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New Frontiers in Gravitational Collapse and Spacetime Singularities

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Preface

Relativistic gravitational collapse provides a theoretical framework within the general theory of relativity (GR) to explain how space-time singularities and black holes originate from massive astrophysical objects. It may be argued that relativistic gravitational collapse lies at the foundation of black hole physics and our belief that such objects exist in the universe. In fact, the 2020 Nobel Prize in Physics was awarded, among others, to Roger Penrose for his work on space-time singularities that originate from gravitational collapse.

The first analytical model for relativistic collapse was proposed in 1939 by J. R. Oppenheimer and H. Snyder. They studied how pressureless, homogeneous matter collapsing under its own gravity produces what they called a 'frozen star'. Around the same time, B. Datt obtained the same results independently, and today the analytical solution for homogeneous dust is generally known as the Oppenheimer-Snyder-Datt (OSD) model. These works show how a matter cloud may collapse continually under its own gravity to form a black hole. However, at that time, space-time singularities and horizons were not yet fully understood and thus the final fate of such a collapse was not clear. In such a scenario, Oppenheimer and Snyder used the term 'frozen star' to describe the collapsing object as it reaches its final stages, as they did not make use of a coordinate system that allows us to see how particles cross the horizon and what happens afterwards.

In fact, the very nature of the event horizon in the Schwarzschild solution was not yet clear when the OSD model came to light. It was D. Finklestein in 1958 who showed that the horizon is a 'one way membrane' that can be crossed only in one direction. Around the same time, people also realised that the continued gravitational collapse of the OSD model produces a curvature singularity at the centre of symmetry of the system in a finite proper time. Similar singularities were studied and found in cosmology also, which identified with the origin of the universe. The question then arose whether such singularities were generic in gravitational collapse and cosmology within the framework of general relativity. The expectation was that singularities would disappear once restrictive constraints such as spherical symmetry were relaxed.

Instead, as it often happens, nature turned out to be more nuanced than expected. R. Penrose, S. Hawking and R. Geroch proved that space-time singularities in gravitational collapse and cosmology are inevitable, within GR, when certain natural physical conditions are met. When black holes form in gravitational collapse, they hide the curvature singularities at their centre. This led to a variety of new research directions aimed at answering questions such as

- Can the singularities forming at the end of collapse in GR be visible to far away observers?
- How is the singularity formation scenario affected by modifications of GR?
- Do quantum effects ensure that singularities do not form?
- What are the properties of the final endstate of collapse in these modified scenarios? Are they distinguishable from black holes?

Susequently, detailed analytical and numerical models for gravitational collapse were developed, which have been at the heart of modern black hole physics. Formation of event horizons as well as visibility or otherwise of the space-time singularities have been examined. For decades these have been fruitful areas of research on many fronts, ranging from astrophysics to the search for a theory of quantum gravity, from exact solutions in GR and alternative theories to numerical simulations.

Today more than ever analytical and numerical models of gravitational collapse provide an ideal tool to probe into the nature of Einstein's equations, explore the limits of the theory and the implications for astrophysical phenomena and for quantum-gravitational effects.

Many alternative ideas, improvements and modifications of the OSD model have been proposed. For example, from the point of view of astrophysics, the study of collapse of inhomogeneous fluids with astrophysically relevant equations of state provides insights into how black holes or singularities not covered within horizons may be born from massive dying stars. On the other hand, from the point of view of fundamental aspects of gravity, the study of collapse of scalar fields and attempts towards a quantization of collapse models provide insights into the features that a viable theory of quantum gravity should have.

Such new models help us towards a better understanding of gravity in extreme regimes and potentially may have consequences for astrophysical phenomena and experimental searches of quantum-gravity signatures.

In addition, recent experimental results, such as the detection of gravitational waves from binary black hole inspirals, and the first images of the shadow of supermassive ultracompact objects at galactic centres, prelude to the possibility of testing the validity of the various theoretical proposals that have been put forwards over the years. These are exciting times and there are still many unanswered questions that are driving research forward. The aim of this book is to take the pulse of the current state of research in gravitational collapse, the issue of formation and causal nature and structure of space-time singularities. Our hope is that it will be useful for new researchers starting to work in the field as well as experienced researchers interested in reviewing the current research trends.

Astana, Kazakhstan — Daniele Malafarina
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Chapter 1
After Collapse: On How a Physical Vacuum Can Change the Black Hole Paradigm

Julio Arrechea, Carlos Barceló, and Valentin Boyanov

1.1 Introduction

General Relativity (GR) is the best theory we currently have for describing gravitational phenomena. It is successful not only in terms of fitting observational data extremely well [118], but also due to the elegance of its formulation and the depth of its implications, which have given rise to mathematical and philosophical concepts that are still being developed to this day (see e.g. [32, 48]). However, it is generally accepted that GR is not the final theory of gravity. Much like Newtonian gravity before it, GR has limits in its range of applicability. This becomes apparent in problems which involve either strong gravitational fields on small scales (i.e. large densities and curvatures, or configurations involving black-hole horizons), or large-scale astrophysical and cosmological structures (i.e. the dark energy and, potentially, dark matter puzzles).

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For strong fields in compact regions—this being the topic of the present chapter—the problem comes from the seemingly inevitable appearance of singularities when matter is compressed beyond certain limits [91, 105], at least when this matter satisfies certain classical energy-positivity conditions [57]. However, neither singularities themselves, nor a purely classical description of matter at such densities, can be considered physically reasonable. Indeed, the quantum nature of matter is generally thought to be directly related to the potential resolution of the singularity issue, along with a possible quantum behaviour of spacetime itself.

With these general considerations in mind, it is tempting to search for a new theory of gravity which is valid in these strong-field regimes. One possibility is to attempt to directly work out a quantum gravitational theory from first principles; examples of this are the Loop Quantum Gravity and String Theory programs. However, thus far these approaches have had major difficulties in making a clear connection between theory and phenomenology. A more conservative approach involves the search for improved theories of gravity that retain the idea of a classical spacetime geometry as an appropriate effective notion. In other words, maintaining an effective metric as one of the dynamical variables to be solved for, and only changing the dynamical equations it satisfies.

Taking the latter approach, however, suggests keeping in mind its potential limitations, given that a field theory with no pathologies whatsoever is an unlikely outcome. For instance, the Einstein field equations are rather unique in that they do not by themselves generate shockwaves or, in more generic terms, weak solutions (these only appear when the source fields themselves produce them [99], or when a full-blown curvature singularity is approached [75]). Modifying the Einstein field equations might achieve improved behaviours in certain situations, such as a removal of the formation of singularities in simple models of gravitational collapse, but this can potentially come at the cost of pathological behaviours in other situations. Within this effective spacetime perspective, the philosophy is therefore that of finding progressive improvements which are also compatible with previous successful theories, but not that of finding the *final* dynamical theory for spacetime. Ultimately, the success of these improvements is then to be judged by how they accommodate the observed phenomenology.

The standard approach in modified gravity searches is to analyse sets of possibilities in theory space and classify them based on the phenomenology they present. The large variety of observed and measured gravitational phenomena then allows for the direct elimination of a substantial amount of these theories, while others have their free parameters constrained. Given the vastness of theory space, to even begin such an analysis calls for some formal or physical arguments which allow for a selection of a specific theory or set of theories. Different researchers have different tastes and criteria for such a selection, resulting in the dendritic exploration of theory space currently being carried out [28, 82, 96].

New theories can be broadly categorised according to two non-mutually-exclusive structural features: (a) those which modify how matter behaves in strong gravity situations while retaining the form of the Einstein equations, or (b) those which modify directly the gravitational equations, without necessarily modifying the matter sources. Within the latter set, proposals can be found that change the geometry significantly only in the surroundings of would-be classical singular regions, and others that can lead to alterations of the geometry well beyond such regimes. There are numerous interesting proposals out there, some of them discussed in the chapters of this book.

This chapter in particular is centred around one modification of GR which can be considered as part of category (b). The idea is the following: we assume that the zero-point fluctuations of the quantum fields permeating spacetime gravitate by some amount determined by the very deviation from flatness of the spacetime, generally in a non-local manner (we note that at this stage we will neglect the possible presence of a global contribution in the form of a cosmological constant); then, even in regions where no classical matter exists, there would be an average vacuum energy (as well as pressure, fluxes, etc.) which would be a source of gravity. As this new source ultimately depends on the geometry, we can interpret it as part of the geometric side of Einstein equations, maintaining that the only real source on the right-hand side is the classical stress-energy tensor. An alternative way of describing this approach is by saying that the standard vacuum used in GR is, in a sense, "too empty"; this approach instead aims to characterise a more physical vacuum as the stage on which all gravitational phenomena take place. Before we describe this theory in more detail, let us first make a brief digression and discuss the notion of a theory of quantum gravity.

It is clear that Nature has a way of melding together both the quantum behaviour that we observe for matter in the atomic and subatomic regimes and the gravitational behaviour that we observe in the macrocosm. A theory of quantum gravity would be a model that consistently incorporates both of these behaviours, furthering our understanding of each. Thus far, we have had no clear indication of how these two regimes come together; however, theoretical considerations do provide some clues. For instance, in the direction of *how gravity affects matter* there are strong indications that the causality provided by a geometric gravitational description affects the behaviour of quantum matter (see for example [81]). Under this hypothesis, analyses of quantum field theory over fixed curved background spacetimes have lead to some of the most important results in modern theoretical physics. On the one hand, the idea that a quantum particle is an observer-dependent notion leads to analysis of particle production in cosmology [55, 87], and subsequently to the idea of how an inflationary regime could lead to a primordial spectrum of fluctuations in the early universe [80, 108]. On the other hand, a calculation of particle production associated to the formation of a black hole (BH) lead Hawking to his famous result that BHs should evaporate [58, 59].

In the opposite direction of *how matter affects gravity*, GR tells us that the average effect of the stress-energy contained in a macroscopic lump of matter is to bend causality in particular, well-defined ways. However, we know nothing of how a single quantum lump of matter affects causality, or even whether a classical notion of causality would be valid at these scales. In fact, it would not be surprising if it turned out that gravity as we know it is not a relevant notion until matter starts behaving classically due to its aggregation [93, 94]. Indeed, such regimes are where

GR is less known and less clearly tested, to say nothing of when matter enters into full Planckian-density regimes. Putting aside the precise resolution of these issues, it is at least reasonable to believe that there is a regime in which an effective classical spacetime makes sense, with curvature being sourced by the average energy contained in quantum states. Then, aside from the intrinsic excitations of quantum fields, which we usually define as matter, one must also take into account the energy which can be generated by the very presence of curvature in the spacetime. Since on large scales matter can, for the most part, be described classically, in this regime the total stress-energy tensor in the Einstein equations should contain a classical contribution, where actual matter is localised, plus another source term taking into account the average energy of the vacuum fluctuations (as mentioned before, this last term can also be moved to the geometric side of the equations). This description would be self-consistent whenever the fluctuations of the energy around the average (or expectation value) are small [68], as one might expect to be the case for a vacuum state in standard situations, or in quasi-classical states.

When performing calculations within standard relativistic quantum field theory in Minkowski spacetime, there are no observables that couple to an absolute notion of energy—they only couple to differences of energy between different states. Thus, one can always subtract from all energy measurements an arbitrary reference value. In fact, a first naive calculation indicates that any quantum state has an infinite energy, as is shown in any introductory text on the subject [95]. However, there is the freedom to renormalise the energy of all states in such a way that the lowest energy state, i.e. the vacuum state, has zero energy. By subtracting the "same" infinite value from the energy of all states, we obtain finite definite values for finite-particle states (with respect to the chosen zero point). This can be done easily in flat spacetime because there is a unique natural notion of a vacuum state (the so-called Minkowski vacuum), and subtracting its expectation values from other states is a procedure which preserves the symmetry of the background.

However, the theory changes in an essential way in the presence of spacetime curvature, as there is no longer a natural notion of vacuum state. Indeed, extensions of the flat spacetime theory only lead to conclude that different observers can have a different perception of what the true vacuum state is. There is generally no longer a "vacuum" state which can be used for renormalisation by subtraction, as such a procedure would break the symmetries of GR. However, given that gravity is sensitive to the total energy of a state, rather than just energy differences, renormalisation is of particular importance. Extreme care is required in selecting how and what to subtract in search of sensible results.

With these observations in mind, the requirements for renormalisation indeed change quite a bit. Particularly, one preferably needs a local subtraction prescription which does not make use of knowledge of either the global structure of spacetime, or of the particular vacuum and particle states chosen for quantisation. Prescriptions of this sort do exist, but they generally leave a non-homogeneous, finite vacuum energy residue, which depends on the characteristics of the spacetime and of the chosen vacuum state [27, 114]. The subtraction can be made consistent with recovering asymptotic Minkowski spacetime when going arbitrarily far from the classical sources of curvature, or it can be made to leave an offset in the form of a cosmological constant. As in the present chapter we are not considering cosmological scenarios, we will neglect such terms, and set the cosmological constant to zero.

The central hypothesis of this chapter is that the theory of classical matter plus vacuum fluctuations sourcing gravity is applicable in the astrophysical scenarios we analyse below. Formally, this theory is constructed with the effectively classical stress-energy tensor (SET), and the residual vacuum energy, or zero-point fluctuations, of the quantum fields as sources in the Einstein field equations,

$$G_{\mu\nu} = 8\pi G(T^{\rm C}_{\mu\nu} + T^{\rm ZP}_{\mu\nu}), \tag{1.1}$$

were $T^{\rm C}_{\mu\nu}$ is the classical and $T^{\rm ZP}_{\mu\nu}$ the zero-point stress-energy source term. Equivalently, from a modified gravity perspective the equations can formulated as

$$G_{\mu\nu} - 8\pi G\, T^{\rm ZP}_{\mu\nu} = 8\pi G\, T^{\rm C}_{\mu\nu}. \tag{1.2}$$

These formal equations of motion constitute what is commonly referred to as *semi-classical gravity*. Indeed, the presence of a non-trivial zero-point SET is one of the most soundly motivated modifications of gravity we currently have. Though we have no direct observational evidence of this gravitational vacuum polarisation, we can straightforwardly make an analogy with the case of electromagnetism, where we know that a similar notion of charge polarisation does exist [70].

Up to this point we have talked about vacuum energy in a deliberately vague manner. If we knew the exact expression of $T^{\rm ZP}_{\mu\nu}$ in terms of simple analytic functions of the metric, we would just have to solve the new system of gravitational equations and analyse its phenomenology, testing against observations. However, things are not so simple, as we do not possess an indisputable prescription for calculating $T^{\rm ZP}_{\mu\nu}$ in generic scenarios. The best reasoned method we have for obtaining an appropriate $T^{\rm ZP}_{\mu\nu}$ is through the expectation value of a SET operator for different fields in a given vacuum state,

$$T^{\rm ZP}_{\mu\nu} = \langle \Psi_0 | \hat{T}^{\rm QF}_{\mu\nu} | \Psi_0 \rangle. \tag{1.3}$$

Ideally, $T^{\rm ZP}_{\mu\nu}$ would describe the SET operator associated with the complete standard model of particle physics, with all its interacting fields, together with any yet undiscovered fields, such as the possible constituents of dark matter. Additionally, $T^{\rm ZP}_{\mu\nu}$ would incorporate any fluctuating energy offset that might be contained within the gravitational field itself. Neither of these idealised requirements is realistically feasible as of yet. On the one hand, given the absence of a theory of quantum gravity, we are bound to hope that the gravitational contribution to these fluctuations is small enough to be negligible with respect to the ones associated to the standard model fields.[1] On the other hand, even the standard model zero-point SET is nearly

[1] It has been suggested that this assumption becomes more accurate when the number of quantum fields N is sufficiently large [1, 27].

impossible to calculate. Such difficulty arises from two fundamental aspects: (a) it is unknown how to treat interacting field theories beyond the S-matrix perturbative approach; (b) as the SET operator contains products of field operators at the same point, it is not a well defined operator in the quantum theory; therefore, to make sense of this object one has to resort to regularising and renormalising this expectation value, which poses its own difficulties in curved spacetimes. Faced with these problems, as a proxy to the qualitative form that $T_{\mu\nu}^{\rm ZP}$ might have, researchers have opted for calculating the renormalised SET (RSET) for free field theories (often as a one-loop approximation to interacting theories) in simple backgrounds. Among the test fields useful to understand semiclassical gravity, the free scalar field is the simplest, and indeed the most used in past and present literature. The term *semiclassical gravity* is typically used to refer to any theory that incorporates the effect of vacuum fluctuations, even in these simplified test field scenarios.

Even calculating the RSET of a free scalar field in simple, highly-symmetric geometries is not a trivial task; in fact, it can typically only be done numerically and with great difficulties. The complete problem of solving the semiclassical Einstein equations self-consistently and exactly is therefore not feasible at present. This problem has a large body of work addressing it, and we will briefly and non-exhaustively review it in the next section. For now let us just say that given the difficulties mentioned, there are essentially two strategies that can be adopted: (1) to develop incremental improvements in the method of calculating the RSET, or very close approximations thereof, and study its effects in physically relevant situations in an approximate perturbative manner; (2) to prescribe less precise but analytically simpler approximations to the RSET, such that one can readily investigate more complicated and realistic geometric situations, understanding meanwhile that the information extracted only provides a qualitative preview of what a full exact solution may look like. In the following sections of this chapter we present an application of the second strategy to gravitational collapse and compact object geometries.

Particularly, we will present work on two closely related physical problems. Firstly, we will address the question of whether semiclassical gravity allows for qualitatively different configurations of stellar equilibrium as compared to those present in GR. Then, we will look into the problem of gravitational collapse with a bit more scrutiny under a semiclassical lens. In other words, we will present results for both (meta-)stable configurations of semiclassical gravity which allow the existence of stellar objects of higher compactness than their classical counterparts [9, 10], as well as for a revised dynamical process of gravitational collapse which may lead to their formation [15, 17, 18].

Even considering the approximations and hypotheses involved, our philosophy with these analyses is to progressively build up and develop a modified gravitational model which is consistently treatable and comparable with GR on an equal footing. In this endeavour, both of the above-mentioned strategies can greatly contribute. While such a theory would by no means be a complete and exact treatment of matter at high densities (due to the approximations involved, as well as due to neglecting the fact that even classical matter at such densities would likely behave differently from what is known), it is our best attempt to push gravity onto the next stage of development.

The outline of this contribution is the following. In the next section we briefly review the status of research into calculating the RSET. Then, Sect. 1.3 will describe two qualitative approximation to the RSET which are analytically manageable: the Regularised Polyakov approximation and the Order-Reduced Anderson-Hiscock-Samuel tensor. Armed with these two RSETs, in Sect. 1.4 we will present an analysis of how the vacuum-induced changes in the equations of hydrostatic equilibrium lead to new families of stellar solutions absent in GR. Subsequently, in Sect. 1.5 we will analyse the dynamics of gravitational collapse, paying close attention to possible modifications to the standard picture due to horizon-related effects. Finally, we will summarise our findings and conclude with some final remarks.

1.2 Semiclassical Gravity: In Search of an Appropriate RSET

Within the semiclassical approach, the central and most important problem is the search for non-ambiguous and feasibly calculable RSETs. At the present stage, calculations are usually performed for simple test fields, as they provide an invaluable glimpse into the potential behaviours hidden within the semiclassical theory. Let us consider in particular the free scalar field. This is typically used in the literature [68] as the starting point for considering more complicated fields and interactions, which, while bringing about additional contributions, are not expected to lead to fundamental changes in the vacuum dynamics of semiclassical systems. For instance, particle creation processes in cosmology and gravitational collapse occur in a qualitatively robust way for a variety of different fields [27, 53, 85], showing that test-field semiclassical analyses suffice for a qualitative analysis, and even for some quantitative estimates [58]. Hereafter, the tool used for the entirety of the discussion will be the scalar field.

In general terms, we can define the RSET of a scalar field in some vacuum state as the result of applying a regularisation and renormalisation procedure $\mathscr{P}$ to the ill-defined object $\langle 0|\hat{T}_{\mu\nu}|0\rangle$. Symbolically we can write

$$\langle \hat{T}_{\mu\nu}\rangle_{\text{ren}} := \mathscr{P}\left[\langle 0|\hat{T}_{\mu\nu}|0\rangle\right] \quad (1.4)$$

Several regularisation techniques have been developed in the literature to find expressions for $\langle \hat{T}_{\mu\nu}\rangle_{\text{ren}}$. Examples of that are: covariant point separation (or point-splitting) [43, 44]; Hadamard regularisation [33]; dimensional regularisation [29]; Riemann ξ-function regularisation [60]; Pauli-Villars regularisation [90]; proper time [104] and adiabatic regularisation [89]. The choice of an appropriate method depends on the particularities of the system under consideration, but the underlying logic is common in all of them: the subtraction of local divergences is carried out in a way which preserves general covariance. Under some reasonable assumptions, it turns out that different regularisation procedures lead to RSETs that differ at most in locally constructed conserved quantities, as proven by Wald [115, 116]. After the divergent terms have been subtracted, one can find a well-defined RSET with the appropriate physical characteristics expected from a source term in the Einstein equations [114].

The task of identifying divergent terms in generic spacetimes so that they can be subtracted was completed by the end of the seventies [44]. Nonetheless, the problem of how to calculate the remaining finite terms in the most efficient and accurate manner has remained. The expectation values involved are constructed through a spectral decomposition of the field, as quantisation and the definition of a vacuum state themselves rely on choosing and obtaining a specific basis of modes which satisfy the field equation of motion. It is at this stage where we encounter the main difficulty of the problem: field modes cannot be calculated in closed form for most of the spacetimes of interest, let alone for generic spacetimes.

Given this situation, the approaches used for calculating the RSET are split into two categories: on the one hand, there are those which look for appropriate approximation schemes that are as accurate as possible, and on the other, those which implement progressively more efficient numerical schemes. For instance, the first methods used to calculate the RSET in Schwarzschild spacetime made use of Wentzel-Kramers-Brillouin (WKB) approximations to the modes. The WKB based method was originally devised in [35, 66, 67], and has been improved over the years [2]. The epitome of applying this method can perhaps be found in the work by Anderson-Hiscock-Samuel [3]. They were able to give expressions for the RSET in arbitrary static, spherically-symmetric spacetimes in the Hartle-Hawking vacuum state (there are results also for the Boulware vacuum state in [71]) for a scalar field with arbitrary mass and coupling to curvature. The RSET they found is split into two terms, each of which is conserved separately. The first has an analytic closed form, and the second is in general only obtainable numerically. Additionally, they showed that for massless fields the analytic part on its own constitutes a good approximation to the total RSET. In fact, for the conformally invariant field this approximate RSET was found before by Page [84] and later on by Frolov and Zelnikov [54].

However, this analytical approximation to the RSET, which we will refer to as the AHS-RSET, has a number of shortcomings for its implementation in the field equations of gravity. Chief among them is the fact that it depends on up to fourth order derivatives of the metric functions. Firstly, this makes the solutions of the semiclassical Einstein equations depend on too many boundary conditions, with no clear physical interpretation (such as initial momentum, reflective boundaries, etc.). Secondly, when analysing self-consistent solutions of the system of equations, spurious solutions connect to seemingly tame initial conditions, making it so that even Minkowski spacetime can be unstable under small perturbations, as was found even before this approximation to the RSET was obtained [65]. This is reminiscent to what occurs with the Lorentz-Abraham-Dirac self-force of classical electrodynamics, which contains self-accelerated solutions [101, 102]. In that case we can track the origin of these solutions and improve the system of equations introducing some integro-differential operator [101], or applying an order-reduction procedure [52, 106].

Another issue with the AHS-RSET is that the WKB expansion it relies on breaks down at the horizon of BH geometries. This results in the approximate RSET having logarithmic divergences at the horizon, present even in the supposedly regular Hartle-Hawking vacuum state. These divergences have been shown to disappear when one adequately adds the numerical part of the tensor, as outlined in [3] (alternatively, one can also deal with them by treating the first modes of the expansion separately [13]). However, these issues make it so that using the AHS-RSET directly to find self-consistent solutions is rather complicated and the results are somewhat untrustworthy. Nonetheless, certain physical scenarios do allow for the approximation as such to be useful, such as for the wormhole solutions found in [62].

In more recent times, two new methods for obtaining the RSET have been devised. One is the so-called *pragmatic mode-sum method* pioneer by Levi and Ori [73]. The other is the *extended-coordinates method* proposed by Taylor and Breen [109]. The first is a completion and generalisation of a method developed by Candelas [35]. It does not make use of WKB expansions in the corresponding Euclidean sector. In fact, it does not use WKB expansions at all, since in the Lorentzian sector high-order WKB approximations are very cumbersome to use, as they involve using asymptotic matching techniques to approximate the modes at turning radii at which the approximation breaks down. Instead, the idea is to construct generalised integrals in frequency which directly incorporate a subtraction of the divergences (based on high frequency information in Christensen counterterms [44]), in such a way that the convergence of the integral is efficient. The advantage of this method is that it can be applied to dynamical situations (e.g. Hawking evaporation) and also to axisymmetric configurations (e.g. in [72] it was used to calculate the RSET on a Kerr background).

The extended coordinate method [109, 110] is performed in the Euclidean sector, and so it is not suitable for dynamical configurations. It introduces some new useful coordinates to decompose the Hadamard parametrix into multipoles and Fourier components. This allows for a mode by mode subtraction in a manner that results in a numerically very efficient algorithm. Summing up a few tens of modes gives quite accurate results and the method can be applied equally well to higher dimensional spacetimes [31, 78, 79, 111].

These developments are part of an exciting progress trend and motivated on excellent grounds. Due to recent computational advances, it is expected that progress in the efficiency of calculating the RSET in scenarios of greater phenomenological interest will carry on. Nonetheless, it is difficult to foresee how these RSETs could be used to search for self-consistent solutions. The main obstruction here owes to the complexity of simultaneously finding the geometry of spacetime and the field modes propagating on (and being sources of) that very same spacetime.[2] For this reason, we believe it is worth considering approximations to the RSET which, despite being

[2] In this regard, we could aim for developing efficient grid-search algorithms that converge to a metric satisfying the semiclassical equations, but even this possibility seems to us computationally discouraging.

less accurate, prove better suited for finding self-consistent solutions in semiclassical gravity.

1.3 Approximate RSETs

Analytic approximations to renormalised stress-energy tensors are all based on a similar rationale of finding trade-offs between the functional complexity of these RSETs and the accuracy of the physics they encode. As mentioned above, we will use a free scalar field throughout the remainder of the chapter. This field obeys the equation of motion

$$\Box\phi - \left(m^2 + \xi R\right)\phi = 0, \tag{1.5}$$

where $\Box$ is the d'Alembertian operator, m is the field mass, and ξ the coupling to the Ricci scalar R. A commonly used method for obtaining analytic, approximate expressions for the RSET is based on fixing the field parameters (the mass and curvature coupling) and restricting analyses to particularly simple spacetimes. For instance, for conformally invariant fields ($m = 0,\ \xi = 1/6$) on conformally flat backgrounds, the RSET is determined by the local trace anomaly [33, 84]. This leads to the existence of explicit analytic expressions for the RSET in a variety of situations, such as Friedmann-Lemaître-Robertson-Walker cosmologies and stellar interiors of constant density [88] (for the particular quantisations which are formulated in accordance with the conformal symmetry), and even for fields of higher spin [54].

However, for the situations we will analyse below, such as the formation and subsequent evaporation trapped regions or the existence of ultracompact stellar configurations in equilibrium, spacetime is not conformally flat. In such cases the RSET includes contributions which are non-local in curvature, and that depend on the vacuum state under consideration. The simplest RSET that captures these state-dependent effects in spherical symmetry is the Polyakov approximation, which incorporates the essential features of the propagation of a massless minimally coupled scalar ($m = 0,\ \xi = 0$) in four spacetime dimensions via two-dimensional model, described by the line element

$$ds^2_{(2\text{D})} = -\mathscr{C}(u, v)dudv, \tag{1.6}$$

where $u = t - r^*,\ v = t + r^*$ are radial null coordinates, with r^* the tortoise coordinate obtained by integrating $dr^*/dr = [h(r)/f(r)]^{1/2}$. The analogy between scalar field propagation in four and two spacetime dimensions becomes clear when one expands the field in spherical harmonics and restricts the analysis to the $l = 0$ (or s-wave) mode, which typically dominates long-distance effects. By considering the propagation of the s-wave over BH spacetimes and taking the near-horizon limit in the wave equation (1.5), the part of the equation that can be identified as a gravitational potential vanishes, and the (t, r) sector reduces to the two-dimensional free wave equation

$$\partial_u \partial_v \phi = 0. \tag{1.7}$$

This equation is manifestly conformally invariant, and admits an analytic basis of solutions in the form of plane waves. In fact, there are infinitely many such bases, one for each pair of possible null coordinates, and each can be used for performing quantisation [19]. Using any one of these quantisations, the two-dimensional RSET can be obtained in closed analytic form [47], in particular,

$$\begin{aligned}
\langle \hat{T}_{uu} \rangle^{(2D)} &= \frac{1}{24\pi}\left(\frac{\mathscr{C}_{uu}}{\mathscr{C}} - \frac{3\mathscr{C}_u^2}{2\mathscr{C}^2}\right), \\
\langle \hat{T}_{vv} \rangle^{(2D)} &= \frac{1}{24\pi}\left(\frac{\mathscr{C}_{vv}}{\mathscr{C}} - \frac{3\mathscr{C}_v^2}{2\mathscr{C}^2}\right), \\
\langle \hat{T}_{uv} \rangle^{(2D)} &= \langle \hat{T}_{vu} \rangle^{(2D)} = -\frac{R^{(2D)}}{96\pi}\mathscr{C},
\end{aligned} \tag{1.8}$$

where $R^{(2D)}$ is the two-dimensional Ricci scalar. The difference between the results obtained for the modes corresponding to one pair of null coordinates or another, i.e. between different choices of vacuum state, comes in the form of uu and vv flux terms,

$$\begin{aligned}
\langle \hat{T}_{uu} \rangle^{(2D)} &\xrightarrow[\text{vac. change}]{} \langle \hat{T}_{uu} \rangle^{(2D)} + \langle :\hat{T}_{uu}: \rangle, \\
\langle \hat{T}_{vv} \rangle^{(2D)} &\xrightarrow[\text{vac. change}]{} \langle \hat{T}_{vv} \rangle^{(2D)} + \langle :\hat{T}_{vv}: \rangle, \\
\langle \hat{T}_{uv} \rangle^{(2D)} &\xrightarrow[\text{vac. change}]{} \langle \hat{T}_{uv} \rangle^{(2D)}
\end{aligned} \tag{1.9}$$

The terms $\langle :\hat{T}_{\mu\nu}: \rangle$ incorporate all the dependence on the state in the RSET. For instance, in the case of BHs, when one switches between the Boulware and Unruh states, they contain the fluxes across horizons responsible for the phenomenon of Hawking evaporation [51]. They are obtained through the Schwarzian derivative between the null coordinates which encode the different quantisations [49, 51].

Having chosen a particular quantisation and obtained the two-dimensional RSET, the next step in the Polyakov approximation consists in defining a four-dimensional RSET from the components (1.8) through the relations

$$\langle \hat{T}^{\mu}_{\nu} \rangle^{\mathrm{P}} = \frac{F(r)}{4\pi}\delta^{\mu}_{a}\delta^{b}_{\nu}\langle \hat{T}^{a}_{b} \rangle^{(2D)} + (T_{\mathrm{AC}})^{\mu}_{\nu}, \tag{1.10}$$

where Greek and Latin indices take 4 and 2 values, respectively, P stands for Polyakov RSET, $F(r)$ is a radial function that up-scales the tensor to four dimensions, and $(T_{\mathrm{AC}})^{\mu}_{\nu}$ is a term which encodes angular pressures not contained in the two-dimensional theory. The standard Polyakov approximation is obtained with the choice $F(r) = 1/r^2$, which mimics the four dimensional behaviour of spherical modes, and $(T_{\rm AC})^{\mu}_{\nu} = 0$. This already suffices to reproduce the behaviour of the RSET at BH horizons [46, 51], and is generally expected to work well far away from the origin.

The Polyakov RSET has been extensively used to study the semiclassical backreaction problem in a variety of situations, from dynamical BH formation and evaporation to static stars in equilibrium [37, 42, 50, 86]. The fact that this RSET is analytic, simple and has only up to second derivatives of the metric functions allows one to pose an evolution problem that is not too different from that of general relativity.

However, due to the divergence at the origin for this standard choice of $F(r)$, one can instead use a regularised version of it when dealing with systems which include $r = 0$. On its own, this comes at the price of breaking conservation, but this can be compensated by the introduction of an appropriate $(T_{\rm AC})^{\mu}_{\nu}$ term. We will refer to this tensor as the *regularised* Polyakov RSET (RP-RSET), which we will use for stellar configurations in the next section. The $F(r)$ function can be fixed freely in static configurations (though the $1/r^2$ form should be retained far away from the origin in order to recover the s-wave behaviour), as the conservation equations

$$\nabla_{\mu}\langle \hat{T}^{\mu}_{r}\rangle^{\rm P} = \partial_r \langle \hat{T}^{r}_{r}\rangle^{\rm P} + \frac{2}{r}\left(\langle \hat{T}^{r}_{r}\rangle^{\rm P} - \langle \hat{T}^{\theta}_{\theta}\rangle^{\rm P}\right) + \frac{f'}{2f}\left(\langle \hat{T}^{r}_{r}\rangle^{\rm P} - \langle \hat{T}^{t}_{t}\rangle^{\rm P}\right) = 0, \quad (1.11)$$

can be satisfied with an appropriate choice of $(T_{\rm AC})^{\mu}_{\nu}$. In dynamical situations, however, such a regularisation is not as straightforward, and one needs a more thorough deformation of the RSET components near the origin. In the following we only use the RP-RSET in static scenarios, while in dynamical ones we simply steer clear of the origin for now, focusing instead on the vicinity of horizons.

For completeness let us also mention that it is possible to incorporate into the Polyakov RSET the backscattering effects of the gravitational potential by considering a two-dimensional scalar coupled to a dilaton field (we refer the reader to [49, 51] for details on this approach). This method has a similar issue to the RP-RSET, in the sense that the resulting two-dimensional RSET is not conserved; this is again compensated with angular components in the four-dimensional tensor. We avoid the use of this approach in the following analyses, since the RSET it gives exhibits similar problems as the standard Polyakov RSET at $r = 0$, while at the same time containing terms with third order spatial derivatives of the metric functions.

The simplicity of the RP-RSET grants it a sort of malleability that makes it applicable to a variety of static scenarios. The static and spherically-symmetric geometries we will work with have a metric which can be written as

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2. \quad (1.12)$$

Leaving F unspecified and choosing the quantisation which respects staticity (corresponding to the Boulware vacuum), we arrive, through (1.11), at the components of the RP-RSET,

$$\langle \hat{T}_t^t \rangle^{\rm P} = \frac{F}{96\pi^2 h}\left[\frac{2f'h'}{fh} + 3\left(\frac{f'}{f}\right)^2 - \frac{4f''}{f}\right],$$
$$\langle \hat{T}_r^r \rangle^{\rm P} = -\frac{F}{96\pi^2 h}\left(\frac{f'}{f}\right)^2,$$
$$\langle \hat{T}_\theta^\theta \rangle^{\rm P} = -\frac{(2F + rF')}{192\pi^2 h}\left(\frac{f'}{f}\right)^2. \tag{1.13}$$

This will be the RSET we will use for the majority of the next section, where we analyse self-consistent semiclassical solutions. Before this, let us make some final remarks on other existing RSET approximations in static spacetimes.

Approximate RSETs based on dimensional reduction have been extensively used despite their problematic behaviour at $r = 0$. These problems are absent from RSET approximations that consider the four-dimensional field dynamics from the start, like the analytic approximation derived by Anderson, Hiscock and Samuel [3]. Although this approximation gives a well-behaved RSET at $r = 0$, the AHS-RSET has additional problems for massive fields. For example, in flat spacetime it contains non-zero contributions that cannot be renormalised away [11] nor be identified with a cosmological constant [76].

Some of the complications associated with the AHS-RSET can be circumvented by considering just the zero-mass case and applying to it a reduction-of-order procedure (see [11, 88]). After the reduction the resulting system of differential equations is again second order a thus adequate for backreaction analyses [6, 11]. In the next section we will use mostly the Regularised Polyakov approximation, but also this Order-Reduced AHS-RSET to find self-consistent solutions of the semiclassical equations both for classically empty spacetimes and for stars of constant density.

1.4 Stellar Equilibrium on a Physical Vacuum Soil

Equipped with a suitable RSET satisfying the desired properties of analyticity, regularity, conservation and low derivative order, the semiclassical equations (1.1) can be solved in a full, self-consistent manner. Since the RSET is a function of the spacetime metric and its derivatives, self-consistent solutions to the semiclassical equations will be those in which classical spacetime configurations are everywhere corrected by the backreaction of quantum vacuum polarisation.

Treating semiclassical gravity as a modified theory of gravity generates situations in which semiclassical corrections overcome their $\mathscr{O}(\hbar)$ suppression. When there is not classical matter this, for example, can lead to geometries that cannot be smoothly deformed into their general relativistic counterparts in the $\hbar \to 0$ limit. Notwithstanding the absence of a unique prescription to obtain analytic RSET approximations, one advantage of the semiclassical approach is that self-consistent solutions exhibit robust properties that appear to be independent of the RSET approximation adopted.

Before passing to the description of the solutions found, let us make two remarks regarding the selection of Boulware state and the surpassing of the Buchdahl limit.

1.4.1 The Importance of the Boulware Vacuum

Semiclassical gravity discloses its non-perturbative phenomenology at event horizons. In static situations, the natural vacuum state for the field is the Boulware vacuum [30], which reduces to the Minkowski vacuum at radial infinity, a characteristic that is consistent with the asymptotic flatness of spacetime. This modes defining the state are manifestly singular at the event horizon, such behaviour spreading to the RSET. In fact, the RSET has a physical divergence at $r = r_{\rm H}$ if the energy density measured by a freely-falling observer diverges there [74]. Let us illustrate this divergence in the Polyakov RSET. For a metric adopting the form

$$f(r) = h(r)^{-1} = \frac{r - r_{\rm H}}{r_{\rm H}} + \mathscr{O}\left(\frac{r - r_{\rm H}}{r_{\rm H}}^2\right) \tag{1.14}$$

near the event horizon, then the quantity

$$\mathscr{E} = \frac{\langle \hat{T}_r^r \rangle^{\rm P} - \langle \hat{T}_t^t \rangle^{\rm P}}{f} \propto -\frac{l_{\rm P}^2}{r_{\rm H}^2 (r - r_{\rm H})^2} + \mathscr{O}\left(\frac{r - r_{\rm H}}{r_{\rm H}}\right)^0, \tag{1.15}$$

is infinite (here $l_{\rm P} = 1/\sqrt{12\pi}$). A similar divergence is found for the AHS-RSET [11].

As a consequence of its divergent behaviour at the event horizon,[3] the Boulware vacuum is commonly dismissed as plainly non-physical, deemed as the natural state only for horizonless stellar configurations instead. This argument is based on the assumption that the background spacetime is unaffected by vacuum polarisation. At the level of the semiclassical equations, when the RSET is allowed to backreact on the spacetime, the Boulware vacuum is a perfectly self-consistent vacuum state, since its characteristic divergence gets absorbed by the background spacetime. As a consequence, the event horizon gets destroyed by the backreaction of vacuum polarisation. When there is no classical matter, this carries along additional pathologies that are absent in classical vacuum solutions, such as the presence of curvature singularities that are not concealed by event horizons. This is avoided in the presence of classical matter, which makes the geometry akin to stellar configurations.

At this stage, it is interesting to advance an intriguing difference between GR and its semiclassical counterpart [12]. In GR the eternal Schwarzschild (or Kerr) vacuum solution captures all the relevant aspects of the more realistic situation in which a BH is formed from the collapse of a previous stellar configuration. This can be taken

[3] This divergence is rooted to the choice of plane-wave mode solutions to Eq. (1.7). The null Eddington-Finkelstein coordinates diverge at even horizons making the modes to acquire infinite frequencies.

as suggesting that the behaviour of matter is not that relevant for analysing the end point of gravitational collapse (beyond imposing that matter does not violate energy conditions). On the contrary, in semiclassical GR vacuum eternal (or static) solutions exhibit more clearly their pathological nature. As we will see, in order to obtain physically reasonable configurations, some classical matter needs to be included in the spacetime. It is this classical material that seeds how vacuum polarisation is in turn excited.

1.4.2 Surpassing the Buchdahl Limit

We follow this line of thought to its ultimate consequences, assuming the existence of an additional material modelled as a classical SET describing an isotropic perfect fluid of constant density in equilibrium. Through this simple assumption, a window opens towards the possibility that the (on average repulsive) effects of vacuum polarisation generate new configurations in equilibrium that are more compact than those allowed by classical general relativity. We define the compactness function as

$$C(r) \equiv \frac{2m(r)}{r} = 1 - h(r)^{-1}, \tag{1.16}$$

where $m(r)$ is the Misner-Sharp mass [77]. The value $C(r) = 1$ denotes the compactness of the BH event horizon and $C(r) = 8/9$ denotes the largest surface compactness attainable by hydrostatic equilibrium configurations in general relativity, or Buchdahl limit [34, 112]. The Buchdahl compactness bound applies to stars satisfying the following: (i) the star has a Schwarzschild exterior, (ii) internal pressures in the angular directions do not surpass the pressure in the radial direction, and (iii) a density profile that is non-increasing outwards. Self-consistent semiclassical gravity has the potentiality to violate all three possibilities: the exterior spacetime is no longer Schwarzschild, RSETs are anisotropic by construction, and they have negative energy densities that can revert the tendency of the total density to be non-increasing outwards. In consequence, this theory stands out as a promising place in which to seek for new stages of stellar equilibrium that can solve the pathologies posed by vacuum solutions.

By seeking for RSET approximations that are adapted to describe stellar structures, it is possible to find families of RSETs whose backreaction effects support stars that overcome the Buchdahl limit. Once the Buchdahl limit is surpassed, these stars can have a surface lying extremely close to their gravitational radius (both surfaces being separated just by few Planck lengths). Their large interior redshifts makes them easily mistaken for BHs through electromagnetic observations [40]. Nonetheless, the presence of a surface inside their photon sphere could produce distinct gravitational-wave echoes [41]. This way we evidence that, by considering a more physical vacuum than the one from general relativity, together with the simplest material contents, semiclassical gravity allows for the existence of ultracompact alternatives to BHs.

These exotic compact objects, that we denoted as relativistic semiclassical stars, are realised in two ways: through exploring families of RP-RSETs where the regulator function $F(r)$ is distorted within some central stellar core, and through order-reduced versions of the AHS-RSET. Our philosophy in here is not to argue for a particular approximation scheme as the best one; it is more to put all the possibilities on the table to see what they can offer. Our point of view concerning semiclassical theories is more heuristic and closer to the phenomenological philosophy underneath modified theories of gravity: motivating a possible form for some modifications of general relativity and then analysing the new equations without caring how these equations might show up hierarchically from an even deeper description of spacetime. The existence of common features in the solutions to semiclassical equations sourced by unrelated RSETs evidences the robustness of semiclassical analyses.

1.4.3 Solutions with No Classical Matter

We now turn towards deriving the semiclassical counterpart to the Schwarzschild BH solution. By semiclassical counterpart, we refer to the solution that incorporates the effects of vacuum polarisation in a self-consistent way through the backreaction of the RSET. We will use two qualitative approximations to the RSET: the Regularised Polyakov approximation and an Order Reduced AHS approximation.

1.4.3.1 The Regularised Polyakov Approximation

For the metric (1.12), the tt and rr components of the semiclassical Einstein equations in vacuum are, respectively,

$$\begin{aligned} \frac{h(1-h)-rh'}{h^2r^2} &= 8\pi\hbar\langle\hat{T}_t^t\rangle^{\mathrm{P}}, \\ \frac{rf'f-fh}{fhr^2} &= 8\pi\hbar\langle\hat{T}_r^r\rangle^{\mathrm{P}}, \end{aligned} \tag{1.17}$$

where the RSET is described by the Regularised Polyakov approximation (1.13) with

$$F(r) = 1/\left(r^2+\alpha l_{\mathrm{P}}^2\right), \ \alpha > 1. \tag{1.18}$$

This simple choice of $F(r)$ acts as a cutoff to the magnitude of the Polyakov RSET, which becomes finite on regular spacetimes.

Equation (1.17) can be integrated as a boundary value problem from radial infinity (in practice, a distant referential radius) assuming the metric takes the asymptotic form

$$f(r) = h(r)^{-1} = 1 - \frac{2M}{r}, \quad M > 0, \tag{1.19}$$

which is consistent with the way the RSET components (1.13) decay at infinity in the Boulware vacuum [7]. Due to the presence of an additional source in the right-hand side of the semiclassical equations, the metric no longer obeys $f(r) = h(r)^{-1}$. As the semiclassical equations are integrated inwards, the spacetime geometry progressively deviates from the Schwarzschild solution. This deviation amounts to a redshift function $f(r)$ modified with respect to its Schwarzschild form (1.19) and a Misner-Sharp mass $m(r)$ that acquires a dependence on the radial coordinate, as if the whole spacetime was surrounded by an inhomogeneous cloud of negative mass. Notice however that at any macroscopic distance from the gravitational radius this modification is absolutely negligible (it is of order $\hbar$) leaving unchanged any gravitational test related to the Schwarzschild exterior metric. The magnitude of the RSET increases inwards until a special surface $r_{\rm B} > 2M$ where quantum corrections become non-perturbative is encountered. At $r_{\rm B}$, $h(r_{\rm B}) \to \infty$ and we find a coordinate singularity that corresponds to a minimal surface for r. This is demonstrated by adopting a change to the proper radial coordinate l,

$$\frac{dr}{dl} = \pm\frac{1}{\sqrt{h}}, \tag{1.20}$$

where the $\pm$ signs denote the two branches of the radial coordinate at each side of the minimal surface. Assuming the following behaviours for the metric functions

$$\begin{aligned} f(l) &= f_{\rm B} + f_1\,(l - l_{\rm B}) + \mathscr{O}\,(l - l_{\rm B})^2\,, \\ r(l) &= r_{\rm B} + r_1\,(l - l_{\rm B}) + r_2\,(l - l_{\rm B})^2 + \mathscr{O}\,(l - l_{\rm B})^3\,, \end{aligned} \tag{1.21}$$

and replacing them in the semiclassical equations we find

$$f_1 = \frac{2 f_{\rm B}\sqrt{r_{\rm B}^2 + \alpha l_{\rm P}^2}}{l_{\rm P} r_{\rm B}}, \quad r_1 = 0, \quad r_2 = \frac{\left(r_{\rm B}^2 + \alpha l_{\rm P}^2\right)^2 + \alpha l_{\rm P}^4}{2 r_{\rm B}\left[r_{\rm B}^2 + (\alpha - 1)\,l_{\rm P}^2\right]\left(r_{\rm B}^2 + \alpha l_{\rm P}^2\right)}, \tag{1.22}$$

where $f_{\rm B}$ and $r_{\rm B}$ are positive constants baring a non-analytic relation to the ADM mass M and the parameter α. The functions (1.22) make the metric explicitly regular at $l = l_{\rm B}$. The radial function $r(l)$ is symmetric around the minimal surface $r_{\rm B}$, while the redshift function is positive and asymmetric, showing there is a wormhole neck that connects the asymptotically flat region of the spacetime with a new region of different characteristics. Figure 1.1 shows these metric functions in terms of l for an example integration of the semiclassical equations.

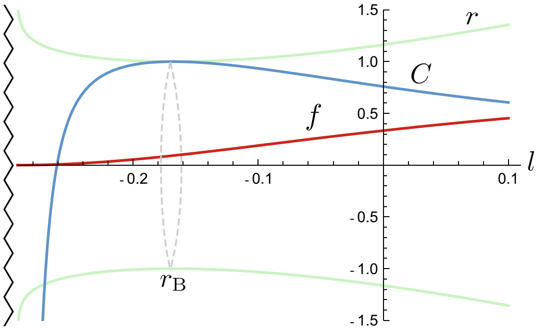

Fig. 1.1 Numerical plot of the semiclassical counterpart of the Schwarzschild vacuum geometry. The horizontal axis is the proper coordinate l while the above and below curves in green represent the radial coordinate r. The behavior of the redshift function, in red, and the compactness, in blue, are shown. The right side of the wormhole is asymptotically flat whereas the other is asymptotically singular. Both regions are joined by a minimal surface of radius $r = r_B$. We have chosen $M = 0.1$ and $\alpha = 1.01$ for illustrative purposes. Geometrical characteristics are identical for larger ADM masses

Below the wormhole neck, vacuum polarisation enters into a runaway regime that makes the metric approach a null singularity at infinite r, but finite l. Through an asymptotic analysis of the semiclassical equations [7], the form of the metric nearing the singularity is found to be, in Schwarzschild coordinates,

$$\begin{aligned} ds^2 \simeq & \left(\frac{r}{l_P}\right)^{1-4\alpha} e^{-\frac{2r^2}{l_P^2}} \cdot \\ & \cdot \left\{-a_0\left(1-\frac{l_P^2}{8r^2}\right)dt^2 + \frac{2\chi_0 r^2}{l_P^2}\left[1-\frac{(9-32\alpha)l_P^2}{8r^2}\right]dr^2\right\} + r^2 d\Omega^2, \end{aligned} \tag{1.23}$$

where a_0 and χ_0 are dimensionless positive constants. The vanishing of the conformal factor as $r \to \infty$ manifests the null character of this singularity. The Ricci scalar, defined as

$$\mathscr{R} = \frac{2}{r^2}\left(1-\frac{1}{h}\right) + \frac{2}{hr}\left(\frac{h'}{h} - \frac{f'}{f} + \frac{rf'h'}{4fh}\right) + \frac{1}{2h}\left[\left(\frac{f'}{f}\right)^2 - 2\frac{f''}{f}\right], \tag{1.24}$$

becomes negatively divergent at the singularity, i.e.,

$$\mathscr{R} \simeq -\frac{e^{2r^2/l_P^2}(2\alpha-1)}{l_P^2\chi_0}\left(\frac{r}{l_P}\right)^{-5+4\alpha}. \tag{1.25}$$

Furthermore, this singular region is located at a finite proper distance $l_S < l_B$ from the throat, as shown by integrating the quantity

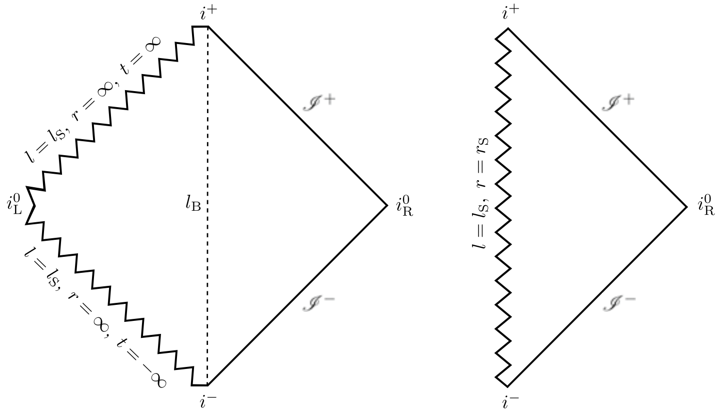

Fig. 1.2 Left panel: Penrose diagram corresponding to the singular wormhole solution for the RP-RSET. The dashed lines denote the location of the wormhole neck. To their right, the asymptotically flat portion of spacetime is depicted alongside its asymptotic regions. The left hand side of the diagram shows the internal past and future null singularities, which are located at finite proper distance from the neck $l_S - l_B$. The point i^0_L is singular as well, and is reached in finite proper time by spacelike geodesics. Right panel: Penrose diagram associated with the vacuum solution for the OR-RSET. In this case, the singularity is timelike and constitutes a naked singularity. While differences in the modelling of the semiclassical source result in singularities of different sorts, both models agree on the absence of event horizons

$$\left(\frac{dl}{dr}\right)^2 = 2\chi_0\,(r/l_P)^{3-4\alpha}\, e^{-\frac{2r^2}{l_P^2}}\left[1 - \frac{(9-32\alpha)l_P^2}{8r^2}\right]. \tag{1.26}$$

Near asymptotically flat regions, far from any source of gravity, semiclassical corrections amount to extremely weak, thus perturbative, corrections. As the surface $r = 2M$ is approached, however, vacuum polarisation builds up and destroys the event horizon, generating instead a wormhole neck. While, as shown below, the specific features of the region of non-perturbative semiclassical corrections depend on the particular modelling of the RSET (see Fig. 1.2), the replacement of the event horizon by a singularity appears as a robust characteristic of the RSET not depending on the approximation, as it is a consequence of evaluating the RSET in the Boulware state. Similar characteristics are displayed by the semiclassical counterpart to the Reissner-Nordström sub-extremal BH [8].

The regularisation scheme we have adopted for the Polyakov RSET [see Eq. (1.18)] amounts to a cutoff to the magnitude of its components, whose strength is modulated by α. Increasing α brings the singularity at l_S closer to the wormhole neck l_B. Spacetime regions near $r = 0$, while unexplored in vacuum solutions, are present in stellar configurations. We will return to exploring more elaborate regularisation schemes for the Polyakov RSET later.

The results here presented are consistent with previous works [25, 50, 61]. A clear extension of this work is to consider the backreaction effects of an RSET approximation that does not rely on dimensional reduction. The Order Reduced version of the AHS-RSET stands on equal footing with the RP-RSET, in the sense that it is a quantity covariantly conserved, analytic, without higher-derivative terms, and well-defined at $r = 0$. We now briefly sketch the derivation of this RSET approximation and the characteristics of the corresponding vacuum solutions in the minimally coupled case.

1.4.3.2 The Order Reduced AHS-RSET

We start with the tt and rr components of the vacuum semiclassical equations (1.1), now sourced by the AHS-RSET,

$$\begin{aligned} \frac{h(1-h) - rh'}{h^2 r^2} &= 8\pi\hbar \langle \hat{T}_t^t \rangle^{\rm AHS}, \\ \frac{rf'f - fh}{fhr^2} &= 8\pi\hbar \langle \hat{T}_r^r \rangle^{\rm AHS}, \end{aligned} \tag{1.27}$$

where the right-hand side contains higher-derivative terms. The concrete and lengthy form of the AHS-RSET, which is not very illustrative, can be seen in [3]. To obtain a set of equations of the same derivative order as the classical ones, we subject the AHS-RSET to a perturbative reduction of order. The first step in this procedure consists in neglecting terms $\mathscr{O}(\hbar)$ in Eq. (1.27), leading to

$$\begin{aligned} \frac{h(1-h) - rh'}{h^2 r^2} &= \mathscr{O}(\hbar), \\ \frac{rf' + f - fh}{fhr^2} &= \mathscr{O}(\hbar). \end{aligned} \tag{1.28}$$

These expressions can be differentiated consecutively to derive recursion relations between f, h, and their higher-order derivatives $\{f^{(n)}\}_{n=1}^{\infty}$ and $\{h^{(n)}\}_{n=1}^{\infty}$. For h, said relations are obtained by solving the tt equation directly, which can then be used to derive the f relations from the rr equation. The resulting relations are

$$\begin{aligned} h^{(n)} &= (-1)^n \frac{n! h^n}{r^n} (h-1) + \mathscr{O}(\hbar), \\ f^{(n)} &= (-1)^{n+1} \frac{n! f}{r^n} (h-1) + \mathscr{O}(\hbar). \end{aligned} \tag{1.29}$$

Relations (1.29) are now inserted in the AHS-RSET components $\langle \hat{T}_t^t \rangle^{\rm AHS}$ and $\langle \hat{T}_r^r \rangle^{\rm AHS}$ until they only depend on f and h. After a lengthy but straightforward calculation using symbolic computation software, we arrive at