its complex dS counterpart, we can establish a criterion that resolves all uncertainties in dS QFTs and allows us to identify preferred vacuum states. These preferred vacuum states, despite exhibiting thermal characteristics as defined by Gibbons-Hawking [54, 55], align with the corresponding Minkowski vacuum representations when the curvature tends to zero. Their original approach keeps from the Minkowskian case the idea that the analytic continuation properties of the QFT in the complexified spacetime are directly related to the energy content (in particular to the spectral condition) of the model considered. Technically, in order to apply this appealing idea to dS QFTs, they have put forward a genuine, global dS-Fourier type calculus, realized by the introduction of (coordinate-independent) *dS plane waves*[7] in their tube domains. On this basis, they have shown that, for instance, in the simplest cases, i.e., linear dS QFTs which are of interest in the present study, the spectral condition is substituted by a certain geometric Kubo-Martin-Schwinger (KMS) condition [59, 60], equivalent to a precise thermal manifestation of the associated vacuum states (known in the literature under the name of Euclidean [54] or Bunch-Davies [61] vacuum states).[8]

Consequently, within the pages of this manuscript, we embark on a journey that involves both the dS group representation theory alongside its Wigner interpretation, and, in parallel,

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[6] For closely related discussions, we also recommend referring to Refs. [49–53] and the references therein.

[7] Such waves are the dS counterparts of the standard plane waves in Minkowski spacetime. They are well adapted to the dS group representations and also allow to control in a very suggestive way the null-curvature limit of dS QFTs to their Minkowskian counterparts [40, 47, 48]. Furthermore, they offer the potential for establishing a coherent holographic correspondence within dS spacetime, as discussed in Ref. [56] (also see Refs. [57, 58]).

[8] For this, except the references cited above, see also Refs. [62–68].

the Wightman and Gärding axioms enriched with the necessity of analyticity within the complexified dS manifold (as delineated by Bros et al.). It is within this interdisciplinary framework that we delve into the QFT formulation of elementary systems in the expansive canvas of dS spacetime.

Ultimately, the pivotal concern arises: how to find a universally applicable alternative to the concept of mass within dS relativity? This imperative leads us to embrace a coherent and definitive definition of mass in dS spacetime, as initially put forth by Garidi in 2003 [69]. This Garidi definition, founded upon the invariant parameters characterizing the dS UIRs, impressively confers meaningfulness upon terms like "massive" and "massless" fields in the dS relativity context, harmonizing them with their Minkowskian counterparts derived through group contraction procedures. Furthermore, this definition holds the remarkable advantage of encompassing all mass formulations introduced within the framework of dS.

1.3.1 Limitations of Analyticity-Based Quantization in dS Fields

It is imperative to underscore that the quantization approach, hinging on the earlier-discussed criterion of analyticity, encounters limitations when applied to specific fields within the dS framework. Notably, this constraint manifests itself in scalar fields associated with the scalar discrete series representations, as well as the dS graviton field linked to the discrete massless spin-2 representation. In both cases, these fields exhibit a distinctive form of gauge invariance, which becomes anomalous at the quantum level. This gauge anomaly disrupts the consistency of the theory, necessitating resolution at any cost. A direct outcome of this inconsistency is the unavailability of established dS-invariant Euclidean/Bunch-Davies vacuum states for these specific fields.

To address this anomaly, a departure from the established framework becomes essential, necessitating the adoption of an alternative approach grounded in a *Krein structure (endowed with an indefinite inner product)*, diverging from the conventional *Hilbertian* approach; the *Krein QFT construction* versus the *Hilbertian one*. This alternative approach remarkably assures that the theory embodies all the anticipated attributes of a free field in dS spacetime characterized by high symmetry, including the positivity of the norm of all physical states, adherence to causality, (full) covariance, and the positivity of the energy operator across all physical states. For a more comprehensive grasp of these concepts, we guide readers to Refs. [70–78].

In our current study, we narrow our focus to the most straightforward manifestation of these fields, specifically the dS minimally coupled scalar field, which corresponds to the lowest case within the scalar discrete series representations. With meticulous attention, we embark on a comprehensive exploration of the corresponding Krein QFT formulation.

1.4 Reading Guide and Conventions

The remainder of this manuscript is structured as follows:

In part I, we discuss $1 + 1$-dimensional dS relativity to provide a foundation for better understanding the mathematical concepts. Interestingly, despite its mathematical transparency, this simplified form of dS relativity contains all the essential ingredients found in the realistic $1 + 3$-dimensional dS relativity.

Moving on to Part II, we shift our focus to the latter scenario, $1 + 3$-dimensional dS relativity, at both the group/algebra and representation levels. In other words, we explore it from the perspectives of classical and quantum mechanics, respectively. Subsequently, we proceed with the corresponding formulation of QFT. Lastly, we engage in a comprehensive discussion on the concept of mass within the realm of dS relativity. In order to establish a comparative framework, we also explore the concept of mass in AdS relativity.

The main conventions of our notations are:

- Throughout this manuscript (unless noted otherwise), for the sake of simplicity, we consider the units $c = 1 = \hbar$, where c and $\hbar$ are respectively the speed of light and the Planck constant.
- To differentiate between $1 + 1$-dimensional dS spacetime and its $1 + 3$-dimensional counterpart, we introduce the subscripts '2' (dS_2) and '4' (dS_4), respectively. Additionally, to distinguish the relevant entities, particularly those that do not explicitly exhibit spacetime indices, we draw a line underneath the ones that pertain specifically to dS_4 relativity.
- We use the letters $a, b, c, \ldots$ for the indices 0, 1, 2, the letters $\mu, \nu, \rho, \ldots$ for 0, 1, 2, 3, the letters $A, B, C, \ldots$ for 0, 1, 2, 3, 4, the letters $A', B', C', \ldots$ for 0, 1, 2, 3, 5 (the number 4 is left apart!), the letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ for 0, 1, 2, 3, 4, 5, the letters $\mathtt{A}, \mathtt{B}, \mathtt{C}, \ldots$ for 1, 2, 3, 4, and finally the letters $i, j, k, \ldots$ for 1, 2, 3.

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