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Wolfgang Lück
Tibor Macko

Surgery Theory

Foundations

With contributions by Diarmuid Crowley

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Preface

Surgery theory was created in the sixties to solve classification problems for manifolds and since then has led to an enormous number of striking results. It has many interactions with other areas of mathematics, such as algebra, differential geometry, geometric group theory, algebraic K-theory, number theory, and the theory of operator algebras. Surgery theory also promises to be a major tool in geometry and topology in the future.

Surgery theory was initiated by Kervaire and Milnor in their paper [216] on the classification of homotopy spheres. Surgery theory for simply connected closed manifolds was developed systematically in Browder's book [55]. The book of Wall [414] established surgery theory for arbitrary fundamental groups. It also contains numerous new results on the classification of closed manifolds. The main tool in surgery theory is the surgery exact sequence due to Browder, Novikov, Sullivan, and Wall. It combines the homotopy theory of manifolds with the L-theory of quadratic forms over the group rings of their fundamental groups in order to obtain classification results about manifolds. The work of Kirby and Siebenmann [219] made it possible to do surgery also in the topological category. Quinn [337] gave a description of the surgery exact sequence as the long exact sequence of homotopy groups of a fibration and identified one of its maps as the so-called assembly map. Ranicki [344] developed a chain complex version of algebraic L-theory, answering a request by Wall [414, Chapter 17G], and later provided an algebraic description of the assembly map [348]. The Farrell–Jones Conjecture [150] about the assembly maps and its proofs for a large class of groups allow computations of L-groups of infinite groups and open the door to many applications of surgery theory for compact manifolds with infinite fundamental groups.

The goal of this book is to present an accurate, comprehensible, complete, and detailed introduction to surgery theory, which is useful for various groups of readers, such as experts in surgery theory, experienced mathematicians, who may not be experts in surgery, but just want to learn or use it, and also, of course, advanced undergraduate and graduate students. This is quite a challenge since surgery theory is sophisticated and complicated, requiring a large amount of previous knowledge, and since a lot of material has been accumulated, but not all details are well documented. We tried to find a reasonable compromise between the intention to present many details in full generality, to fix some bugs in the literature, to motivate the constructions, theorems, and proofs, and the desire to allow the reader to just browse through the book to get a first impression, or find a solution to a specific problem or an answer to a specific question, without necessarily going through all of the text.

Throughout the book we rely on the basics of the surgery theory developed in recent decades. None of the main theorems or concepts are new, but there are places where our approach to certain details is novel.

Bonn, February, 2024

Wolfgang Lück
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Contents

Chapter 1
Introduction

1.1 Some Classical Problems that Can Be Attacked by Surgery Theory

In this section we give a list of concrete, classical, and prominent problems that have been (partially) solved by surgery theory. The list shall illustrate the high potential of surgery theory and the (partial) solutions of these problems will constitute the contents of this book.

The following two problems represent the prototype of surgery problems, which, however, cannot be solved in full generality.

Problem 1.1 (Recognising manifolds) Let X be a connected finite CW-complex. Under which conditions is X homotopy equivalent to a closed manifold that is topological, PL (= piecewise linear), or smooth?

Problem 1.2 (Classifying manifolds) Let M and N be closed manifolds that are both topological, PL, or smooth. Under which conditions can one decide whether they are homeomorphic, PL homeomorphic, or diffeomorphic respectively. What are possible obstructions and under which conditions are they sufficient?

Exact formulations of both of these problems may vary. We sometimes consider modifications and use slightly different descriptions. Problem 1.1 may also be called an "existence problem" because we are asking whether there exists a manifold in the homotopy type of X. The variation of Problem 1.2 where we assume to begin with that M and N are homotopy equivalent may be called a "uniqueness problem" since we are asking how unique the manifolds in a given homotopy type are.

The next conjecture, a special case of Problem 1.2 in the topological category, is known to be true for all $n \geq 1$. Its proof uses surgery theoretic methods, except in dimension 3 where the proof relies on Ricci flow.

Conjecture 1.3 ((Generalised) Poincaré Conjecture) If M is a closed topological manifold homotopy equivalent to the standard n-sphere S^n, then M is homeomorphic to S^n.

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The following problem, a special case of Problem 1.2 in the smooth category, triggered the development of surgery theory and will be discussed in Chapter 12.

Problem 1.4 (Homotopy spheres) Classify all oriented homotopy spheres, that means closed oriented smooth manifolds homotopy equivalent to the standard n-sphere, up to orientation preserving diffeomorphism.

Another problem, which is completely solved, is the following.

Problem 1.5 (Fake complex projective spaces) Classify all fake complex projective spaces, that means topological manifolds homotopy equivalent to the standard n-dimensional complex projective space, up to homeomorphism.

One can ask the corresponding question for other prominent manifolds, for example for fake real projective spaces or fake lens spaces. There are solutions in many interesting cases, as we will see in Chapter 18.

The next conjecture about aspherical manifolds, that means connected manifolds whose universal covering is contractible, will be treated in Chapter 19. It is known to be true if the fundamental group is contained in a large class of groups, which encompasses hyperbolic groups, CAT(0)-groups, solvable groups, and lattices in almost connected Lie groups, but is open in general. It is the topological version of Mostow rigidity.

Conjecture 1.6 (Borel Conjecture) Let M and N be closed aspherical topological manifolds. Then:

(i) The fundamental groups $\pi_1(M)$ and $\pi_1(N)$ are isomorphic if and only if M and N are homeomorphic;
(ii) Any homotopy equivalence $f\colon M \to N$ is homotopic to a homeomorphism;
(iii) Any map $f\colon M \to N$ inducing an isomorphism between the fundamental groups is homotopic to a homeomorphism.

The next problem, which is essentially solved, triggered surgery theory for non-simply connected manifolds, see Section 3.6. It is a kind of generalisation of the Space Form Problem asking which finite groups occur as fundamental groups of closed Riemannian manifolds with constant positive sectional curvature.

Problem 1.7 (Spherical Space Form Problem) Which finite groups can act freely and topologically or smoothly respectively on a standard sphere, or, equivalently, occur as fundamental groups of closed manifolds whose universal covering is homeomorphic or diffeomorphic respectively to a standard sphere.

1.2 Overview of the Contents of this Book

Chapters 2 to 11 contain the core of surgery theory that leads to a general method for solving Problems 1.1 and 1.2, while Chapter 12 illustrates the method on the most prominent example of homotopy spheres. The following Chapters 13 to 17 contain additional theoretical tools that are needed to effectively solve Problems 1.1 and 1.2 in other cases, in particular in the topological category. Chapters 18 and 19 illustrate how all this is applied to the concrete examples from the list in the previous section. Finally, Chapter 20 is about an alternative approach called modified surgery.

Although it may not be obvious at first glance, Problems 1.1 and 1.2 are closely linked. The general slogan is that Problem 1.2 is a relative version of Problem 1.1. The general approach to both of these problems is to split them into several steps and to treat the steps separately. Both of these ideas are explained with more details at the appropriate places in the following brief survey of the individual chapters. In the book itself, each chapter has its own more detailed introduction.

In the overview below we will often only treat the smooth category. Nearly all notions and statements carry over to the PL and the topological category.

Chapter 2: The s-Cobordism Theorem

We state and prove the s-Cobordism Theorem 2.1. Roughly speaking, it says that a compact smooth cobordism W from the closed smooth manifold M_0 to the closed smooth manifold M_1, for which the inclusion $M_i \to W$ is a simple homotopy equivalence for $i = 0, 1$ and $\dim(M_0) \geq 5$ holds, is diffeomorphic relative M_0 to the cylinder $M_0 \times [0, 1]$. This implies that M_0 and M_1 are diffeomorphic. Hence the s-Cobordism Theorem is highly relevant for the solution of Problem 1.2, namely, it is a cornerstone in the Surgery Program, see Remark 2.9, designed to solve Problem 1.2 by splitting it into three steps. Roughly, first find a (simple) homotopy equivalence, then construct a cobordism compatible with the (simple) homotopy equivalence, and finally improve this cobordism to an s-cobordism. If we get a positive answer in all three steps, then by the s-Cobordism Theorem we obtain a diffeomorphism.

The s-Cobordism Theorem 2.1 (in the topological category) implies the (Generalised) Poincaré Conjecture 1.3 in dimensions ≥ 5.

Chapter 3: Whitehead Torsion

We give a systematic treatment of Whitehead torsion, which is the obstruction for a homotopy equivalence of finite CW-complexes to be a simple homotopy equivalence. This is relevant since it appears in the s-Cobordism Theorem 2.1. We also explain the classification of lens spaces by their Reidemeister torsion in Section 3.5. This yields a solution of Problem 1.2 in a very specific case where it can exceptionally be achieved without surgery theory from later chapters.

Chapter 4: The Surgery Step and ξ-Bordism

This chapter contains the first step towards a solution of Problem 1.1. Namely, we solve the following Problem 4.2: Given a map $f\colon M \to X$ from a closed n-dimensional smooth manifold M to a CW-complex X of finite type, can we modify it, without changing the target, to a map $f'\colon M' \to X$ with a closed n-dimensional smooth manifold as source such that f' is k-connected where $n = 2k$ or $n = 2k + 1$? The basic idea is to modify the procedure of making a map of finite CW-complexes highly connected by attaching cells to a finite sequence of so-called surgery steps, so that the source still remains a closed smooth manifold, a procedure commonly called "surgery below middle dimension". We will see in the process that the presence of certain bundle data is desirable in order to make this work. These bundle data will be formalised in the technical notion of a normal map (also called a surgery problem) in Chapter 7.

Chapter 5: Poincaré Duality

We explain the notion of a finite Poincaré complex. This is relevant for Problem 1.1 since any finite CW-complex that is homotopy equivalent to a closed manifold is a finite Poincaré complex.

Chapter 6: The Spivak Normal Structure

Recall that a closed smooth manifold has a normal bundle, which is unique up to stable isomorphism. Its underlying sphere bundle is a spherical fibration unique up to stable fibre homotopy equivalence. In this chapter we show that any finite Poincaré complex possesses a Spivak normal structure, which is a spherical fibration coming with a certain collapse map and is unique up to stable fibre homotopy equivalence. Since this structure is homotopy invariant, we discover another obstruction for a finite Poincaré complex X to be homotopy equivalent to a closed manifold. Namely, its Spivak normal structure must have a vector bundle reduction, that means it must come from a vector bundle since the Spivak normal structure of a closed smooth manifold comes from its normal vector bundle by the Pontrjagin–Thom construction.

Chapter 7: Normal Maps and the Surgery Problem

We define the notion of a normal map of degree one motivated by the previous sections. Roughly speaking, a normal map of degree one is a map $f\colon M \to X$ from a closed smooth manifold M to a finite Poincaré complex X of degree one, which comes with bundle data, namely, a bundle map from the normal bundle of M to a vector bundle reduction ξ of the Spivak normal structure on X. The surgery problem, see Problem 7.40, now asks whether we can modify it by surgery to a normal map whose underlying map $f'\colon M' \to X$ is a homotopy equivalence. We also show that the set of smooth normal bordism classes of smooth normal maps with the target a fixed finite Poincaré complex X (also called the set of smooth normal invariants) can be identified with the set of homotopy classes of maps from X to a certain space G/O, see Theorem 7.34. Analogous statements hold in the PL category and the topological category, see Theorem 11.24.

Summarising the development of the chapters so far, we see that the solution of Problem 1.1 is split into three steps as formulated in the Surgery Program for recognising manifolds 7.47. Roughly speaking, the first step is to check the necessary homotopical and homological condition on X, that means it must be a finite Poincaré complex. The second step is to find a normal map of degree one from some closed smooth manifold M to X, which exists if the Spivak normal fibration has a vector bundle reduction. In the third step one tries to improve the map to a homotopy equivalence. Surgery below middle dimension from Chapter 4 yields first improvements towards the third step.

Now the slogan that Problem 1.2 is the relative version of Problem 1.1 becomes more apparent, see Remark 7.46. Moreover, the steps in Surgery Program 2.9 and in Surgery Program for recognising manifolds 7.47 correspond to each other as explained in the discussion after Remark 7.47.

Chapter 8: The Even-Dimensional Surgery Obstruction

Not every surgery problem can be solved; there are so-called surgery obstructions. In this chapter we construct the even-dimensional L-groups and the surgery obstructions taking values in them and show that in dimension ≥ 5 the vanishing of the surgery obstruction is equivalent to the existence of a solution of the surgery problem. The L-groups are defined in terms of quadratic forms over the group ring of the fundamental group of X. If the dimension n is divisible by four and X is simply connected, then the surgery obstruction group is $\mathbb{Z}$ and the surgery obstruction is the difference of the signatures of M and X. In this particular case the surgery obstruction is independent of the bundle data, but that is not true in general. If the dimension n is even but not divisible by four and X is simply connected, the surgery obstruction group is $\mathbb{Z}/2$ and the surgery obstruction is the so-called Arf invariant, which definitely depends on the bundle data. This chapter yields the final step in the solution of Problem 1.1 in the even-dimensional case.

Chapter 9: The Odd-Dimensional Surgery Obstruction

We construct the odd-dimensional L-groups and the surgery obstructions taking values in them and show that in dimension ≥ 5 the vanishing of the surgery obstruction is equivalent to the existence of a solution of the surgery problem. The L-groups are defined in terms of automorphisms of quadratic forms, or, equivalently, in terms of formations over the group ring of the fundamental group of X. If X is simply connected and odd-dimensional, the surgery obstruction groups vanish and there are no surgery obstructions. This chapter yields the final step in the solution of Problem 1.1 in the odd-dimensional case.

Chapter 10: Decorations and the Simple Surgery Obstruction

We develop the simple version of the surgery obstruction groups and the surgery obstructions. The difference to the previous constructions is that we want the underlying map $f\colon M \to X$ to be a simple homotopy equivalence, while before we were only aiming at a homotopy equivalence. This is relevant in view of the s-Cobordism Theorem 2.1. In the definition of the surgery obstruction groups we now take finitely generated free modules coming with a basis as the underlying modules of quadratic forms and then consider the Whitehead torsion of the various isomorphisms appearing in the previous constructions.

Chapter 11: The Geometric Surgery Exact Sequence

We introduce the surgery exact sequence in Theorems 11.22 and 11.25, see also Remark 11.23. It is the realisation of the Surgery Program 2.9, and thus yields a general method for solving Problem 1.2. The surgery exact sequence is the main theoretical tool in solving the classification problem of manifolds of dimensions greater than or equal to five.

Its simple topological version for an n-dimensional closed topological manifold N from Theorem 11.25 aims at the computation of the simple structure set $\mathcal{S}^{\mathrm{TOP},s}(N)$. Elements in $\mathcal{S}^{\mathrm{TOP},s}(N)$ are represented by simple homotopy equivalences $f\colon M \to N$ with a closed topological manifold M as source and N as target. Two such elements $f\colon M \to N$ and $f'\colon M' \to N$ represent the same element if there is a homeomorphism $h\colon M \to M'$ such that $f' \circ h$ and f are homotopic. The surgery exact sequence is an exact sequence of abelian groups of the shape

$$\mathcal{N}^{\mathrm{TOP}}(N \times [0,1], N \times \{0,1\}) \to L^s_{n+1}(\mathbb{Z}\pi, w) \to \mathcal{S}^{\mathrm{TOP},s}(N) \\ \to \mathcal{N}^{\mathrm{TOP}}(N) \to L^s_n(\mathbb{Z}\pi, w)$$

where $L^s_n(\mathbb{Z}\pi, w)$ is the simple algebraic L-group of the integral group ring of the fundamental group π of N with the orientation homomorphism w, the normal invariants $\mathcal{N}^{\mathrm{TOP}}(N)$ are given by the surgery problems with target N, and the first and the fourth map are given by taking surgery obstructions. Here one needs to require either $n \geq 5$ or that $n = 4$ and the fundamental group is good in the sense of Freedman, see [157, 158], and Remark 8.30

Note that a closed topological manifold N has the property that any simple homotopy equivalence $M \to N$ from a closed topological manifold to N is homotopic to a homeomorphism if and only if the structure set $\mathcal{S}^{\mathrm{TOP},s}(N)$ consists of precisely one element, namely the one given by id_N.

The surgery exact sequence in the smooth category from Theorem 11.22 is in general not an exact sequence of abelian groups, only of pointed sets, see Section 11.8.

Chapter 12: Homotopy Spheres

This chapter is devoted to the classification of oriented homotopy spheres up to oriented diffeomorphism, where a homotopy sphere is a smooth closed manifold that is homotopy equivalent to S^n. This boils down to calculating the structure set $\mathcal{S}(S^n)$ in the smooth category. The input is the geometric surgery exact sequence in the smooth category from Chapter 11, calculations of the L-groups in the simply connected case from Chapters 8 and 9, and homotopy theoretic results about the so-called J-homomorphisms, which shed light on the normal invariants. The maps in the sequence are determined by studying the signature and the Arf invariant of surgery problems.

Information gained by these calculations yields results about various classifying spaces, which are organised in the so-called Kervaire–Milnor braid, see Section 12.7.

Chapter 13: The Geometric Surgery Obstruction Group and Surgery Obstruction

This chapter is equivalent to the famous Chapter 9 in Wall's book [414]. We give a geometric approach to the L-groups and the surgery obstruction based on bordism theory. This is convenient in some situations where the necessary algebra is hard to analyse or not even available, such as controlled or equivariant surgery.

Chapter 14: Chain Complexes

This chapter has two goals. Firstly, we summarise the sign conventions that we use in the subsequent chapter about algebraic surgery and in fact throughout the book. These have been well thought through, and we hope that they will become standard. Unfortunately, in the literature many different sign conventions are used. The second goal of this chapter is to review some basic homotopy theory of chain complexes. Both of these topics provide background for the next chapter.

Chapter 15: Algebraic Surgery

We introduce a chain complex version of the L-groups and of the surgery obstructions and identify them with the L-groups and surgery obstructions from Chapters 8 and 9, see Theorem 15.3 and Theorem 15.4. So the chapter contains a presentation of Ranicki's theory of algebraic surgery where forms and formations are uniformly generalised to algebraic Poincaré chain complexes and their algebraic cobordism theory. One drawback of Ranicki's presentation in the original sources is that he very often describes certain constructions, such as algebraic surgery, only by writing down formulas without giving any structural insight. In our exposition we try to give certain general chain complex constructions that shall motivate the outcome and lead finally to explicit formulas. Moreover, we always use our sign conventions whereas Ranicki uses different sign conventions in different papers.

Chapter 16: Brief Survey of Computations of L-Groups

We give a brief survey of computations of L-group of group rings $\mathbb{Z}\pi$. For finite π the calculations were mostly done in the previous century and involve using representation theory and number theory. For infinite π nowadays the main tool is the Farrell–Jones Conjecture, which will be extensively treated in the book in preparation [261].

Chapter 17: The Homotopy Type of G/TOP, G/PL and G/O

We review how to determine the homotopy types of the classifying spaces G/PL and G/TOP, see Section 11.9 and Theorem 17.6. This leads to the computation of the set of normal invariants in the topological category in terms of singular cohomology after localising at 2 and in terms of KO-theory after inverting 2, see (11.41) and Theorem 11.24.

Chapter 18: Computations of Topological Structure Sets of some Prominent Closed Manifolds

We discuss how surgery theory and in particular the surgery exact sequence lead to computations of topological structure sets. We treat products of spheres, complex and real projective spaces, lens spaces, and tori. This leads to the classification of closed topological manifolds that are homotopy equivalent to these spaces up to homeomorphism. In particular we completely solve Problem 1.5. We have treated this problem for the standard sphere in the smooth category already in Chapter 12 about homotopy spheres.

Chapter 19: Topological Rigidity

A closed topological manifold N is called topologically rigid if any homotopy equivalence $M \to N$ with a closed topological manifold M as source and N as target is homotopic to a homeomorphism. In this chapter we want to study the question of which closed topological manifolds are topologically rigid. Section 19.4 is devoted to the Borel Conjecture predicting that any aspherical closed manifold is topologically rigid. Examples of non-aspherical closed manifolds that are topologically rigid are discussed in Section 19.5.

In Section 19.6 we briefly discuss the rarity of smooth rigidity in high dimensions.

Chapter 20: Modified Surgery

In this chapter we digress from the main line of the book, which treats the classification of manifolds with a given homotopy type via classical surgery, and discuss aspects of the use of surgery theory to classify manifolds with less homotopy theoretic input. Specifically we discuss variations of the Surgery Program, see Remark 2.9, which were pioneered by Kreck [225] and are often called modified surgery. Modified surgery might not have the general structural impact as surgery has on the classification of manifolds, on prominent conjectures such as those of Borel or Novikov, or on index theory and C^*-algebras, but leads in some special but very interesting cases to better and beautiful results, for instance for complete intersections, homogeneous spaces, and 4-manifolds.

1.3 Outlook

Here is a (not necessarily complete) list (in alphabetical order) of topics that we were not able to treat in this book in detail or at all, but which are very interesting. Some of them could be part of sequels to this book (not necessarily written by the authors of this book). For some items we include references where these topics have already been addressed and where further references can be found.

- Algebraic surgery in the setting of ∞-categories and the relation of algebraic surgery to hermitian K-theory, see [69, 70, 71, 271];
- ANR-homology manifolds and the Quinn obstruction, see [67, 338, 339];

- Applications of surgery theory to knot theory, see [245, 351];
- Applications to questions from differential geometry, in particular to the existence of Riemannian metrics with positive scalar curvature, see [362];
- Automorphism groups of closed manifolds, see [430];
- Computations of the L-groups of group rings of finite groups, see [177];
- Controlled surgery theory [328, 330];
- Equivariant surgery theory [76, 136, 137, 138, 139, 262, 263, 335];
- Finite H-spaces, see [6, 34];
- Full presentation of the proof of the Spherical Space Form Problem, see [120, 277];
- Mapping surgery to analysis, see [184, 185, 186, 440];
- Parametrised surgery, see [163, 162, 195];
- Poincaré surgery, Poincaré embeddings, and LS-groups, see [179],[221], [414, Chapter 11];
- Stratified surgery theory; see [424];
- Surgery in dimension 4, see [37, 159];
- Surgery in the topological category, see [219];
- The Novikov Conjecture, see [153, 154, 226];
- The total surgery obstruction, see [234, 343, 348];
- UNil-terms and splitting obstructions, see [77, 78, 79, 104].

1.4 How to Use this Book

As mentioned before, the potential readers may vary from established experts on surgery theory to advanced students without any previous knowledge about surgery theory. Obviously the various groups of readers have rather different expectations and needs. On the one hand we want to give correct and complete definitions, theorems and proofs, but we also want to allow the reader to browse through the text and get a first impression or a global picture. This leads of course to some tension that we tried to solve as explained below.

A typical example is the notion of a normal map and normal bordism. The definition is quite lengthy, see for instance Definition 7.13 and Definition 7.15. This is actually necessary, as none of the items occurring there can be dropped when one wants to set up the theory and give accurate proofs in full generality. But when one is working or thinking about a problem or wants to get a first impression, one should work with an extract of these definitions as explained for instance in Section 7.2. It can also be useful to make simplifying assumptions, for instance, that all manifolds are orientable, or, equivalently, that the orientation homomorphism $w\colon \pi \to \{\pm 1\}$ is trivial, or even that every manifold is simply connected. Then one can ignore the local coefficient systems and work with ordinary homology, and one does not have to deal with group rings but only with the ring $\mathbb{Z}$ of integers. In daily life one may get as far as to say that a normal map of degree one is a map $f\colon M \to X$ of degree one with connected orientable source and target covered by bundle data, without really memorising what the bundle data are.

Another example of how one can use the book in different ways concerns Chapter 2 on the s-Cobordism Theorem. The minimal approach is to go through the introduction and ignore the rest; that suffices completely to go on with the book. Or one may want to get the full proof and therefore go through all the material of Chapter 2 and at least parts of Chapter 3. This is explained in the Guide of Chapter 2.

The rather long Chapter 15 on algebraic surgery can also be read in rather different ways. One may ignore all the motivations and structural explanations, just concentrate on the formulas, completely leave out the proofs, and just browse through the definitions and main theorems. Or one may want to understand all the details and get an insight into why the definitions and proofs are set up as they stand, and therefore read everything. Again here a reader should first go through the introduction and the guide at its end (and then through the Overview given in Section 15.2), before she or he decides which parts of the chapter she or he wants to read in which reading modus.

One can apply surgery theory in the smooth, PL or topological category. In other words one may consider smooth compact manifolds and try to classify them up to diffeomorphism, compact PL manifolds and try to classify them up to PL homeomorphism, or compact topological manifolds and try to classify them up to homeomorphism. When we explain some technical constructions, such as the surgery step, the bundle data, and so on, we will work in the smooth category since there all the notions such as tangent bundle, normal bundle, transversality and so on are well documented. All this carries over to the PL category and the topological category. We will not go into the sophisticated details of how this can be done since it would go beyond the scope of this book. For the topological category the seminal work of Kirby–Siebenmann [219] is needed. A good reference for the PL category is Rourke–Sanderson [367]. So basic tools such as the surgery exact sequence do exist in all three categories.

The classification results do of course depend on whether we are working in the smooth, PL, or topological category. It turns out that the nicest results occur in the topological category. The reader will have to accept the fact that we develop surgery theory in detail only in the smooth category, but will also apply it to the topological category without further explanations.

Very often we will make the assumption that the dimensions of the manifolds under consideration are greater than or equal to 5. The problem is that the so-called Whitney trick applies only under this dimension assumption. The problem with the Whitney trick can be solved and hence surgery can also be carried out in dimension 4, provided that we work in the topological category and the fundamental group π is good in the sense of Freedman, see Remark 8.30. All of the results presented in this book with the dimension condition ≥ 5 extend to dimension 4 in the topological category if the fundamental group is good. The reader has to live with the fact that we do not explain what is behind these ideas of Freedman, but refer for instance to [37, 159].

The book contains a number of exercises. They come in two flavours. A few of them contain additional information or a computation that may be needed later. Most of them are not needed for the exposition of the book, but give some illuminating insight. Moreover, the reader may test whether she or he has understood the text, or improve her or his understanding by trying to solve the exercises. Note that hints to the solutions of the exercises are given in Chapter 21.

Readers wishing to find a specific topic are advised to first look at the Overview of the Contents of this Book in Section 1.2, in order to find the right chapter and then that chapter's introduction. Each introduction to a chapter concludes with a guide, which may help the reader to figure out how to access the contents of that chapter. The extensive index at the end of the book can also be used to find the right spot for a specific topic. The index contains an item Theorem, under which all theorems with their names appearing in the book are listed, and analogously there is an item Conjecture.

We have successfully used parts of this book for seminars, reading courses, and advanced courses for students.

The reader may also consult other monographs on surgery theory such as [55, 93, 252, 352, 414]. Further surveys article or more information can be found for instance in [72, 73, 145, 146, 153, 154, 219, 226, 348, 424, 425].

1.5 Prerequisites

We require that the reader is familiar with the basics of the following concepts and notions. Readers can learn these topics from the suggested references, but there are many more books and monographs available.

- CW-complexes, Cellular Approximation Theorem, Whitehead Theorem, see [45, 178, 399];
- Covering theory, universal coverings, see [45, 178, 399];
- Homology, cohomology, cup and cap-product, signature, characteristic classes [45, 178, 198, 308, 399];
- Homotopy groups, fibrations, cofibrations, Hurewicz Theorem, see [45, 178, 399];
- Topological and smooth manifolds, see [45, 49, 224, 237, 241];
- Vector bundles, normal and tangent bundle of a smooth manifold, see [45, 49, 189, 224, 308];
- Classifying spaces for groups and for vector bundles, fibre bundles, and fibrations, see [198, 279, 289, 308];
- Transversality, regular values, immersions, and submersions, see [45, 49, 189, 224];
- Group rings, modules and chain complexes over a non-commutative ring, see [59, 238, 327, 421];
- Bordism of manifolds, bordism ring, see [189, 308].

1.6 Acknowledgement

The origin of this book dates back to 2001 when the Summer school on Topology of high-dimensional manifolds was held at ICTP in Trieste. Wolfgang Lück gave a series of lectures and the resulting notes [252] were published as *A basic introduction to surgery theory* in the Proceedings of the school.

However, the idea that the notes could be expanded into a book that would provide a more comprehensive tour of the foundations of the subject persisted. In 2011 Wolfgang Lück approached Diarmuid Crowley and Tibor Macko with an offer to contribute to this endeavour. At the time all three were based in Bonn, and the two younger mathematicians joined the project.

After his moves to Aberdeen and later to Melbourne, Diarmuid Crowley left the team working on the book. The authors are grateful to him for his contributions.

Diarmuid Crowley was scientifically involved with the material covered in Chapters 4 to 9, and 11 and 12, and specifically with versions of these chapters prior to the introduction of the intrinsic fundamental class. His involvement was most significant in Chapters 6, 8, and 9. He also contributed material which persists in Chapters 4, 11 and 12. Specific contributions of his that we would like to mention are:

- Material on the construction of Poincaré complexes without vector bundle reduction in Section 6.6 and the proof of the uniqueness of the Spivak normal fibration covered in Sections 6.9 and 6.10;
- Material on the groups $L_{2k}(\mathbb{Z})$ in Sections 8.5.2 and 8.5.3, the role of the bundle data in the surgery obstruction for highly connected normal maps in Section 8.7.7 and the discussion of surgery kernels for maps of pairs and triads in Sections 8.6.2, 8.8.2, and 8.8.4;
- Chapter 9, in particular the proof that $L_{4k+1}(\mathbb{Z}) = 0$ in Section 9.2.4;
- Material on the spaces G/O, G/PL and G/TOP in Section 11.9 and on the algebraic properties of surgery exact sequences in Section 11.8.

His insights, through our many conversations over time, have contributed to our understanding of the subject and many of its subtleties.

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Chapter 2
The s-Cobordism Theorem

2.1 Introduction

In this chapter we want to discuss and prove the following theorem (in the smooth category).

Theorem 2.1 (s-Cobordism Theorem) *Let M_0 be a closed connected smooth manifold with* $\dim(M_0) \geq 5$ *and fundamental group* $\pi = \pi_1(M_0)$. *Then:*

(i) *Let* $(W; M_0, f_0, M_1, f_1)$ *be a smooth h-cobordism over* M_0. *Then W is trivial over* M_0 *if and only if its Whitehead torsion* $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ *vanishes;*
(ii) *For any* $x \in \mathrm{Wh}(\pi)$ *there is a smooth h-cobordism* $(W; M_0, f_0, M_1, f_1)$ *over* M_0 *with* $\tau(W, M_0) = x \in \mathrm{Wh}(\pi)$*;*
(iii) *The function assigning to a smooth h-cobordism* $(W; M_0, f_0, M_1, f_1)$ *over* M_0 *its Whitehead torsion yields a bijection from the diffeomorphism classes relative* M_0 *of smooth h-cobordisms over* M_0 *to the Whitehead group* $\mathrm{Wh}(\pi)$.

The analogous statements hold in the PL *category and in the topological category.*

Here are some explanations. In the sequel we work in the smooth category unless explicitly stated otherwise. An *n-dimensional cobordism* (sometimes also just called a bordism) $(W; M_0, f_0, M_1, f_1)$ consists of a compact n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0 and M_1, a disjoint decomposition $\partial W = \partial_0 W \coprod \partial_1 W$ of the boundary ∂W of W, and diffeomorphisms $f_0 \colon M_0 \to \partial_0 W$ and $f_1 \colon M_1 \to \partial_1 W$. If we want to specify M_0, we say that W is a *cobordism over* M_0. If $\partial_0 W = M_0$, $\partial_1 W = M_1$ and f_0 and f_1 are given by the identity or if f_0 and f_1 are obvious from the context, we briefly write $(W; \partial_0 W, \partial_1 W)$. Note that the choices of the diffeomorphisms f_i do play a role, although they are often suppressed in the notation. Two cobordisms $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0, f_0', M_1', f_1')$ over M_0 are *diffeomorphic relative* M_0 if there is a diffeomorphism $F \colon W \to W'$ with $F \circ f_0 = f_0'$. We call a cobordism $(W; M_0, f_0, M_1, f_1)$ an *h-cobordism* if the inclusions $\partial_i W \to W$ for $i = 0, 1$ are homotopy equivalences. We call an h-cobordism over M_0 *trivial* if it is diffeomorphic relative M_0 to the trivial h-cobordism $(M_0 \times [0,1]; M_0 \times \{0\}, M_0 \times \{1\})$. We will discuss the Whitehead group in Sections 2.5 and 3.2.

W. Lück, T. Macko, *Surgery Theory*, Grundlehren der mathematischen Wissenschaften 362,
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