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Manuel J. Castro
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Hyperbolic Problems: Theory, Numerics, Applications. Volume II

HYP2022, Málaga, Spain, June 20–24, 2022

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To the memory of Professor Antonio VALLE SÁNCHEZ (1930–2012), founder of the EDANYA group.

Preface

The present volume contains selected papers issuing from the *XVIII International Conference on Hyperbolic Problems: Theory, Numerics, and Applications* (HYP2022) which was held in June 20–24, 2022 in Málaga (Spain). The conference proceedings have been divided into two volumes: this one collects some of the contributions focusing on numerical aspects and applications.

This series of conferences constitute an international event of reference in the field of Hyperbolic Partial Differential Equations. Their objective is to bring together scientists with interests in the theoretical, applied, and computational aspects of hyperbolic partial differential equations (systems of hyperbolic conservation laws, wave equations, etc.) and of related mathematical models (PDEs of mixed type, kinetic equations, nonlocal or/and discrete models, etc.). The first conference was held in 1986 in St. Etienne (France) and has been organized since then biennially at different locations. The last few meetings were held at 2018 Penn State (USA), 2016 Aachen (Germany), 2014 Rio de Janeiro (Brazil), 2012 Padua (Italy), 2010 Beijing (China), 2008 College Park (USA), 2006 Lyon (France).

The eighteenth edition of this series of conferences should have been held in June 2020 in Málaga, but the situation due to the COVID-19 pandemic led the Organizing and Scientific Committees to postpone the Conference. In order to avoid a 4-year period without any activity related to the HYP series, an online activity, the *HYP2020/21 day*, took place in July 2, 2021. This event included the talks by the first *Peter Lax Awardee*, Jacob Bedrossian (University of Maryland), and the first *James Glimm Lecturer*, Constantine Dafermos (Brown University). These special lectures, which will be part of the program in every future edition of the HYP series, were instituted by the Scientific Committee to distinguish, respectively, a young researcher (at most 10 years after the Ph.D.) and a senior one for their contributions to the field of hyperbolic PDEs. The names of these two distinguished talks honor the fundamental ideas and contributions of two outstanding researchers, Peter Lax and James Glimm, who were present at the HYP2020/21 day, what makes this event unforgettable for all the attendees. The program of the HYP2020/21 day was completed with two talks given by Min Tang (Shanghai Jiaotong University) and Manuel J. Castro (University of Málaga).

The second Peter Lax Awardee and James Glimm Lecturer were Maria Colombo (EPFL, Switzerland) and Benoît Perthame (Sorbonne-Université, France), respectively. Besides their distinguished lectures, the program of HYP2022 included five plenary talks by speakers Eduard Feireisl (Inst. Math. Prague, Czech Republic), Jan S. Hesthaven (EPFL Lausanne, Switzerland), Denis Serre (ENS Lyon, France), Eleuterio F. Toro (U. Trento, Italy), and Tong Yang (Hong Kong PolyU). It also included eight invited talks by Benjamin Gess (U. Bielefeld, Germany), Kenneth H. Karlsen (U. Oslo, Norway), Qin Li (U. Wisconsin-Madison, USA), Raphaël Loubère (U. Bordeaux, France), Giovanni Russo (U. Catania, Italy), Konstantina Trivisa (U. Maryland, USA), Emil Wiedemann (U. Ulm, Germany), and Yao Yao (Georgia Tech., USA). Finally, 190 contributed talks (66 of them by Ph.D. students) were given and 19 posters were presented. Despite mobility restrictions in place in June 2022 due to the pandemic and international conflicts, which made it impossible for many colleagues to travel to Málaga, 287 researchers from 27 different countries attended the conference.

One of the main goals of HYP2022 was to promote the attendance of Ph.D. students, many of whom had never before had the opportunity of attending an international conference in person due to COVID-19. This goal was largely achieved: 86 attendees were Ph.D. students. Among the measures taken to stimulate their participation, more than 30 grants that covered the registration and accommodation fees were given (with priority for female students) and a recognition to the best presentations by Ph.D. students in the three fields of the Conference, the Springer Awards, was given. The awardees in the fields Theory, Numerics, and Applications were, respectively, William Golding (University of Texas at Austin, USA), Alessia del Grosso (Université de Versailles Saint-Quentin-en-Yvelines, France), and Kathrin Hellmuth (University of WÃ¼rzburg, Germany). The awardees received a certificate and a book voucher from Dr. Francesca Bonadei, Executive Editor of Springer.

The conference proceedings contain 69 chapters, 64 of which correspond to contributed talks or posters. The 40 chapters of this volume are grouped in two categories: Numerics (20 chapters) and Applications (20 chapters).

We would like to address our warmest thanks and gratitude to all who have made this book possible: first of all, to all the speakers of HYP2020/21 day and HYP2022 for their valuable contributions and, very especially, to those who accepted our invitation to contribute to this volume. This book has undergone a rigorous peer-review process: we are grateful for the work of the anonymous referees who, in a disinterested way, have helped the authors to improve the quality of their manuscripts. We would also like to thank the members of the Scientific Committee for their support and help in the speakers selection and those of the Organizing Committee for ensuring the smooth running of the event. We would like to thank the sponsors, without whom HYP2022 would not have been possible: we are really grateful to the University of Málaga and the Sociedad Española de Matemática Aplicada (SEMA). The financial support of the Office of Naval Research (ONR) of the United States allowed us to increase the number of grants for Ph.D. students: we thank Dr. Reza Malek-Medani for his interest and his help. We also thank the Springer staff for their help and support during the edition process, and especially Dr. Francesca Bonadei. Finally,

we are very grateful to the Editorial Board of the SEMA/SIMAI Springer series for having accepted this volume and to the Editor-in-Charge, María Elena Vázquez Cendón, for her helpful comments.

Málaga, Spain
July 2023

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Contents

Numerics

A Discrete Velocity Numerical Scheme for the 2D Bitemperature Euler System

Denise Aregba-Driollet, Stéphane Brull, and Corentin Prigent

Abstract This paper is devoted to the numerical approximation of the bidimensional bitemperature Euler system. This model is a nonconservative hyperbolic system describing an out of equilibrium plasma in a quasi-neutral regime, with applications in Inertial Confinment Fusion (ICF). We present a second order numerical scheme based on a discrete BGK relaxation model and we compare the results with the ones obtained by using a conservative model in the case of the triple point problem.

Keywords Nonconservative products · Euler equations · Discrete BGK · Finite volumes

1 Introduction

This paper is devoted to the numerical resolution of the two dimensional bitemperature Euler system by using a relaxation model under the form of a discrete BGK type approximation.

The bitemperature Euler system is a nonconservative hyperbolic system with a source term. It describes a mixture of electrons and ions in a quasi-neutral regime and in a thermal nonequilibrium. This system is constituted by two conservative equations for mass and momentum and two nonconservative equations on electronic and ionic energies. Several techniques have been proposed to define and approximate solutions with shocks in such a context [9, 12, 13]. For the present system physical assumptions lead to systems of conservation laws [8, 17] or hyperbolic-parabolic systems [18].

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Here we generalize the ideas of [3], where we derived our system as a fluid limit starting from a Vlasov-BGK model coupled with Ampère and Poisson equations in a quasi-neutral regime when the inter species collisions are dominant. Entropy dissipation properties were proved, allowing to define admissible weak solutions as weak entropy dissipative ones. A discrete BGK scheme was developed in 1D. This scheme has been validated in one space dimension and first order by comparison with the numerical results of the underlying Vlasov-Maxwell system discretized at the fluid level [3] and then at the kinetic level by a DVM method [6].

Discrete BGK models have been introduced in a conservative setting in [11] for scalar conservation laws. The method was next generalized for systems in [2]. Entropy properties are studied in [5]. In [3], those models are generalized in order to handle the nonconservative terms of the 1D bitemperature Euler system. In [1], we introduce a 2D second order generalization. In particular we explain how to adapt the ideas of [14, 15] in this nonconservative context.

This paper is organised as follows. In Sect. 2, the bitemperature model is introduced with the discrete BGK model that is associated. In Sect. 3, the numerical scheme is presented. We focus on the specificities of the second order in space due to nonconservativity. Finally we study in the last part the triple point problem proposed in [17]. Comparisons between conservative and nonconservative models are done.

2 Underlying Discrete BGK Model for a Nonconservative Euler System

Superscripts e and i respectively denote electronic and ionic quantities. We denote by ρ^e and ρ^i the electronic and ionic densities, $\rho = \rho^e + \rho^i$ the total density, m^e and m^i the related masses, c^e and c^i the mass fractions. These variables satisfy

$$\rho^e = m^e n^e = c^e \rho, \quad \rho^i = m^i n^i = c^i \rho, \quad m^e > 0, \quad m^i > 0, \quad c^e + c^i = 1. \quad (1)$$

Quasineutrality is assumed, so that the ionization ratio $Z = n^e/n^i$ is a constant. This implies that the electronic and ionic mass fractions are constant and given by

$$c^e = \frac{Zm^e}{m^i + Zm^e}, \quad c^i = \frac{m^i}{m^i + Zm^e}. \quad (2)$$

Electronic and ionic velocities u^e, u^i are assumed to be equal in the model: $u^e = u^i = u$, where u denotes mixture velocity. The pressure of each species satisfies a gamma-law with its own γ exponent :

$$p^e = (\gamma^e - 1)\rho^e \varepsilon^e = n^e k_B T^e, \quad p^i = (\gamma^i - 1)\rho^i \varepsilon^i = n^i k_B T^i, \quad \gamma^e > 1, \quad \gamma^i > 1, \quad (3)$$

where k_B is the Boltzmann constant ($k_B > 0$), ε^α and T^α represent respectively the internal specific energy and the temperature of species α for $\alpha = e, i$.

Denoting by $|\cdot|$ the euclidean norm in $\mathbb{R}^D$, the total energies for the particles are defined by

$$\mathcal{E}^\alpha = \rho^\alpha \varepsilon^\alpha + \frac{1}{2}\rho^\alpha |u|^2 = c^\alpha \Big(\rho\varepsilon^\alpha + \frac{1}{2}\rho|u|^2\Big), \quad \alpha = e, i. \tag{4}$$

We denote by $\nu^{ei} \geq 0$ the interaction coefficient between the electronic and ionic temperatures. The model consists of two conservative equations for mass and momentum and two nonconservative equations for each energy:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u + (p^e + p^i)\mathrm{I}) = 0, \\ \partial_t \mathcal{E}^e + \mathrm{div}(u(\mathcal{E}^e + p^e)) - u \cdot \nabla \left(c^i p^e - c^e p^i\right) = \nu^{ei}(T^i - T^e), \\ \partial_t \mathcal{E}^i + \mathrm{div}(u(\mathcal{E}^i + p^i)) + u \cdot \nabla \left(c^i p^e - c^e p^i\right) = -\nu^{ei}(T^i - T^e), \end{cases} \tag{5}$$

where I represents the identity matrix in $\mathbb{R}^D$. In the following we denote

$$\mathcal{U} = (\rho, \rho u, \mathcal{E}^e, \mathcal{E}^i), \quad U^\alpha = (c^\alpha \rho, c^\alpha \rho u, \mathcal{E}^\alpha). \tag{6}$$

The system (5) is hyperbolic, diagonalisable and owns 3 eigenvalues $\lambda_- = u \cdot \omega - a$, $\lambda_0 = u \cdot \omega$ (with multiplicity $D + 1$ where D is the space dimension), $\lambda_+ = u \cdot \omega + a$ where $a = \sqrt{\sum_{\alpha=e,i} \frac{\gamma^\alpha p^\alpha}{\rho}}$. The fields related to $\lambda_\pm$ are genuinely nonlinear, while the field related to λ_0 is linearly degenerate. Due to nonconservativity, some information has to be added in order to define physically realistic shocks. In [3] we defined jump admissibility via an entropy dissipation property and we proved that our approximation satisfies this property. We did not prove that this admissibility condition provides a unique jump.

Let us now present the discrete BGK approximation. Denoting $f^\alpha(x,t) \in (\mathbb{R}^4)^L$ the distribution function for α species ($\alpha \in \{e, i\}$), $U^{\alpha,\varepsilon} = P f^{\alpha,\varepsilon}$, q^e and q^i the electronic and ionic charges, and $E(x,t)$ the electric field, the discrete BGK system for (5) is as follows ($1 \leq l \leq L$):

$$\begin{cases} \partial_t f_l^{e,\varepsilon} + \displaystyle\sum_{d=1}^{2} v_{d,l} \partial_{x_d} f_l^{e,\varepsilon} + \frac{q^e}{m^e} N(E^\varepsilon) f_l^{e,\varepsilon} = \frac{1}{\varepsilon}\left(M_l^e(U^{e,\varepsilon}) - f_l^{e,\varepsilon}\right) + B_l^{ei}(f^{e,\varepsilon}, f^{i,\varepsilon}), \\ \partial_t f_l^{i,\varepsilon} + \displaystyle\sum_{d=1}^{2} v_{d,l} \partial_{x_d} f_l^{i,\varepsilon} + \frac{q^i}{m^i} N(E^\varepsilon) f_l^{i,\varepsilon} = \frac{1}{\varepsilon}\left(M_l^i(U^{i,\varepsilon}) - f_l^{i,\varepsilon}\right) + B_l^{ie}(f^{e,\varepsilon}, f^{i,\varepsilon}). \end{cases} \tag{7}$$

Note that despite a formal resemblance with the usual, physically meaningful BGK equations, each unknown $f_l^{\alpha,\varepsilon}$ is a vector valued function, see [2] for details. The "maxwellian function" M^α satisfies the compatibility conditions

$$\forall U^\alpha = (\rho^\alpha, \rho^\alpha u^\alpha, \mathcal{E}^\alpha), \quad \sum_{l=1}^{L} M_l^\alpha(U^\alpha) = U^\alpha, \quad \sum_{l=1}^{L} v_{d,l} M_l^\alpha(U^\alpha) = F_d^\alpha(U^\alpha)$$

where F^α is the flux function of the monotemperature Euler equations with γ^α pressure law. The source terms $B^{\alpha\beta}$ model the interactions between ions and electrons, see [3].

The novelty of the 1D paper [3] was the treatment of the nonconservative terms, analogous to $E \cdot \nabla_v f$ in kinetic models. In 2D it writes as

$$\forall \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \quad N(E)\varphi = -(0, \varphi_1 E, \varphi_2 \cdot E).$$

The quasineutrality condition and the fact that the magnetic field is neglected give the additionnal conditions

$$u^e = u^i = u, \quad \rho^\alpha = c^\alpha \rho, \quad \alpha \in \{e, i\}.$$

When ε tends to 0, if a limit (f^e, f^i, E) exists, then we have formally:

$$f^\alpha = M^\alpha(U^\alpha), \qquad \alpha = e, i.$$

By summing over l the equations of (7) and taking the limit $\varepsilon \to 0$, it comes, for $\alpha = e, i$:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho c^\alpha u) + \mathrm{div}(\rho c^\alpha u \otimes u) + \nabla p^\alpha - \dfrac{q^\alpha}{m^\alpha} E \rho c^\alpha = 0, \qquad \alpha \in \{e, i\}, \\ \partial_t \mathcal{E}^e + \mathrm{div}(u(\mathcal{E}^e + p^e)) - \dfrac{q^e}{m^e} E \rho c^e u = \nu^{ei}(T^i - T^e), \\ \partial_t \mathcal{E}^i + \mathrm{div}(u(\mathcal{E}^i + p^i)) - \dfrac{q^i}{m^i} E \rho c^i u = -\nu^{ei}(T^i - T^e). \end{cases}$$

The moment equations give

$$\frac{\rho^i q^i}{m^i} E = -\frac{\rho^e q^e}{m^e} E = -c^i \nabla p^e + c^e \nabla p^i.$$

and then we obtain (5). Stability (subcharacteristic) conditions have to be imposed in order to prove that the obtained solutions are entropy-admissible [1].

3 Numerical Approximation

In this section, we use the discrete BGK model presented in the previous section to design a finite volume scheme for system (5). We restrict ourselves to a cartesian grid. Denote Δx_1 and Δx_2 the space steps, Δt the time step, and $j = (j_1, j_2) \in \mathbb{Z}^2$. Denoting $e_1 = (1, 0)$, $e_2 = (0, 1)$, and for any unknown $v(x_1, x_2, t)$, v_j^n denotes its approximate value at time t^n in cell $C_j =]x_{1,j_1-\frac{1}{2}}, x_{1,j_1+\frac{1}{2}}[\times]x_{2,j_2-\frac{1}{2}}, x_{2,j_2+\frac{1}{2}}[$. We denote x_j the center of the cell C_j.

The whole construction of the scheme is described in [3] in 1D and first order, and in [1] in 2D. Essentially one has to approximate a set of linear bidimensional transport equations and then to take into account the force terms $N(E) f_l^{\alpha,\varepsilon}$ and the source terms $B_l^{\alpha\beta}$. The second order in time is reached by Heun's method.

Here, we focus our attention to second order in space. Like in [16], a piecewise affine reconstruction is used to determine intermediate values in subcells, but here this viewpoint leads to practical computations that are not required in the conservative case. Let us first recall the viewpoint for a one-dimensional system of conservation laws

$$\partial_t U + \partial_x F(U) = 0.$$

Assume that a first-order conservative scheme has been chosen:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

with $F_{j+\frac{1}{2}}^n = \mathcal{F}(U_j^n, U_{j+1}^n)$ and $\mathcal{F}(U, U) = F(U)$. Define a piecewise affine reconstruction:

$$\forall x \in C_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[, \qquad U^n(x) = U_j^n + \sigma_j^n (x - x_j), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}). \tag{8}$$

Once the reconstruction has been chosen, the values at the interfaces are

$$U_{j+\frac{1}{2}}^+ = (U^n(x_{j+\frac{1}{2}}))^+ = U_{j+1}^n - \sigma_{j+1}^n \frac{\Delta x}{2}, \qquad U_{j+\frac{1}{2}}^- = (U^n(x_{j+\frac{1}{2}}))^- = U_j^n + \sigma_j^n \frac{\Delta x}{2}. \tag{9}$$

Taking $U_{j-\frac{1}{2}}^+$ in C_j^- and $U_{j+\frac{1}{2}}^-$ in C_j^+ as initial values at time t_n, one gets on half cells:

$$U_j^{n+1,-} = U_{j-\frac{1}{2}}^+ - \frac{2\Delta t}{\Delta x}\left(\mathcal{F}(U_{j-\frac{1}{2}}^+, U_{j+\frac{1}{2}}^-) - \mathcal{F}(U_{j-\frac{1}{2}}^-, U_{j-\frac{1}{2}}^+)\right)$$

$$U_j^{n+1,+} = U_{j+\frac{1}{2}}^- - \frac{2\Delta t}{\Delta x}\left(\mathcal{F}(U_{j+\frac{1}{2}}^-, U_{j+\frac{1}{2}}^+) - \mathcal{F}(U_{j-\frac{1}{2}}^+, U_{j+\frac{1}{2}}^-)\right).$$

Then one sets $U_j^{n+1} = \frac{1}{2}\left(U_j^{n+1,-} + U_j^{n+1,+}\right)$:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U^-_{j+\frac{1}{2}}, U^+_{j+\frac{1}{2}}) - \mathcal{F}(U^-_{j-\frac{1}{2}}, U^+_{j-\frac{1}{2}})\right). \tag{10}$$

This procedure is extended in the case of a two-dimensional triangular mesh in [16]. More developments, particularly on the limitation procedure can be found in [4, 7, 14]. It is important to note that the effective computation of the numerical fluxes at the interface of two subcells is not needed in the conservative case. It is just useful to interpretate the scheme as a combination of first order schemes. One can also add others subcells in order to realize positivity requirements, but without additional computational cost, see [4].

To treat the nonconservative case, we want to use the same ideas. We treat directly the case of the two-dimensional cartesian grid. Contrarily to the conservative case, this algorithm necessitates the computation of the numerical fluxes at the interface of two subcells. This is a key point that leads us to detail our procedure. $(\mathcal{U}_j^n)_j$ being considered as a piecewise constant function on the cartesian mesh, we cut each C_j along its principal diagonals into four triangles $T_j^{(i)}$ $(i = 1, 2, 3, 4)$ and define a piecewise affine function $\mathcal{U}^n(x) = \mathcal{U}_j^n + (x - x_j)\cdot\sigma_j^n$ for $x \in C_j$. To compute the slopes σ_j^n in each direction, the minmod limitation procedure is applied in order to preserve the extrema and avoid oscillations. Then we define four constant states

$$\mathcal{U}_j^{(1)} = \mathcal{U}_j^n - \frac{\Delta x_1}{2}\sigma_{1,j}^n, \qquad \mathcal{U}_j^{(2)} = \mathcal{U}_j^n - \frac{\Delta x_2}{2}\sigma_{2,j}^n,$$
$$\mathcal{U}_j^{(3)} = \mathcal{U}_j^n + \frac{\Delta x_1}{2}\sigma_{1,j}^n, \qquad \mathcal{U}_j^{(4)} = \mathcal{U}_j^n + \frac{\Delta x_2}{2}\sigma_{2,j}^n.$$

The state $\mathcal{U}_j^{(i)}$ is the initial value at time t_n in the subcell $T_j^{(i)}$ of C_j. We denote $T_\mu = T_j^{(i)}$, $\mathcal{U}_\mu = \mathcal{U}_j^{(i)}$. We set

$$U_\mu^{\alpha,n} = (c^\alpha \rho_\mu^n, c^\alpha \rho_\mu^n u_\mu^n, \mathcal{E}_\mu^{\alpha,n}), \quad f_\mu^{\alpha,n} = M^\alpha(U_\mu^{\alpha,n}), \quad \alpha \in \{e, i\}.$$

Then we solve the set of transport equations $\partial_t f_l^\alpha + \sum_{d=1}^{2} v_{d,l}\,\partial_{x_d} f_l^\alpha = 0$ by the upwind scheme. This scheme is monotone under the CFL condition

$$\Delta t \max_{1\le d\le 2} \frac{\lambda_d}{\Delta x_d} \le \frac{1}{4}.$$

For a triangle T_μ, the adjacent triangles are denoted T_{μ_1}, T_{μ_2}, T_{μ_3}, the outward unit normal vector from T_μ to T_{μ_k} is denoted n_k, the edge between T_μ and T_{μ_k} is denoted Γ_k. The upwind scheme then writes as

$$f_{\mu,l}^{\alpha,n+\frac{1}{2}} = f_{\mu,l}^{\alpha,n} - \frac{\Delta t}{|T_\mu|}\sum_{k=1}^{3}\left((V_l\cdot n_k)^+ f_{\mu,l}^n - (V_l\cdot n_k)^- f_{\mu_k,l}^n\right)|\Gamma_k|, \quad l\in\{1,2,3,4\} \tag{11}$$

which can be rewritten

$$f_{\mu,l}^{\alpha,n+\frac{1}{2}} = f_{\mu,l}^{\alpha,n} - \Delta t\sum_{k=1}^{3}\Phi_{k,l}(f_{\mu,l}^{\alpha,n}, f_{\mu_k,l}^{\alpha,n}, n_k),$$

where for $f, g\in\mathbb{R}^4$ and $n\in\mathbb{R}^2$,

$$\Phi_{k,l,\mu}(f,g,n) = \left((V_l\cdot n)^+ f - (V_l\cdot n)^- g\right)\frac{|\Gamma_k|}{|T_\mu|}.$$

The CFL condition is

$$\Delta t\max_{1\le d\le 2}\frac{\lambda_d}{\Delta x_d}\le\frac{1}{4}.$$

The remaining steps for the subcell T_μ are the same as in the cartesian case. Macroscopic fluxes for species α can be defined as

$$\forall (U,V)\in\mathbb{R}^4,\qquad \mathcal{F}_{k,\mu}^{\alpha}(U,V,n_k) = \sum_{l=1}^{4}\Phi_{k,l,\mu}(M_l^\alpha(U), M_l^\alpha(V), n_k)$$

and we obtain

$$\begin{cases}
\rho_\mu^{n+1} = \rho_\mu^n - \Delta t\displaystyle\sum_{k=1}^{3}\mathcal{F}_{k,\mu,1}^n,\\
\rho_\mu^{n+1}u_\mu^{n+1} = \rho_\mu^n u_\mu^n - \Delta t\displaystyle\sum_{k=1}^{3}\mathcal{F}_{k,\mu,2}^n,\\
\mathcal{E}_\mu^{e,n+1} = \mathcal{E}_\mu^{e,n} - \Delta t\displaystyle\sum_{k=1}^{3}\mathcal{F}_{k,\mu,3}^n + \Delta t u_\mu^{n+1}\cdot\sum_{k=1}^{3}\delta_{k,\mu}^n + \Delta t\nu^{ei}(T_\mu^{i,n+1} - T_\mu^{e,n+1}),\\
\mathcal{E}_\mu^{i,n+1} = \mathcal{E}_\mu^{i,n} - \Delta t\displaystyle\sum_{k=1}^{3}\mathcal{F}_{k,\mu,4}^n - \Delta t u_\mu^{n+1}\cdot\sum_{k=1}^{3}\delta_{k,\mu}^n - \Delta t\nu^{ei}(T_\mu^{i,n+1} - T_\mu^{e,n+1}),
\end{cases}$$

where

$$\mathcal{F}_{k,\mu,1}^n = \sum_\alpha\mathcal{F}_{k,\mu,1}^\alpha(U_\mu^{\alpha,n}, U_{\mu_k}^{\alpha,n}, n_k),\qquad \mathcal{F}_{k,\mu,2}^n = \sum_\alpha\mathcal{F}_{k,\mu,2}^\alpha(U_\mu^{\alpha,n}, U_{\mu_k}^{\alpha,n}, n_k),$$
$$\mathcal{F}_{k,\mu,3}^n = \mathcal{F}_{k,\mu,3}^e(U_\mu^{e,n}, U_{\mu_k}^{e,n}, n_k),\qquad \mathcal{F}_{k,\mu,4}^n = \mathcal{F}_{k,\mu,3}^i(U_\mu^{i,n}, U_{\mu_k}^{i,n}, n_k),$$

and

$$\delta_{k,\mu}^{n} = -c^{i}\mathcal{F}_{k,\mu,2}^{e}(U_{\mu}^{e,n}, U_{\mu_k}^{e,n}, n_k) + c^{e}\mathcal{F}_{k,\mu,2}^{i}(U_{\mu}^{i,n}, U_{\mu_k}^{i,n}, n_k) \in \mathbb{R}^2.$$

The partial energies can be computed explicitly by the resolution of a linear 2×2 system.

Finally, denoting $\mathcal{U}_j^{(i),n+1}$ the value obtained in subcell number $T_j^{(i)}$, solution at time t^{n+1} is defined by:

$$\mathcal{U}_j^{n+1} = \frac{1}{4}\sum_{i=1}^{4} \mathcal{U}_j^{(i),n+1}.$$

4 A Numerical Experiment: The Triple Point Problem

This test has been presented in [10, 17]. The domain $\Omega = [0, 7] \times [-3, 3]$ is divided into three different subdomains as indicated in Fig. 1. Ω_1 contains a high density and low pressure fluid, Ω_2 is composed of a high density and high pressure fluid, and Ω_3 contains a low density and low pressure fluid. Initially $T^i = T^e$ and the velocity u is equal to zero. The triple points are the two points where the three domains intersect. We set $\nu^{ei} = 0$. As already noted in the cited articles, ionic and electronic temperatures do not remain equal as time evolves. On Fig. 2 we compare the electronic temperature computed by first and second order schemes for a 1000×1000 mesh.

Then, in order to compare our approach with the conservative system proposed in [10], we have implemented a discrete BGK scheme for their system of conservation

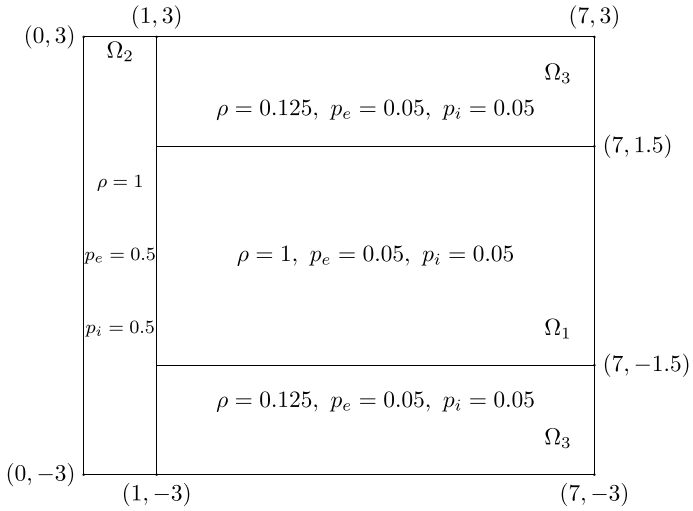

Fig. 1 Triple point test case: the computational domain and the data

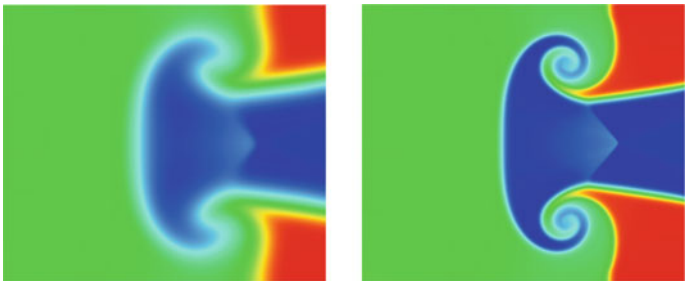

Fig. 2 Triple point problem computation with a grid 1000 by 1000: electronic temperature at time $T = 5$ s. Left: first order scheme. Right: second order scheme

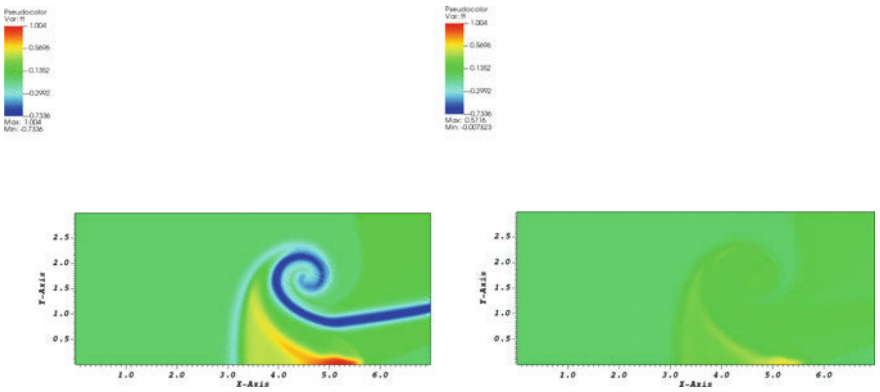

Fig. 3 Triple point problem computation with a grid 500 by 214: $\frac{T_i - T_e}{T_e}$ at time $T = 5$ s. Left: conservative model. Right: our bitemperature model

laws. For symmetry reasons, the computational domain is the half domain $[0, 7] \times [0, 3]$ meshed with 500×214 points. The difference between temperatures is higher in the conservative case than in the non conservative one. In Fig. 3 we depicted the isovalues of $\frac{T_i - T_e}{T_e}$ in both cases, with the same scale. In Fig. 4 we depicted the isovalues of the same quantities in the nonconservative case, with its own scale. We observe that the maximal difference occurs around the x-axis for both models but not with the same amplitude.

We recall that to obtain the conservative model, the authors of [10] assume that the electron entropy is conserved across shocks. They attribute the fact that electronic and ionic temperature do not remain equal to this assumption. In the nonconservative model, there is no reason why electronic and ionic temperatures should remain equal and actually they do not.

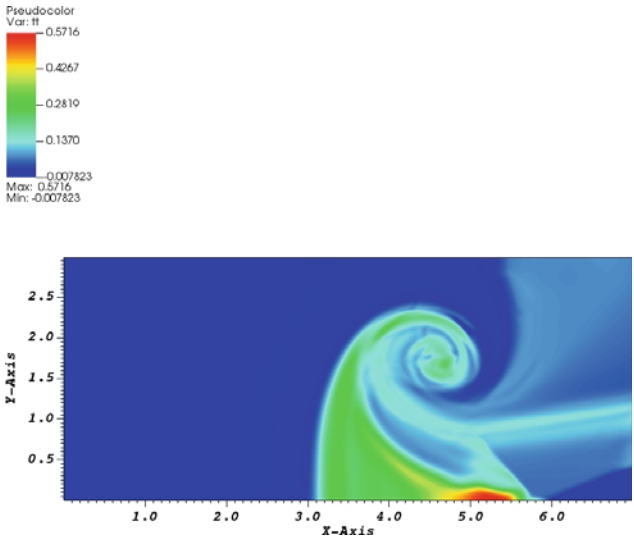

Fig. 4 Triple point problem computation with a grid 500 by 214 for our bitemperature model at time $T = 5\,\mathrm{s}$ at time $T = 5\,\mathrm{s}$. $\frac{T_i - T_e}{T_e}$

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Finding an Approximate Riemann Solver via Relaxation: Concept and Advantages

Claudius Birke and Christian Klingenberg

Abstract We first explain the general concept of relaxation models using the Jin-Xin model for scalar conservation laws. Then we consider the Suliciu model specifically for the homogeneous Euler equations. In a third step, using a two-speed relaxation model for the Euler equations with gravity as an example, we show how the construction of the relaxation system can endow the resulting method with useful properties. Finally, these properties are verified in numerical tests.

Keywords Euler equations · Finite volume methods · Relaxation · Well-balancing · Low Mach · Asymptotic-preserving · Entropy satisfying · Checkerboard modes · Positivity preserving

1 Introduction

Finite volume methods are a popular way to solve systems of partial differential equations in fluid dynamics. In 1959 Godunov introduced the revolutionary idea to solve Riemann problems in order to compute solutions for non-linear hyperbolic conservation laws [10]. In his approach, Godunov computes the exact solution of the Riemann problems which makes his scheme rather cumbersome and computationally inefficient. Later, Roe pointed out that the Riemann problems need not be solved exactly, but that an approximate solution is sufficient in many cases. The result of his work was the Roe solver [14]. Since the Riemann solver is the core part in Godunov type methods, the efficiency of the scheme can be greatly increased by the use of approximate Riemann solvers. While with Roe's solver it was possible to compute quite accurate solutions in an efficient way, the solver still lacked some important

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properties. A major flaw of Roe's solver is that it does not necessarily satisfy a discrete entropy inequality. Building on Roe's ideas, other approximate Riemann solvers were developed in the following years to overcome this problem. One class of such solvers are the Harten-Lax-van Leer (HLL) solvers. The first solver of this kind is presented in [11]. The key idea is to assume that the solution to the Riemann problem consists of three constant states separated by two waves. By using the integral form of the conservation law over the Riemann fan, it is possible to determine the intermediate state in the solution. This approach yields accurate results for systems with only two equations, such as the shallow water equations, but performs poorly for larger systems like the full Euler equations because it focuses only on the outer waves and ignores intermediate waves. Newer solvers like HLLC for the Euler equations also consider intermediate waves and therefore lead to more accurate solutions [18].

In the 1990s, the concept of relaxation schemes emerged [2, 8, 12]. The basic idea is to construct a new enlarged relaxation system, including a relaxation term on the right-hand side, that is an approximation of the original system. The numerical scheme then solves the relaxation system in two steps:

1. First solve the left-hand side of the relaxation system, which consists of a linear transport and is therefore numerically easy to solve.
2. Then project the solution of the first step back onto the equilibrium variables, i.e. use only the variables of the original system to solve the next time step.

The resulting numerical method is thus simple and yet leads to rather accurate results. Furthermore, the approximate Riemann solver naturally satisfies a discrete form of the entropy inequality, which results in an increased robustness of the method. Since there is a certain degree of freedom in how to construct the relaxation system, it is possible to equip the approximate Riemann solver with additional desirable properties.

2 Concept of Relaxation

In order to familiarize ourselves with the concept of relaxation, let us first consider the simple case of the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{1}$$

To solve this equation, Jin and Xin [12] introduced the relaxation system

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, \\ \partial_t v^\varepsilon + a^2 \partial_x u^\varepsilon &= \frac{1}{\varepsilon}\left(f(u^\varepsilon) - v^\varepsilon\right), \end{aligned} \tag{2}$$

with a constant relaxation speed a and relaxation parameter ε. This system is a diffusive approximation of the original scalar conservation law in (1). This can be illustrated by a Chapman-Enskog expansion [6]. For this procedure we consider a formal expansion of v in terms of ε

$$v^\varepsilon = v_0^\varepsilon + \varepsilon v_1^\varepsilon + \mathcal{O}(\varepsilon^2) \tag{3}$$

and insert this expansion into the system (2)

$$\partial_t u^\varepsilon + \partial_x \left(v_0^\varepsilon + \varepsilon v_1^\varepsilon\right) = 0, \tag{4}$$

$$\partial_t \left(v_0^\varepsilon + \varepsilon v_1^\varepsilon\right) + a^2 \partial_x u^\varepsilon = \frac{1}{\varepsilon}\left(f(u^\varepsilon) - v_0^\varepsilon - \varepsilon v_1^\varepsilon\right). \tag{5}$$

From collecting all terms of order $\mathcal{O}(1/\varepsilon)$, we can determine

$$v_0^\varepsilon = f(u^\varepsilon). \tag{6}$$

For the terms with order $\mathcal{O}(1)$, on the other hand, we gain the system

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v_0^\varepsilon &= 0, \\ \partial_t v_0^\varepsilon + a^2 \partial_x u^\varepsilon &= -v_1^\varepsilon. \end{aligned} \tag{7}$$

We can reformulate the second equation using both (6) and the chain rule

$$v = f(u) - \varepsilon \left(a^2 - f'(u)^2\right) \partial_x u. \tag{8}$$

This expression can be plugged into the first equation of (2) and we derive

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\left(a^2 - f'(u)^2\right) \partial_x u\right). \tag{9}$$

Clearly this equation is diffusive as long as the stability criterion

$$-a \leq f'(u) \leq a \tag{10}$$

is satisfied. This criterion is called subcharacteristic condition [12]. The Chapman-Enskog expansion shows that the relaxation system is a suitable approximation of the original conservation law. Therefore it is sufficient to determine the solution of the relaxation system. We do that by applying the following splitting approach. In a first step we solve the left-hand side of (2)

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, \\ \partial_t v^\varepsilon + a^2 \partial_x u^\varepsilon &= 0. \end{aligned} \tag{11}$$

All eigenvalues of this system are linearly degenerate so that it is easy to find the solution to the associated Riemann problem. In the second step, the projection step, we solve the system in the limit $\varepsilon \to 0$, i.e.

$$\begin{aligned} \partial_t u^\varepsilon &= 0, \\ \partial_t v^\varepsilon &= \frac{1}{\varepsilon} \left(f(u^\varepsilon) - v^\varepsilon \right). \end{aligned} \tag{12}$$

In practice, we simply take the solution of the first step for u^ε and use this as the initial value when calculating the solution at the next time step. This projection step was first introduced in [4].

3 The Suliciu Relaxation Model

While the Jin-Xin relaxation system consists of twice as many equations as the original conservation law (this also applies to systems of conservation laws), it is also possible to construct relaxation systems with fewer additional equations. For the compressible Euler equations

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0, \\ \partial_t E + \partial_x((E + p)u) &= 0, \end{aligned} \tag{13}$$

with density ρ, velocity u, total energy E and pressure p, one such system is the so-called Suliciu relaxation model [2, 7, 16]. The main idea of this approach is to derive an evolution equation for the pressure from the continuity equation in (13)

$$\partial_t(\rho p) + \partial_x(\rho u p) + \rho^2 p'(\rho) \partial_x u = 0. \tag{14}$$

In this equation one replaces the pressure p by a relaxation variable π and the sound speed $\rho\sqrt{p'(\rho)}$ by a positive constant relaxation speed a

$$\partial_t(\rho\pi) + \partial_x(\rho u \pi + a^2 u) = \rho \frac{p - \pi}{\varepsilon}. \tag{15}$$

Finally, this new relaxation equation is added to the original Euler equations and the pressure p is replaced in all equations by π so that the resulting Suliciu relaxation system has the form

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi) &= 0, \\ \partial_t E + \partial_x((E + \pi)u) &= 0, \\ \partial_t(\rho\pi) + \partial_x(\rho u \pi + a^2 u) &= \rho \frac{p - \pi}{\varepsilon}, \\ \partial_t(\rho a) + \partial_x(\rho a u) &= 0. \end{aligned} \tag{16}$$

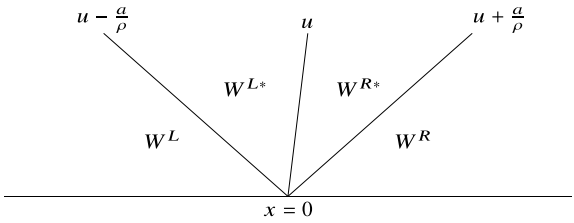

Fig. 1 Structure of the solution to the Riemann problem associated with the Suliciu relaxation model (16)

A Chapman-Enskog expansion with similar steps as for the Jin-Xin relaxation leads to the following subcharacteristic stability condition

$$a \geq \rho c, \tag{17}$$

where c represents the sound speed. The eigenvalues of system (16) are given by

$$\lambda^- = u - \frac{a}{\rho}, \quad \lambda^u = u, \quad \lambda^+ = u + \frac{a}{\rho}. \tag{18}$$

It can easily be checked that all eigenvalues are linearly degenerate. This is of great advantage because it allows us to solve the Riemann problem exactly. As shown in Fig. 1, the solution has four constant states separated by three waves. The intermediate states can then be computed with the help of the Riemann invariants, which are constant across the corresponding wave. For a large enough relaxation speed a, the resulting approximate Riemann solver preserves positivity of density and internal energy and satisfies a discrete entropy inequality [2].

4 A Two-Speed Relaxation System

In this section we want to present a relaxation system that approximates the full Euler equations with a gravitational source term given by

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= -\rho \partial_x \Phi, \\ \partial_t E + \partial_x((E + p)u) &= -\rho u \partial_x \Phi, \end{aligned} \tag{19}$$

where Φ represents the gravitational potential. Although the Suliciu scheme in combination with a discretization of the source term is a suitable approximation to (19), it does not provide accurate solutions to certain problems. For example, if the Mach number M is very small, a method based on the classic Suliciu system leads to very inaccurate solutions, since the diffusion on the velocity in the approximate Riemann solver increases with decreasing Mach number. A second weak point becomes apparent in problems close to hydrostatic equilibrium