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Towards Ulam Type Multi Stability Analysis

A Novel Approach for Fuzzy Dynamical Systems

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To our family

Preface

The main target of this monograph is to present a new concept of Ulam type stability, i.e., Multi Stability, through the classical, well-known special functions and to obtain the best approximation error estimates by a different concept of perturbation stability including the fuzzy approach for uncertainty considerations. This stability allows us to obtain diverse approximations depending on various special functions that are initially chosen and to evaluate maximal stability and minimal error which enable us to obtain a unique optimal solution of functional equations, inequalities, and fractional equations. Stability analysis in the sense of the Ulam and its different kinds has received considerable attention from the researchers. However, how to effectively generalize the Ulam stability problems and to evaluate optimized controllability and stability are new issues. The multi stability not only covers the previous concepts but also considers the optimization of the problem and provides a comprehensive discussion of optimizing the different types of the Ulam stabilities of mathematical models used in the natural sciences and engineering disciplines with the fuzzy attitude.

Besides, this book also deals with nonlinear differential equations with various boundary conditions or initial value problems, based on the matrix Mittag–Leffler function, fixed point theory, as well as Babenko's approach to study uniqueness and existence of solutions.

In general, the benefits for the readers can be concluded as follows:

1. Evaluates maximal stability with minimal error to get a unique optimal solution.
2. Discusses an optimal method of the alternative to study existence, uniqueness, and different types of Ulam stabilities under special consideration of the fuzzy approach.

3. Delves into the new study of boundary value problems of fractional integro-differential equations with integral boundary conditions and variable coefficients.

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Contents

Acronyms

$\mathbb{R}$	The real numbers
$\mathbb{C}$	The complex numbers
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integrals
$\mathbb{R}_+$	Positive real numbers
$\mathbb{R}_-$	Negative real numbers
$\mathbb{Z}^+$	Positive integer numbers
$\mathbb{Z}^-$	Negative integer numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}_0$	$\mathbb{Z} \cup \{0\}$
$\Gamma(.)$	Gamma function
$\underbrace{\varphi_i}_{1 \le i \le n}$	Special control function
$\underbrace{\mathrm{AG}_i}_{1 \le i \le n}$	Aggregation maps
$\bigotimes_{\mathrm{TN}}$	Triangular norm
$\bigotimes_{\mathrm{GTN}}$	Generalized triangular norm
t-norm	Triangular norm
MVFN space	Matrix valued fuzzy normed space
MVFB space	Matrix valued fuzzy Banach space
MVFN-algebra	Matrix valued fuzzy normed algebra
MVFB-algebra	Matrix valued fuzzy Banach algebra
MVFC–$\diamond$–algebra	Matrix valued fuzzy C–$\diamond$–algebra
MVFB–$\diamond$–algebra	Matrix valued fuzzy Banach–$\diamond$–algebra

Chapter 1
Introduction

The study of functional equations has a long history. In 1791 and 1809, Legendre [1] and Gauss [2] attempted to provide a solution of the following functional equation:

$$f(x + y) = f(x) + f(y),$$

for all $x, y \in \mathbb{R}$, which is called the Cauchy functional equation. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called an additive function if it satisfies the Cauchy functional equation. In 1821, Cauchy [3] first found the general solution of the Cauchy functional equation, that is, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous additive function, then f is linear, that is, $f(x) = mx$, where m is a constant. Further, we can consider the biadditive function on $\mathbb{R}^2$ as follows:

A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called an biadditive function if it is additive in each variable, that is,

$$f(x + y, z) = f(x, z) + f(y, z),$$

and

$$f(x, y + z) = f(x, y) + f(x, z),$$

for all $x, y, z \in \mathbb{R}$. It is well known that every continuous biadditive function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is of the form

$$f(x, y) = mxy,$$

for all $x, y \in \mathbb{R}$, where m is a constant.

Since the time of Legendre and Gauss, several mathematicians had dealt with additive functional equations in their books [4–8] and a number of them have studied Lagrange's mean value theorem and related functional equations, Pompeiu's mean value theorem and associated functional equations, two-dimensional mean value theorem and functional equations as well as several kinds of functional equations.

In 1940, S.M. Ulam [9] proposed the following stability problem of functional equations:

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Given a group G_1, a metric group G_2 with the metric $d(.,.)$ and a positive number ε, does there exist $\delta > 0$ such that, if a mapping $f : G_1 \longrightarrow G_2$ satisfies

$$d\Big(f(xy), f(x)f(y)\Big) \leq \delta,$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \longrightarrow G_2$ exists with

$$d(f(x), h(x)) \leq \varepsilon,$$

for all $x \in G_1$?

Since then, several mathematicians have dealt with special cases as well as generalizations of Ulam's problem. In fact, in 1941, D.H. Hyers [10] provided a partial solution to Ulam's problem for the case of approximately additive mappings in which G_1 and G_2 are Banach spaces with $\delta = \varepsilon$ as follows:

Let X and Y be Banach spaces and let $\varepsilon > 0$. Then, for all $g : X \longrightarrow Y$ with

$$\sup_{x,y\in X} \Big\| g(x+y) - g(x) - g(y) \Big\| \leq \varepsilon,$$

there exists a unique mapping $f : X \longrightarrow Y$ such that

$$\sup_{x\in X} \|g(x) - f(x)\| \leq \varepsilon,$$
$$f(x+y) = f(x) + f(y),$$

for all $x, y \in X$.

This proof remains unchanged if G_1 is an Abelian semigroup. Particularly, in 1968, the following theorem was proved by Forti (Proposition 1, [11]):

Theorem 1.1 *Let $(S, +)$ be an arbitrary semigroup and E be a Banach space. Assume that $f : S \longrightarrow E$ satisfies*

$$\Big\| f(x, y) - f(x) - f(y) \Big\| \leq \varepsilon. \tag{1.1}$$

Then, the limit

$$g(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{2^n}, \tag{1.2}$$

exists for all $x \in S$ and $g : S \longrightarrow E$ is the unique function satisfying

$$\| f(x) - g(x)\| \leq \varepsilon, \quad g(2x) = 2g(x).$$

Finally, if the semigroup S is Abelian, then G is additive.

Note that the proof method generating the solution g by the formula like (1.2) is called a direct method.

If f is a mapping of a group or a semigroup $(S, .)$ into a vector space E, then we call the following expression:

$$Cf(x, y) = f(x.y) - f(x) - f(y),$$

the Cauchy difference of f on $S \times S$. In the case that E is a topological vector space, we call the equation of homomorphism stable if, whenever the Cauchy difference Cf is bounded on $S \times S$, there exists a homomorphism $g : S \longrightarrow E$ such that $f - g$ is bounded on S.

In 1980, Rätz [12] generalized Theorem 1.1 as follows: Let $(X, *)$ be a power associative groupoid, that is, X is a nonempty set with a binary relation $x_1 * x_2 \in X$ such that the left powers satisfy $x^{m+n} = x^m * x^n$ for all $m, n \geq 1$ and $x \in X$. Let $(Y, |\,.\,|)$ be a topological vector space over the field $\mathbb{Q}$ of rational numbers with $\mathbb{Q}$ topologized by its usual absolute value $|\,.\,|$.

Theorem 1.2 *Let V be a nonempty bounded Q-convex subset of Y containing the origin and assume that Y is sequentially complete. Let $f : X \longrightarrow Y$ satisfy the following conditions: for all $x_1, x_2 \in X$, there exist $k \geq 2$ such that*

$$f\left((x_1 * x_2)^{k^n}\right) = f\left(x_1^{k^n} * x_2^{k^n}\right), \tag{1.3}$$

for all $n \geq 1$ and

$$f(x_1) + f(x_2) - f(x_1 * x_2) \in V. \tag{1.4}$$

Then there exists a function $g : X \to Y$ such that $g(x_1) * g(x_2)$ and $f(x) - g(x) \in \overline{V}$, where $\overline{V}$ is the sequential closure of V for all $x \in X$. When Y is a Hausdorff space, then g is uniquely determined.

Note that the condition (1.3) is satisfied when X is commutative and it takes the place of the commutativity in proving the additivity of g. However, as Rätz pointed out in his paper, the condition

$$(x_1 * x_2)^{k^n} = x_1^{k^n} * x_2^{k^n},$$

for all $x_1, x_2 \in X$, where X is a semigroup, and, for all $k \geq 1$, does not imply the commutativity.

In the proofs of Theorems 1.1 and 1.2, the completeness of the image space E and the sequential completeness of Y, respectively, were essential in proving the existence of the limit which defined the additive function g. The question arises whether the completeness is necessary for the existence of an odd additive function g such that $f - g$ is uniformly bounded, given that the Cauchy difference is bounded.

For this problem, in 1988, Schwaiger [13] proved the following:

Theorem 1.3 *Let E be a normed space with the property that, for each function $f : \mathbb{Z} \to E$, whose Cauchy difference $Cf = f(x + y) - f(x) - f(y)$ is bounded for all $x, y \in \mathbb{Z}$ and there exists an additive mapping $g : \mathbb{Z} \to E$, such that $f(x) - g(x)$ is bounded for all $x \in \mathbb{Z}$. Then E is complete.*

Corollary 1.1 *The statement of Theorem 1.3 remains true if $\mathbb{Z}$ is replaced by any vector space over $\mathbb{Q}$.*

In 1950, Aoki [14] generalized Hyers' theorem as follows:

Theorem 1.4 *Let E_1 and E_2 be two Banach spaces. If there exist $K > 0$ and $0 \leq p < 1$ such that*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq K\left(\|x\|^p + \|y\|^p \right),$$

for all $x, y \in E_1$, then there exists a unique additive mapping $g : E_1 \longrightarrow E_2$ such that

$$\|f(x) - g(x)\| \leq \frac{2K}{2 - 2^p} \|X\|^p,$$

for all $x \in E_1$.

In 1978, Rassias [15] formulated and proved the stability theorem for the linear mapping between Banach spaces E_1 and E_2 subject to the continuity of $f(tx)$ with respect to $t \in \mathbb{R}$ for each fixed $x \in E_1$. Thus, Rassias' theorem implies Aoki's theorem as a special case. Later, in 1990, Rassias [16] observed that the proof of his stability theorem also holds true for $p < 0$. In 1991, Gajda [17] showed that the proof of Rassias' theorem can be proved also for the case $p > 1$ by just replacing n by $-n$ in (1.2). These results are stated in a generalized form as follows (see Rassias and Šemrl [18]):

Theorem 1.5 *Let $\beta(s, t)$ be nonnegative for all nonnegative real numbers s, t and positive homogeneous of degree p, where p is real and $p \neq 1$, that is, $\beta(\lambda s, \lambda t) = \lambda^p \beta(s, t)$, for all nonnegative λ, s, t. Given a normed space E_1 and a Banach space E_2, assume that $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \beta(\|x\|, \|y\|),$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $g : E_1 \longrightarrow E_2$ such that

$$\|f(x) - g(x)\| \leq \delta \|x\|^p,$$

for all $x \in E_1$, where

$$\delta := \begin{cases} \dfrac{\beta(1,1)}{2-2^p}, & p < 1, \\ \dfrac{\beta(1,1)}{2-2^p}, & p > 1. \end{cases} \tag{1.5}$$

The proofs for the cases $p < 1$ and $p > 1$ were provided by applying the direct methods. For $p < 1$, the additive mapping g is given by (1.2), while in case $p > 1$ the formula is

$$g(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

Corollary 1.2 *Let $f : E_1 \longrightarrow E_2$ be a mapping satisfying the hypotheses of Theorem 1.5 and suppose that f is continuous at a single point $y \in E_1$, then the additive mapping g is continuous.*

Corollary 1.3 *If, under the hypotheses of Theorem 1.5, we assume that, for each fixed $x \in E_1$, the mapping $t \longrightarrow f(tx)$ from $\mathbb{R}$ to E_2 is continuous, then the additive mapping g is linear.*

Remark 1.1 (1) For $p = 0$, Theorem 1.5, Corollaries 1.2 and 1.3 reduce to the results of Hyers in 1941. If we put $\beta(s, t) = \varepsilon(sp + tp)$, then we obtain the results of Rassias [15] in 1978 and Gajda [17] in 1991.

(2) The case $p = 1$ was excluded in Theorem 1.5. Simple counterexamples prove that one can not extend Rassias' Theorem when p takes the value one (see Z. Gajda [17], Rassias and Šemrl [18] and Hyers and Rassias [19] in 1992).

A further generalization of the Hyers-Ulam stability for a large class of mappings was obtained by Isac and Rassias [20] by introducing the following:

Definition 1.1 A mapping $f : E_1 \longrightarrow E_2$ is said to be ϕ-additive if there exist $\Phi \geq 0$ and a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\lim_{t\longrightarrow\infty} \frac{\phi(t)}{t} = 0,$$

such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \Phi[\phi(\|x\|) + \phi(\|y\|)],$$

for all $x, y \in E_1$.

In [20], Isac and Rassias proved the following:

Theorem 1.6 *Let E_1 be a real normed vector space and E_2 be a real Banach space. Let $f : E_1 \longrightarrow E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed $x \in E_1$. If f is ϕ-additive and ϕ satisfies the following conditions:*

(a) $\phi(ts) \leq \phi(t)\phi(s)$ *for all* $s, t \in \mathbb{R}$*;*

(b) $\phi(t) < t$ *for all* $t > 1$,

then there exists a unique linear mapping $T : E_1 \longrightarrow E_2$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\Phi}{2 - \phi(2)}\phi(\|x\|),$$

for all $x \in E_1$.

Remark 1.2 (1) If $\phi(t) = t^p$ with $p < 1$, then, from Theorem 1.6, we obtain Rassias' theorem [15].

(2) If $p < 0$ and $\phi(t) = t^p$ with $t > 0$, then Theorem 1.6 is implied by the result of Gajda in 1991.

In [21], Diaz and Margolis proved a "theorem of the alternative" for any "contraction mappin" T on a "generalized complete metric space" X. The conclusion of the theorem, speaking in general terms, asserts that: either all consecutive pairs of the sequence of successive approximations (starting from an element x_0 of X) are infinitely far apart, or the sequence of successive approximations, with initial element x_0 converges to a fixed point of T (what particular fixed point depends, in general, on the initial element x_0). The present theorem contains as special cases both Banach's contraction mapping theorem [22] for complete metric spaces, and Luxemburg's contraction mapping theorem [23] for generalized metric spaces.

Following Luxemburg [23], the concept of a "generalized complete metric spac" may be introduced as in this quotation: "Let X be an abstract (nonempty) set, the elements of which are denoted by $x, y, \ldots$ and assume that on the Cartesian product $X \times X$ a distance function $d(x, y)(0 \leq d(x, y) \leq \infty)$ is defined, satisfying the following conditions:

(D1) $d(x, y) = 0$ if and only if $x = y$,

(D2) d(x,y)=d(y,x) (symmetry),

(D3) $d(x, y) \leq d(x, z) + d(z, y)$, (triangle inequality),

(D4) every d-Cauchy sequence in X is d-convergent, i.e. $\lim_{n,m \longrightarrow \infty} d(x_n, x_m) = 0$, for a sequence $x_n \in X (n = 1, 2, \ldots)$ implies the existence of an element $x \in X$ with $\lim_{n \longrightarrow \infty} d(x, x_n) = 0$, ($x$ is unique by (D1) and (D3)).

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space".

Using this notion, one has the following:

Theorem 1.7 ([21]) *Suppose that* (X, d) *is a generalized complete metric space, and that the function* $T : X \longrightarrow X$ *is a "contraction," that is,* T *satisfies the condition: There exists a constant* q*, with* $0 < q < 1$ *, such that whenever* $d(x, y) < \infty$ *one has*

$$d(Tx, Ty) \leq q d(x, y).$$

Let $x_0 \in X$*, and consider the "sequence of successive approximations with initial element* x_0*":* $x_0, Tx_0, T^2x_0, \ldots, T^lx_0, \ldots$ *. Then, the following alternative holds: either*

(A) for every integer $l = 0, 1, 2, \ldots$, *one has*

$$d(T^l x_0, T^{l+1} x_0) = \infty, \quad or$$

(B) the sequence of successive approximations $x_0, Tx_0, T^2x_0, \ldots, T^l x_0, \ldots,$ *is* d*-convergent to a fixed point of* T.

In [24, 25], Cadariu and Radu and then Radu and Mihet presented the Cadariu-Radu theory (for classical spaces) and the Radu-Mihet theory (for fuzzy spaces) derived from the Diaz-Margolis theorem, respectively, as follows:

Theorem 1.8 *Let* $x, y \in X$. *Assume the complete* $[0, \infty]$*-valued metric* d *on* X *and strictly contractive function* T *on* X *with* $d(Tx, Ty) \leq qd(x, y)$, *where* $q < 1$. *If we obtain a* $l_0 \in \mathbb{N}$ *s.t.* $d(T^l x, T^{l+1} x) < \infty$, *for any* $l \geq l_0$, *therefore we get the following:*

- *the fixed point* y^* *of* T *is the convergence point of* $\{T^l x\}$;
- *in* $\{y \in X \mid d(T^{\rho_0} x, y) < \infty\}$, y^* *is the unique fixed point of* T;
- $(1 - q)d(y, y^*) \leq d(y, Ty)$ *for every* $y \in X$.

Since the time the above stated results were proven, several mathematicians have extensively studied stability theorems for several kinds of functional equations in various spaces, for example, Banach spaces, 2-Banach spaces, Banach n-Lie algebras, quasi-Banach spaces, Banach ternary algebras, non-Archimedean normed and Banach spaces, metric and ultra metric spaces, Menger probabilistic normed spaces, probabilistic normed space, p-2-normed spaces, C-*-algebras, C-*-ternary algebras, Banach ternary algebras, Banach modules, inner product spaces, Heisenberg groups, random normed spaces, fuzzy normed space and others. Further, researchers focused on the applications of the Hyers-Ulam-Rassias stability problems, for example, (partial) differential equations, fractional differential and integral equations, Volterra integral equations, group and ring theory, mathematical biology modeling, bending beam problems of mechanical engineering also, some kind of models in population dynamics, and some kinds of equations [26–31].

As mentioned at the beginning, the primary target of this monograph is to provide a new interpretation of the Ulam type stability, i.e., multi stability, with the application of classical, well–known special functions. This stability facilitates us to obtain diverse estimations based on the various special functions that are initially selected and to estimate optimal stability with minimal error which provides a unique optimized solution (see [32–52]).

The monograph is divided into 21 chapters:

Chapters 2–8 present a background to the classical well-known special functions which play an important role in mathematical physics, especially in boundary value and initial condition problems of differential equations. Generally speaking, we call a function "special" when the function, just as logarithmic, exponential, and trigonometric functions (the elementary transcendental functions), belongs to the toolbox of applied mathematicians, physicists or engineers. Usually there are a particular standardized notation, and a number of known properties of the function. This branch of mathematics has a respectable history with great names such as Gauss, Euler, Fourier, Legendre, Mittag, Leffler, Bessel, and Riemann. They all made good contributions to the area. A great part of their work was inspired by physics and driven by differential equations. About 70 years ago, these activities were summarized in the standard work "A Course of Modern Analysis" by Whittaker and Watson, which has had great influence and is still important nowadays. Many special functions appear as solutions of differential equations or integrals of elementary functions. Therefore, tables of integrals usually include descriptions of special functions, and tables of special functions include most important integrals and the integral representation of special functions. The main target of these chapters are to provide the detailed investigations to several newly established special functions involving the Euler gamma function, Pochhammer symbols, Gaussian hypergeometric series, Clausen hypergeometric series, supertrigonometric and superhyperbolic functions via the hypergeometric function, the Wright function, Wright's generalized hypergeometric function, supertrigonometric and superhyperbolic functions via the Wright function, Wright's generalized hypergeometric function, Mittag-Leffler function, supertrigonometric functions and superhyperbolic functions via the Mittag-Leffler function, the truncated the Mittag-Leffler function, Wiman function, supertrigonometric functions and superhyperbolic functions via the Wiman function, the truncated Wiman functions, Prabhakar function, the supertrigonometric and superhyperbolic functions via the Prabhakar function, the truncated Prabhakar functions, and so on.

In Chap. 9, the material can be formally divided into two main parts, which are discussed as follows. At first, we recall some definitions and results which will be used later on in the book. Then, starting from a novel view on the stability problem in the sense of the Ulam, we define a new concept of the multi stability to provide a comprehensive discussion of optimizing the different types of the Ulam stabilities.

In Chaps. 10–14, we use Radu's approach derived from the theorem of Diaz and Margolis to study existence, uniqueness and the multi stability results of mathematical equations in classical spaces.

In Chap. 15, we introduce basic and standard properties often required for fuzzy spaces.

In Chaps. 16–21, we consider both functional and fractional equations containing fuzzy uncertainties and prove their multi stability via the fixed point theory in diverse fuzzy normed spaces.

Chapter 22 is comprised of seven independent and self–contained sections which deal with nonlinear differential equations with various boundary conditions or initial value problems, based on the matrix Mittag–Leffler function, fixed point theory, as well as Babenko's approach.

Throughout the book, we let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of the complex numbers, real numbers, integrals, and natural numbers, respectively. Let $\mathbb{Z}^+$, $\mathbb{R}_+$, $\mathbb{Z}^-$ and $\mathbb{R}_-$, be the sets of the positive integers, positive real numbers, negative integer numbers, and negative real numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup 0$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup 0$. Finally, $\Re(\nu)$ denotes the real part of ν if $\nu \in \mathbb{C}$.

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Chapter 2
The Hypergeometric, Supertrigonometric, and Superhyperbolic Functions

In this chapter, we introduce the Euler gamma function, the Pochhammer symbols, the Gaussian hypergeometric series, the Clausen hypergeometric series and the Supertrigonometric, and Superhyperbolic functions via Gaussian hypergeometric series and Clausen hypergeometric series.

2.1 The Euler Gamma Function and the Pochhammer Symbols

In this section, we present the Euler gamma function and the Pochhammer symbols. We begin with the definition of the gamma function.

Definition 2.1 ([1, 2]) The gamma function first defined by Euler is given by

$$\Gamma(X) = \int_0^\infty e^{-Y} Y^{X-1} dY,$$

where $\Re(X) > 0$ and $X \in \mathbb{C}$.

Definition 2.2 ([1]) Let $\Re(X) > 0$ and $X \in \mathbb{C}$. Then the Euler gamma function satisfies

$$\Gamma(X+1) = X\Gamma(X).$$

The result was discovered by Euler in 1729 [3] and reported by Weierstrass [4], Brunel [5], Gronwall [6], and Olver [7].

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