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FOUNDATIONS *of the* PRICING *of* FINANCIAL DERIVATIVES

Theory and Analysis

ROBERT E. BROOKS • DON M. CHANCE

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Theory and Analysis

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Preface

Teaching graduate students in finance must be the second best job in the world, next to parenting. Finance is such a rich and exciting subject, and it is uncommon to teach a student who is not interested. Now, we did not say they all put in the maximum effort, but they seldom act as if they are bored. After all, everyone wants to know more about money. Finance professors are the envy of their relatives, and they typically command a great deal of attention at family reunions and receptions with everyone hoping to learn just one iota of information that might make them financially better off.

The authors of this book have taught graduate students in finance for many decades (we stopped counting while we were still young!), and we can attest that our students have taught us a great deal, too. But as longtime teachers of advanced masters and doctoral students, we have been concerned that it is possible to get one of these impressive advanced degrees and circumvent, or just never have, the opportunity to get familiar with the principles of financial derivative pricing. Corporate finance and asset pricing are the core of what we do in the academic finance field. But if a corporate finance or asset pricing specialist is not exposed to derivative pricing, they are missing out on a rich body of knowledge that can help them do their jobs. Or at least half-way understand when a derivatives person presents a paper.

This book had its origin in a set of teaching notes that one of the authors (Chance) began writing in 1996. These notes were designed to fill what he saw as a void in instructional material at the advanced level. Well, it was either that or he just wanted it done his way. These notes were posted on the Web and eventually grew in number to almost 60. The notes were short, often less than 10 pages, tightly contained treatments of various topics. He received a great deal of recognition from complete strangers around the world, and in time, the notes were morphed into this book. He thought it would be easy just to turn the notes into a book. But that led to two problems.

The first was that we had to remove the notes from the Web, leading to some disappointment from fans. Although we had given away much of this material, publishers expect that you will protect their investment by not giving it away any longer. The second problem was that a multiplicity of notation was used in the notes. They were never written as a unified whole. As such, we had a great deal of cleaning up to do. And, we suppose, a third problem was that the notes did not cover the entire field, so yes, we had a lot more writing to do.

This book in manuscript form has been class-tested three times. This course was a doctoral seminar that was open to finance masters and doctoral students and also STEM students across the university. All of these students contributed a great deal to catching errors, forcing us to rethink how we said something, and in some cases contributing end-of-chapter problems. This book would be nowhere near ready for prime time were it not for them. Because these students endured rough drafts of this book, I would like to thank them by name: Brecklyn Groce, Dennel McKenzie, Jeremy Vasseur, Paul Mahoney, Tengfei Zhang, Nha Tran, Nur Faisal, Jason Priddle, Mehdi Khorram, Mengmeng Liu, Phuc An Vinh Nguyen, Santoshi Rimal, Cameron Roman, Gillian Sims, Pujan Shrestha, Yuanyi Zhang, Aihuan Zhang, Fouad Hasan, Junior Betanco, Ravi Joshi, and Yingying Guo. We also thank Chance's research assistant, Stephanie Hoskins, for additional comments.

A special note of thanks is extended to Chance's PhD student, Amber Schreve, who carefully read and edited the entire book, catching a number of errors and typos, questioning sentences that might not have made complete sense to the reader, and offering many suggestions for improvements. Amber's background in teaching math to undergraduates challenged us to strive for the highest level of clarity that we could.

We never set out to write this book. As noted, it sprung from Chance's teaching notes. And we also agreed not to attempt to compete with highly technical books on quantitative finance and financial engineering written by STEM scholars. What we wanted to do was create a book that was within reach of PhD and advanced masters' students in finance, many of whom do not have the technical backgrounds of STEM students. So, do not expect to find everything on the subject here. But you will find a broad, relatively technical overview of the most important knowledge you need to know to build a solid foundation for understanding the pricing of financial derivatives.

Like all authors, we think we accomplished that. We know, in all honesty, that there are inevitably failures that even another 10 years of editing and class-testing would not completely eliminate. We accept full responsibility for any such deficiencies and promise to consider them if the book goes into a future edition. If you want to communicate with us on any such matters, please send an email to dchance@lsu.edu and rbrooks@frmhelp.com.

The least likely people in the world to email us with a problem in the book are our families. And they deserve credit for just being there and not complaining that we were writing another book when it seemed like we just finished the last one. Don would like to thank his wife, Jan, and their adult and married daughters, Kim and Ashley, and their families. Robert would like to thank his wife, Ann, and their adult and married six children and similarly their constantly increasing number of grandchildren.

Robert E. Brooks
Tuscaloosa, AL

Don M. Chance
Baton Rouge, LA

CHAPTER 1

Introduction and Overview

Finance is the study of money, something commonly used as a medium of exchange. It involves the measurement and management of money: how much we had, how much we have, and how much we expect to have in the future. Much of what we study and do in finance, however, is about making money, so its focus tends to be on the future. Yet, the future is unknown, and the unknown is about risk. People take risks in order to earn money. Finance is essentially the study of the risk and return of money.

In order to do what we do in finance, we must measure money. In fact, measurement in terms of monetary units, such as US dollars, is a core activity of finance, and there is considerably more to measuring money than just counting it. The challenge in finance is in measuring the value of assets that are not cash but can be expressed in terms of cash. An *instrument* is a generic term that refers to a tool that measures something. A *financial instrument* is a type of instrument defined as a tool that measures something in dollars or other currency units. In our context, positive-valued financial instruments are called *assets* and negative-valued financial instruments are called *liabilities*. In general, assets and liabilities are both categorized as instruments or financial instruments because our focus is on finance. Thus, an instrument can reference either assets or liabilities.

Some instruments trade in a market where we can observe their prices, but does that mean that we do not question whether these prices are good prices, in the sense of fairly and accurately reflecting what something should be worth? If you need to buy a used car and you find a 10-year-old car with 150,000 miles on it selling for $50,000, does that mean you would pay that price? No, it is likely you would believe that price to be too high. Perhaps $5,000 is a better price. What we have just done is a valuation of the car. We may have gotten it from some service, observed the prices of similar cars, or simply said that $5,000 is the amount we would pay, meaning that we would willingly part with the consumption of $5,000 of other goods and services to obtain the car.

Likewise, securities that trade in markets have prices that are observable, but that does not mean that we accept those prices as fair. These securities need to be valued, meaning to assign a number to them that represents what one thinks is a fair price. If the value assigned by the investor exceeds the price at which the security is trading in the market, the security is attractively priced and would suggest that the investor should buy the security. If the value assigned by the investor is below the price at which the security is trading in the market, the security is unattractively priced and would suggest that the investor should sell the security if they own it, short sell it if they do not own it, or simply not trade it at all.

We now explain our motivation for writing *Foundations of the Pricing of Financial Derivatives*.

1.1 MOTIVATION FOR THIS BOOK

Many finance courses focus on valuing stocks and bonds. Yet, there is also another family of financial instruments known as *derivatives*, and valuing derivatives is one of the most technical subjects in finance. It requires not only setting up a model of the prices of assets that trade in the market but also establishing a means by which one can connect the derivative to the asset on which the derivative is based. There is a great deal of technical knowledge that must transfer from instructor to student. Much of that knowledge can seem cryptic and inaccessible, though that could be a bit of an overreaction from the fear of learning something new. However, those who know this subject reasonably well can easily fall into the trap of assuming that those who do not know this subject well should find the subject easy. That is the pitfall of being a scholar. A scholar thinks that material in which they have expertise in is not that difficult, when in fact, it really is quite challenging. What a scholar should do in conveying knowledge, however, is to recall how it was when they were learning it. In other words, putting oneself in the student's shoes and empathizing with the student will result in the most successful learning environment. That is indeed one of the overriding objectives of this book: to teach some seemingly complex material in a very user-friendly way.

For without a doubt, teaching and learning advanced material in finance is challenging to the instructor and to the student. Indeed, one of the greatest challenges for instructors in advanced graduate courses in finance is to cover a large body of highly technical information in a relatively short course of study. Well-prepared students make it a lot easier, but student preparation is often not at the desired level, and classes are frequently filled with students with varying degrees of preparation. In an ideal world, such students would have previously had courses in probability, calculus, linear algebra, coding, stochastic processes, econometrics, numerical analysis, non-parametric statistics, differential equations, microeconomics and macroeconomics, and last but certainly not least, finance. It is common for faculty members and students alike to complain that students are inadequately prepared for the technical rigors of advanced graduate study in finance. This book is an effort to address this problem by leveling the base of preparation.

The degree of preparation of finance students is typically a function of the program in which the student is enrolled. Graduate study in finance can generally be done in one of three types of programs. One is an MBA, which usually comprises two years of study, the first consisting of a core set of business courses, of which a broad survey of finance is typically one component. The second year of an MBA is composed of a few required courses in general business but largely permits students to tailor their programs toward their specialized interests. Many students choose to take second-year courses in finance. The MBA is usually the marquee program at top-tier universities, a large money-maker, and is designed to draw students with degrees from all undergraduate disciplines. Hence, the first-year finance course starts at the very foundations of the subject with such topics as time value of money and discounted cash flow valuation, ultimately moving on to understanding financial markets, the relationship between risk and return, market efficiency, and corporate capital structure and dividend policy. Though some MBA students have technical backgrounds, most do not. Hence, MBA students will often struggle with advanced finance courses that are particularly quantitative.

A second form of masters' level study in finance is the specialized master's degree in finance, often called an MS or master of science, and sometimes MSF for master of science in finance. Such a program provides concentrated graduate study in finance, typically over a period of one to two academic years. There may be certain core or required courses, and students are usually allowed to take electives in their preferred areas of finance. Students in this type of program will almost always have previously studied finance and will tend to have more technical backgrounds than the average MBA student.

The third type of program in finance is the doctor of philosophy or PhD. This degree, requiring a minimum of four years, is an intensive research-oriented program that requires all students to achieve a high level of understanding of theoretical models and empirical research methods.[1] It is here that the greatest problem lies in giving students a sufficient level of technical knowledge without sacrificing the time they need to devote to seminars in the various areas of finance. When students are accepted into PhD programs in finance, they are typically required to have a solid foundation in math. But, the definition of a "solid foundation in math" can vary, and merely taking some math courses and making good grades is not necessarily enough. Most finance PhD students, even those with strong math backgrounds, learn something new about math while in their PhD programs. Students who have been accepted into PhD programs are often advised to take more math courses before starting the program. They usually do, but it does not often help nearly as much as one might think.

Let us be clear. Finance is not mathematics. Mathematics is a set of tools used in finance. But just as one cannot build furniture efficiently without knowledge of how to use tools, one cannot understand finance without having the necessary tools of mathematics.

And, as noted, one of the most technical subjects in finance is derivative pricing theory. Sometimes it can be even difficult to keep terms straight because terms can mean different things in different settings. Throughout this book, we will use price and value interchangeably as is financial industry custom. Technically, the concept of price refers to the monetary amount that is exchanged when something is traded, irrespective of what one thinks the item is worth. Value refers to an instrument's non-observed monetary amount as assigned by a market participant. The individual may or may not be using a formal mathematical model. Thus, technically, formal models in finance, such as the capital asset pricing model (CAPM) and the Black-Scholes-Merton option pricing model (BSMOPM), should have been termed the capital asset valuation model (CAVM) and the Black-Scholes-Merton option valuation model (BSMOVM). Theoretical models are just a means of expressing one's view on value. As we will see later, arbitrage activity typically moves observed market prices to the arbitrageur's value. Hence, we will stick with financial industry custom even though value will always have higher levels of epistemic uncertainty when compared to the market price.

Although the majority of finance faculty and PhD students will not specialize in derivatives, there is no doubt that a solid understanding of derivative pricing theory is an important element of doctoral-level education in finance. Derivative pricing theory, in particular the Black-Scholes-Merton model, has had a tremendous impact on finance. It has provided a framework for understanding not only standard derivatives, such as options, but also it has shown us that derivatives can explain many other topics and relationships in finance, such as callable bonds, convertible bonds, credit risk, and corporate capital structure. The impact of derivative pricing theory has been so great that Nobel Prizes were awarded in 1995 to Myron Scholes and Robert Merton with special recognition to the late Fischer Black for work on this subject. Yet, with the increasing need for students to take so many courses in econometrics and statistics, there is often little room in a doctoral program for such a course.

Our goal is to introduce the vast financial derivatives markets to PhD students and others in hopes that it will stimulate your interest in research related to financial derivatives, as well as aid in your future research agenda, even if your agenda is not explicitly financial derivatives. By way of introduction, we present selected derivatives market prices in Table 1.1.[2] Derivatives market prices are unique and often convey cloaked information. On completion of this book, you will be better able to rightfully interpret what information is and is not conveyed.

Table 1.1 Panel A shows natural gas futures prices. Notice that the key descriptor is the delivery month. We denote the current year as Y1. Thus, the December Y1 last traded futures price is $2.896/MMBtu (million British thermal units). Note that futures prices generally rise for longer maturities with the exception of September.[3] The aggregate open interest is the number of either long or short positions currently outstanding. Each contract is for 10,000 MMBtu. Thus, the natural gas futures market currently represents 12,689,550,000 MMBtus.

Table 1.1 Panel B shows option prices of the SPY, which is the exchange-traded fund of the S&P 500 Index. With options, the key descriptors are more complex requiring both the maturity date expressed in days to maturity here and strike price. For example, the 114-day call price with strike price $280/share is $10.40/share and the corresponding put price is $9.76/share. Notice that with longer maturities, similar strike options have higher prices. Further, call prices decline for higher strike prices, whereas put prices rise for higher strike prices. You will learn why later in this book.

Table 1.1 Panel C shows interest rate swap rates. For example, a five-year swap was quoted at 2.022%. The mid fixed rate is the average of the bid and ask rate for the fixed leg of a fixed-for-floating interest rate swap. Notice that this fixed rate initially declines and then subsequently rises for longer maturities.[4]

Table 1.1 Panel D shows 1-month secured overnight financing rate (SOFR) futures data.[5] The data provided here are rate-based and hence show the implied interest rates rather than quoted prices. The interest rate is simply 100 minus the quoted price. For example, the November Y1 rate of 5.400 implies a quoted price of 94.600. Each SOFR futures contract is for $5,000,000 notional amount on 1-month interest rates. At this point, just notice the aggregate open interest is 704,883. *Open interest* is the number of long contracts outstanding. Thus, the total number of contracts outstanding, representing both long and short positions as this is an exchange-traded contract, is 1,409766 [= 2(704,883)].

TABLE 1.1 Panel A. Selected Derivatives Markets Prices: Selected Natural Gas Futures Prices

Description	Price	Change	Open Interest	Volume
JunY1	2.615	+.033	9,781	2,311
JulY1	2.620	+.036	353,254	22,906
AugY1	2.627	+.034	92,888	4,108
SepY1	2.618	+.031	177,123	2,905
OctY1	2.651	+.030	122,641	2,001
NovY1	2.727	+.027	81,686	1,057
DecY1	2.896	+.026	99,738	878
. . .	. . .	. . .	. . .	. . .

Note: Aggregate Open Interest = 1,268,955; Volume = 38,910.

TABLE 1.1 Panel B. Selected Option Prices on S&P 500 Index Exchange-Traded Fund (SPY)

Maturity (Days)	Strike Price	Call Price	Put Price
30	279	5.97	4.94
30	280	5.43	5.39
30	281	4.85	5.82
51	279	7.39	6.22
51	280	6.80	6.63
51	281	6.18	7.06
79	279	9.33	7.59
79	280	8.67	8.02
79	281	7.93	8.39
114	279	11.04	9.30
114	280	10.40	9.76
114	281	9.96	10.04
. . .	. . .	. . .	. . .

Note: Spot SPY = 280.15.

TABLE 1.1 Panel C. Interest Rate Swaps

Tenor	Mid Fixed Rate
1 Year	2.350
2 Year	2.115
3 Year	2.032
5 Year	2.022
10 Year	2.175
20 Year	2.352
30 Year	2.378

TABLE 1.1 Panel D. Secured Overnight Financing Rate Futures Contracts

Description	Implied Rate	Change	Open Interest	Volume
AugY1	5.305	UNCH	275,240	1,362
SepY1	5.325	+0.005	105,264	2,839
OctY1	5.455	+0.005	94,418	22,034
NovY1	5.400	+0.005	106,863	13,293
DecY1	5.395	UNCH	47,427	4,810
JanY2	5.480	–0.005	32,062	7,439
FebY2	5.300	–0.005	19,308	5,216
. . .	. . .	. . .	. . .	. . .

Note: Aggregate Open Interest = 704,883; Volume = 75,845.

As each contract is for $5,000,000 notional, the aggregate notional amount represented by this one market is $7,048,830,000,000 or over $7 trillion. The *notional amount* or simply *notional* is the implied principal on which interest calculations are based. Thus, without explaining all this data in great detail, clearly the derivatives industry is large and involves numerous interesting complexities worthy of investigation.

The purpose of this book is to provide detailed training in the pricing of financial derivatives in a lean and efficient manner. It endeavors to convey the mathematical foundations of derivative pricing theory in a compact way. And although there is a great deal of mathematical formality, it is far less so than there would be in a more formal mathematical course in the subject. There are many great books of that genre, but their audience is quite a bit different. Indeed, no pure mathematician or financial engineer will likely give this book much praise, and that is of no concern. It is not an attempt to turn the student into a quant. What it does attempt to do, however, is to take a finance PhD student who in all likelihood is not going to specialize in derivatives and give that person the foundational layers that will pay off in a better understanding of the role that derivatives play in finance. This book does not incorporate the latest advanced mathematical knowledge. Its goal is more modest: to lay a solid foundation in a lean and efficient manner.

But enough about the book. Let us now begin to lay that foundation. A good place to start is to define a derivative.

1.2 WHAT IS A DERIVATIVE?

This book is about derivatives, so to get started we need to know just exactly what a derivative is. There is a very basic definition of a *derivative* that goes like this:

> *A derivative is a financial instrument in which the payoff is derived from the value of some other asset.*

The payoff is the unknown future cash payment required by the derivative contract. This definition works relatively well, but there are situations in which it breaks down. It does this by encompassing instruments that most people would not consider derivatives. For example, mutual funds and exchange-traded funds derive their values from the securities they hold, and most people would not consider them derivatives.

The Financial Accounting Standards Board, which is the accounting industry association that sets standards for financial accounting, has given us a good definition. In 1998, it created a new standard, which was called *FAS* (Financial Accounting Standard) 133 and is now called *ASC* (Accounting Standards Codification) 815, which laid out a new set of procedures for derivatives accounting. It provides a reasonably strong and more detailed definition, which is as follows:

> *A derivative is a contract with one or more underlyings and one or more notional amounts. Its value changes as the value of the underlying changes. Its initial value is either zero or an amount smaller than that required by other transactions to obtain the same payoff. At expiration it settles either by delivery or an equivalent cash amount.*

As it turns out, this definition works relatively well but does leave out a bit. Note that the definition says that a derivative is a *contract*. Let us not forget that a contract is a legal document and enforceable by law. Though not mentioned in the definition, a contract has two parties, so perhaps we should add that there are two parties, and we should probably identify them. One party is the buyer of the derivative, sometimes called the *long*, whose wealth benefits when the value of the derivative increases. The long or buyer is also sometimes known as the *holder*, in particular when the derivative is an option. The opposite party is called the *short*, whose position benefits when the value of the derivative decreases. The short is the seller of the contract and is sometimes called the *writer*, particularly if the derivative is an option. These parties to whom we have been referring are also sometimes called *counterparties*. When we prepend the word *counter*, we are implying a relationship. Each party is counter to the other. For the most part, however, we can use the terms *party* or *counterparty* interchangeably.

The *ACS* definition says that the derivative has an *underlying*, which can sometimes be an asset, such as a stock, bond, currency, or commodity, or it can be something else such as an interest rate.[6] It can even have more than one underlying. The statement says that the value of the derivative changes with the value of the underlying, but note that it does not specify whether the value changes linearly or not (we shall see that there are both linear and nonlinear cases). The statement also mentions that the initial value of the derivative is either zero or a smaller amount than the value of the underlying, which implies that derivatives have a tremendous amount of leverage. One either invests no money or a smaller amount than would be required to obtain the equivalent exposure in the underlying. And as mentioned, at expiration, the derivative settles up either by delivering the underlying or by having one party, the long or short, exchange an equivalent cash amount with the other party, the corresponding short or long.

The reference to a notional amount captures the requirement that a derivative is based on a certain amount of the underlying. This amount might be shares of stock, face value of bonds, units of currency, or, say, barrels of oil. We should also add that a derivative provides either a right or an obligation to engage in a transaction at a future date. Options provide a right, and forwards, futures, and swaps provide an obligation. We shall see what these points mean in later chapters.

So, let us now try to rewrite the definition of a derivative. We shall also do a little paraphrasing to keep the accountants at bay.

> *A derivative is a legally enforceable contract between two parties, the buyer or long and the seller or short, who has at least one underlying and a notional amount for each underlying. The value of the derivative changes with the value of the underlying(s). The initial amount of money one must put down to engage in the derivative contract is either zero or a smaller amount than required to obtain equivalent exposure in the underlying(s). When it expires, the derivative either settles by delivery of the underlying(s) or the parties exchange the cash equivalent.*

Alas, it is virtually impossible to give a one-sentence definition, but this four-sentence definition works quite well. As we get into the specifics of certain types of derivatives, we will have to add some features, but for now, we are set.

Now that we have defined a derivative let us go back and contrast it with the market for the underlying. Oftentimes, one would just call this type of market the stock market, bond market, currency market, or commodity market. It is common in derivatives lingo to call the market for the underlying the *spot market*. This term refers to transactions that are done "on the spot," meaning that one pays for it and receives it immediately.[7]

This procedure contrasts with derivatives, which always refer to transactions that will be conducted in the future.

So where are these things called *derivatives*? Well, they are created and traded in markets. There are two types of these markets, though the differences are becoming blurred. First, there are *exchange-listed derivatives*, sometimes called *listed derivatives*, which are standardized instruments that trade on exchanges. An exchange is an entity, usually in the form of a corporation or nonprofit, that provides a physical or electronic facility for trading. When we think of an exchange, we tend to think of something like the New York Stock Exchange, but derivatives exchanges also exist, such as the Chicago Mercantile Exchange and the Chicago Board Options Exchange. Also, derivatives can trade on exchanges that are more known as stock exchanges, such as Euronext and the Korea Exchange.

What we mean by the notion of standardized instruments is that the exchanges have decided on most of the terms and conditions, such as which underlyings will have derivatives available for trade, when these derivatives will expire, and how many units of the underlying are covered by a single derivative contract. There are also other terms that are germane to certain types of derivatives that are specified by the exchange. What the exchange does not specify, however, is the price, which is negotiated between the parties on each trade.

An exchange also has rules as to who can trade on the exchanges, and it provides clearing services, which references a system of bookkeeping that matches the counterparties and ensures that the money passes from one party to the other as appropriate. The exchanges also ensure that the parties that make money will always be paid, with the funds coming from the parties that lose money. The exchange guarantees that if the parties losing money cannot pay the parties making money, the exchange will cover through its clearinghouse. In this way, exchanges provide a guaranty that essentially eliminates credit risk.

The other type of derivative is the *over-the-counter*, sometimes called *OTC*, or customized derivative, which trades in an informal market. This type of transaction is essentially one between any two parties that is not conducted on an exchange. As such, the parties can customize the transaction with any terms and conditions they want, as long as they do not break a law. Corporations commonly create derivatives with their banks to manage various risks they face.

Derivatives markets, whether exchanges or over-the-counter, rely on a set of firms or individuals called *dealers*. Dealers stand ready to take either side of a derivative. They do this by quoting a bid price, the price they are willing to pay, and an ask price, the price they are demanding to sell. The ask is higher than the bid, so the dealer has a profit built into the quote. Thus, the investor has a built-in loss as the investor must buy high at the ask price and sell low at the bid. When a dealer takes on a transaction, it has acquired exposure; however, it does not generally carry that exposure, which would be risking its own survival on market direction. Instead, it lays off the risk by finding an offsetting transaction elsewhere in the market. Dealers are, thus, wholesalers of risk. They can easily do virtually any transaction in any market, quickly, and with low cost.

1.3 OPTIONS VERSUS FORWARDS, FUTURES, AND SWAPS

There are two general classes of derivatives that are distinguished by the fact that one class has payoffs that are linear in the underlying, while the other has payoffs that are nonlinear in the underlying. The former are forwards, futures, and swaps. Their payoffs are linearly related to the underlying in that the relationship between what the derivative pays when it expires and the value of the underlying is the same for any value of the underlying. Note that we are not saying that the payoff is the same for any value of the underlying. We are saying that the relationship is the same. This means that we can write the payoff function of the derivative as a linear function of the value of the underlying.

Derivative instruments whose payoffs are linear—forwards, futures, and swaps—are also known as *symmetric instruments*. A symmetric instrument is one in which the payoff for a given change in the underlying is the same in absolute value as the payoff if the underlying changes by the same magnitude but in the opposite direction. For example, if the underlying moves up by 5 and the derivative payoff is 10, then a move of −5 will result in a payoff of −10. The up-front cost to enter into a forward or swap is zero. You essentially pay for the large gains by bearing the risk of large losses.

Options compose the family known as *nonlinear derivatives*. For them, the payoff of the derivative and the value of the underlying are not related in a linear manner, so we cannot write the payoff function as a linear function. As we shall see, however, they are related in what we call a piece-wise linear manner, meaning that the relationship is linear over one range of the underlying but different over another. You can think of that description as a straight line with one slope connected to another straight line with a different slope. We shall get into the details of what these explanations mean in later chapters.

Derivative instruments with payoffs that are nonlinear—options and other option-like instruments—are also known as *asymmetric instruments*, because the payoffs are not symmetric. That is, the payoff for a given move in the underlying is not equivalent in absolute value to the payoff for the same move in the opposite direction of the underlying. So, for example, if the underlying goes up by 5 and the derivative pays 10, a move of −5 will not result in a payoff of −10 on the derivative. The cost to enter an asymmetric instrument is nonzero. You essentially pay for the large gains by paying the price, also called a *premium*, up front or you bear the risk of large losses by receiving this price or premium up front.

This book will start with options. In fact, most of the book is really about options, because as it turns out, options are far more difficult to model than are forwards, futures, and swaps. Everything we know about options will apply to forwards, futures, and swaps, but we shall not need to go to such mathematical lengths to derive our models for those instruments. For the most part, their models are obtained by simple present value calculations.

1.4 SIZE AND SCOPE OF THE FINANCIAL DERIVATIVES MARKETS

The derivatives market is massive in size and global in scope. Measuring the overall size of the derivatives markets, however, is a bit difficult. Generally, derivatives markets are broken down into the over-the-counter (OTC) markets and exchange-traded markets. The OTC markets are typically measured by either notional amount or gross market value. Recall the notional amount, or notional for short, reflects the characteristic that a derivative is based on a certain amount of the underlying. This amount might be shares of stock, face value of bonds, units of currency, barrels of oil, and so forth. Gross market value is the current absolute value of one side of the derivatives position. The overall gross market value of derivatives is zero because for every long position there has to be a short position. The exchange-traded derivatives markets is generally measured with either open interest or trading volume.

Each measure has its advantages and disadvantages. The notional amount is a number that grossly overstates the amount of money that is transacted or at risk. Derivative payments are based on the notional amount, but most derivatives do not involve paying the notional itself. The notional amount, however, is a fairly accurate number, because it is written into the contracts. Market value, however, must be estimated either from market prices, from educated guesses, or from using models. Yet, market value reflects more accurately the amount of money at risk. Most of this book is about estimating market value.

Figure 1.1 Panel A presents the notional amount of OTC derivatives outstanding approximately the decade before and after the global financial crisis of 2008–2009. Clearly, the global financial crisis starting in 2008 significantly affected the growth of the OTC market. The gross market value illustrated in Figure 1.1 Panel B shows a spike in December 2008 that reflects the significant gyrations in all world markets, and the subsequent declines indicate a move out of OTC derivatives. Figure 1.1 Panel C illustrates the exchange-traded open interest. The rise in recent years reflects the push by regulatory bodies for more centralized clearing in response to the global financial crisis. In summary, the size of the derivatives market, measured in hundreds of trillions of dollars, warrants further investigation.

Table 1.2 Panel A presents annual volume by region expressed as a percentage of total volume for a recent year based on information provided by the Futures Industry Association. North America and Asia Pacific have gone back and forth in ranking between number one and number two for a while, but the Asia Pacific region now dominates. Table 1.2 Panel B shows the volume by category for the same year. Equity and equity index derivatives dominate the rankings. Foreign exchange and interest rates are generally next in the rankings. Thus, the derivatives markets span the globe as well as span numerous underlying instruments.

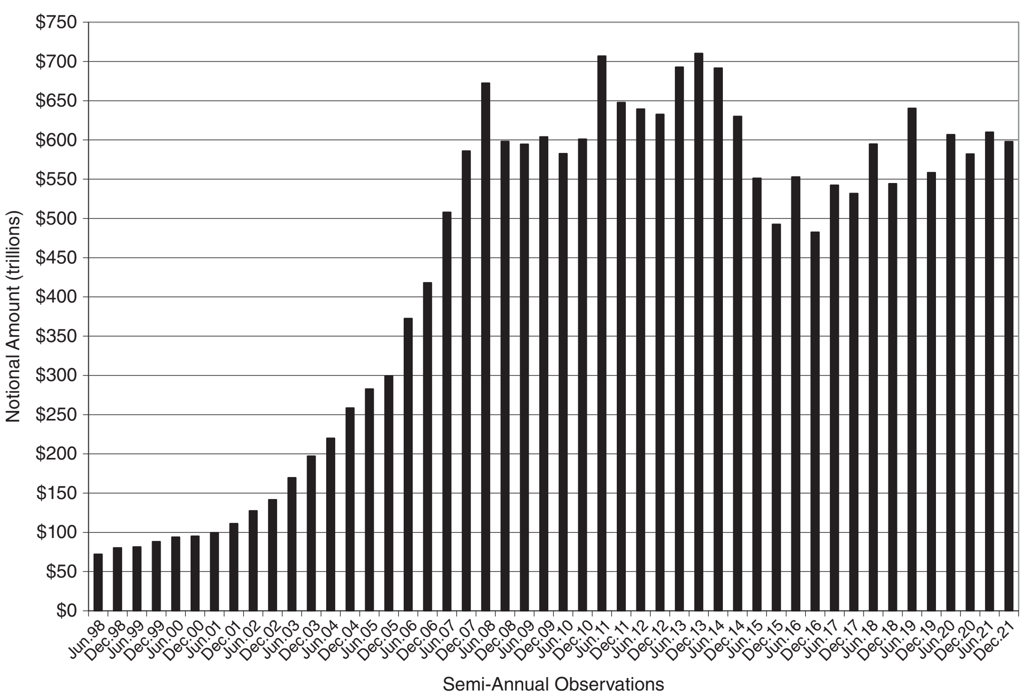

FIGURE 1.1 Panel A. Over-the-Counter Notional Amount Outstanding

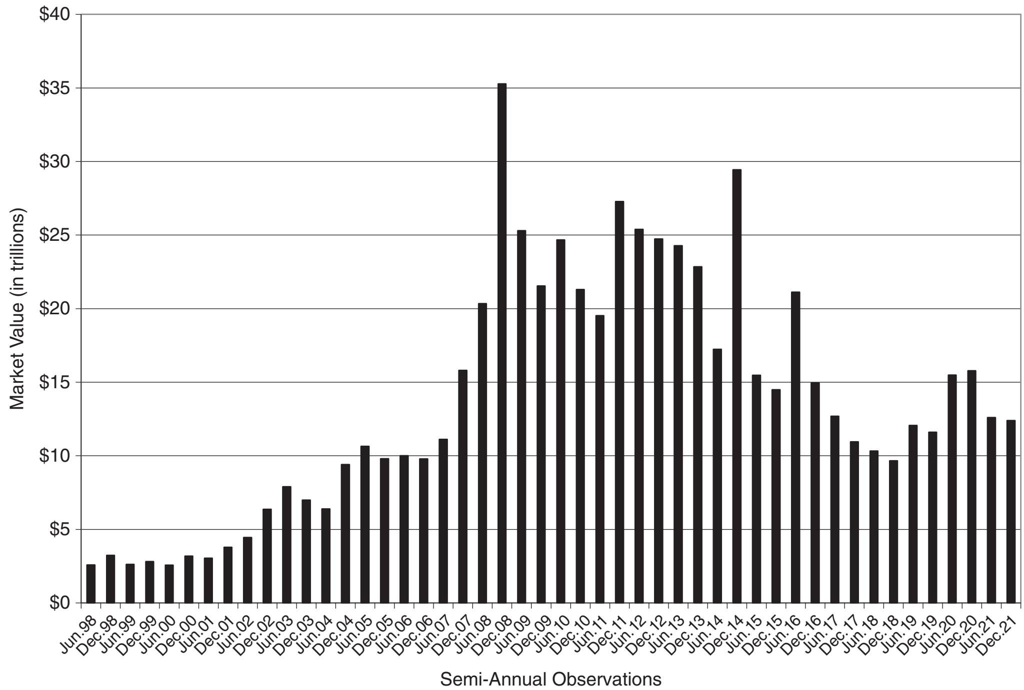

FIGURE 1.1 Panel B. Gross Market Value Outstanding

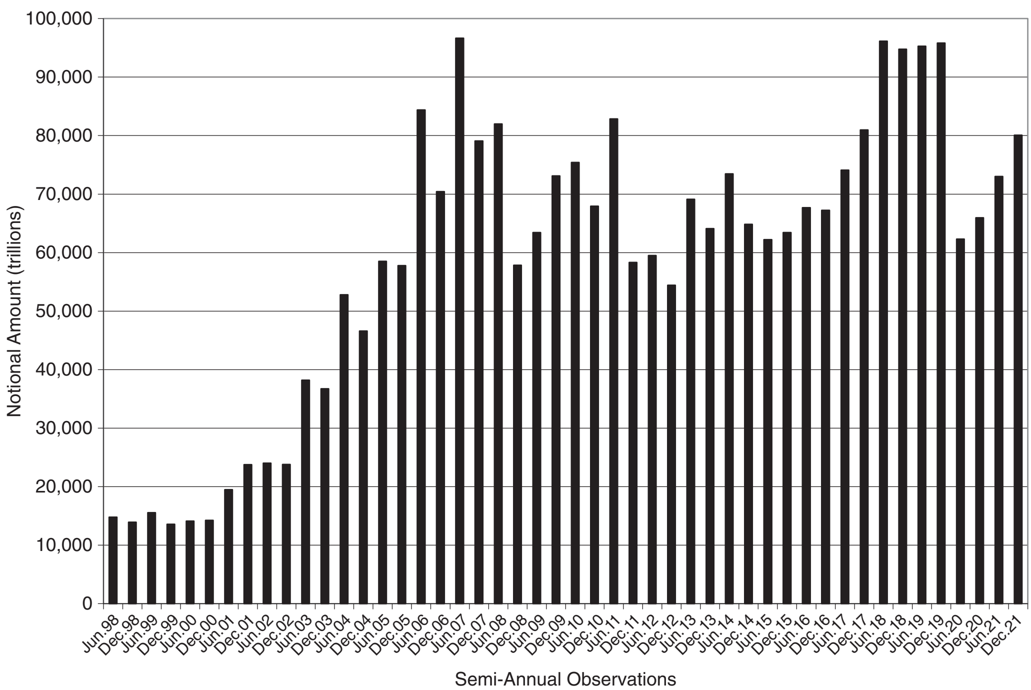

FIGURE 1.1 Panel C. Exchange-Traded Open Interest

TABLE 1.2 Panel A. Global Futures and Options Trading Volume Percentage by Region

Region	Trading Volume (%)
Asia-Pacific	60
North America	20
Latin America	10.2
Europe	5.7
Other	4.1

Note: Total volume = 83,847,697,472.

TABLE 1.2 Panel B. Global Futures and Options by Category

Category	Trading Volume (%)
Equity Index	58
Individual Equity	16
Currency	9.1
Interest Rates	6.1
Energy	2.5
Other	8.3

Note: Total volume = 83,847,697,472.

Thus, the derivatives industry is truly global and massive in size. Techniques to value and manage derivatives have now affected every facet of finance. Within the three broad categories of finance—investments, corporate finance, and financial services—mastery of financial derivatives valuation and management methodologies enhances one's capacities to solve financial challenges. Derivatives are so pervasive that it is common to find financial analysis problems that are more accurately understood from a derivatives perspective. For example, in project finance, you often find embedded options, such as the option to terminate a project or extend it. In investments, you often encounter embedded options in callable and convertible instruments. In fact, a common stock can be analyzed as a call option on the firm. Finally, banks often deal with implicit options, such as from loans that can be prepaid to deposits containing early withdrawal possibilities. Thus, if you hang out in the derivatives world for a while, you will also find it hard to name a finance issue that does not benefit from derivatives knowledge.

1.5 OUTLINE AND FEATURES OF THE BOOK

This book is divided into seven parts, each of which contains between two and seven chapters. Beyond this first chapter, the structure is as follows:

Part I: Basic Foundations for Derivative Pricing

Chapter 2: Boundaries, Limits, and Conditions on Option Prices

Chapter 3. Elementary Review of Mathematics for Finance