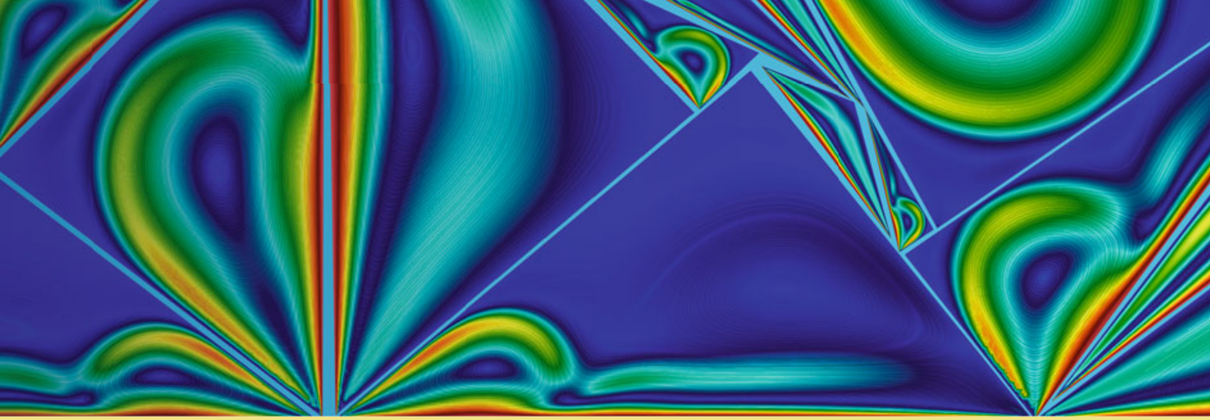

LECTURE NOTES IN COMPUTATIONAL SCIENCE AND ENGINEERING 151

Gianluigi Rozza · Giovanni Stabile · Max Gunzburger · Marta D'Elia *Editors*

Reduction, Approximation, Machine Learning, Surrogates, Emulators and Simulators

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Editors

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Cover image: Composition of triangular cavity flows. Courtesy by Federico Pichi.

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Preface

This book collects contributions from RAMSES workshop held at SISSA, Scuola Internazionale Superiore di Studi Avanzati, in Trieste in December 2021.

It features chapters reviewing recent algorithmic and mathematical advances, as well as developments of new research directions for mathematical approximation via RAMSES–Reduced order modeling, Approximation theory, Machine learning, Surrogates, Emulator, and Simulators—in the setting of parametrized partial differential equations in high-dimensional parameter spaces, including sparse and noisy data.

The volume is made up of 10 selected and peer-reviewed contributions in chapters.

RAMSES was supported by SISSA, International School for Advanced Studies, Trieste, Italy; US Air Force Office of Scientific Research, Computational Mathematics Program; and Florida State University, Department of Scientific Computing, Tallahassee, FL.

More information about the workshop can be found at https://indico.sissa.it/event/43/.

We would like to thank all the SISSA mathLab organizing team, the anonymous reviewers for their careful revision work, as well as Springer Nature, especially Francesca Bonadei and Francesca Ferrari, and LNCSE series editorial board for their support.

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Austin, TX, USA — Max Gunzburger
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An Online Stabilization Method for Parametrized Viscous Flows

Shafqat Ali, Francesco Ballarin, and Gianluigi Rozza

Abstract The purpose of this work is to investigate the inf-sup stability of reduced basis (RB) method applied to parametric Stokes problem. While performing the Galerkin projection on the reduced space, the inf-sup approximation stability has always been a challenge for the RB community, even if the construction of reduced basis is done using a stable high-fidelity method. In this work we propose a new online stabilization strategy for RB approximation of parametrized Stokes problem. In this strategy, a stable high-fidelity method is used to construct the RB spaces, and then, online solution is improved by a post processing based on rectification method [8, 13, 16]. This approach involves the computation of less expensive (but less consistent) FE approximation during the online stage and hence the improvement of online solutions using a RB-based rectification method. The consistency of the RB solution is also improved. We compare this approach with existing *offline-online stabilization* approach presented in our earlier work [2]. All the numerical simulations are carried out using RBniCS [4, 14], an open-source reduced order modelling library, built on top of FEniCS [15].

1 Introduction

Reduced basis (RB) methods [14] has been extensively used to compute rapid and reliable approximations of solutions of complex problems involving physical and geometrical parameters. A motivational study to apply the reduced basis method for parametrized PDEs can be found in [5, 20]. These methods depend on the parametric structure of the model. When the parameters vary, the solutions manifold

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can be approximated by n-dimensional spaces. The performance and efficiency of the RB methods depends on the Kolmogorov n-width [11] of the manifold of all the possible solutions. Thus, when the Kolmogorov n-widths decay rapidly with the space dimension, the manifold of all possible solutions is approximated by a low-dimensional space, the RB space. This RB space is made of particular solutions of the parametrized problem with well chosen parameter values [6, 12]. The main advantage of RB techniques is the decomposition of the computational work into offline and online stages. During the offline stage the reduced basis functions are computed, as well as all parameter-independent quantities. This is done only once, whereas parameter-dependent quantities are computed during the online stage.

Continuing our investigations [1, 2] on the stability of RB methods, we propose in this work a new approach to deal with the problem of instabilities (inf-sup) appearing during the RB approximation of parametrized Stokes problems. These instabilities are a classic problem, whatever the discretization is used to construct the basis functions. Some treatment by adding appropriate stabilization terms [2], or using a supremizer approach [3] is successfully implemented. In the case of RB, we can rely on a set of N bases functions, obtained by a classical FE technique [7], such as the SUPG method [17, 18]. However, the combination of these functions through a method of pure Galerkin is not sufficient to ensure the stability of the RB problem when N increases, hence the stabilization terms are needed, appropriate to the RB level. The reduced space is independent of the classical method used to generate the RB functions.

Our new strategy in this work is online stabilization strategy based on rectification method [8–10]. In this method, in order to retrieve the same accuracy as the high-fidelity model, we first project every solution into the reduced space and then further improve them via post-processing based on a rectification technique. The aim of this paper is to provide tests to validate and generalize our method for parametrized Stokes problems.

This paper is organized as follows. In Sect. 2, first we define advection-diffusion problem and give overview of SUPG stabilization method for advection-dominated case [17]. A brief overview (recall) of the rectification method applied to advection-dominated problem [16] is presented with numerical tests and discussion on the results. The main focus would be the Stokes problem, but before that the reason of recalling the existing rectification approach for advection-diffusion problem is, to make it easy for the readers. In Sect. 3, first we recall the formulation of the Stokes problem [2], and is followed by the introduction of rectification method for the parametrized Stokes problem. Finally, two numerical tests are performed, starting with the benchmark parametrized cavity flow problem, and then, a slightly difficult T-bypass [21] test to check the validity of rectification method. Finally, Sect. 4 concludes the main findings of this work.

2 Rectification Method for Advection-Diffusion Problem

In this section we give a brief review of the rectification method [16] to recall the concepts of rectification for the case of scalar advection-diffusion problem. We start with the definition of scalar advection-diffusion problem as:

$$\begin{cases} L(\mu)u := -\mu\Delta u + \mathbf{b}\cdot\nabla u = f & \text{in } \Omega = (0,1)^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mu \in [10^{-6}, 1]$ denotes the diffusion coefficient and $\mathbf{b} = (1, 1)^T$, the constant transport field. The SUPG-stabilization method [7] in the offline stage to get the basis matrix Z [14] is implemented. The weak form of problem (1) is: for any $\mu \in \mathbb{P}$, find $u(\mu) \in V$ such that

$$a(u(\mu), v; \mu) = f(v) \qquad \forall v \in V, \tag{2}$$

where $V = H_0^1(\Omega)$ and

$$a(u(\mu), v; \mu) = \mu \int_\Omega \nabla u \cdot \nabla v d\boldsymbol{x} + \int_\Omega \boldsymbol{b}\cdot\nabla u v d\boldsymbol{x}, \qquad f(v) = \int_\Omega f v d\boldsymbol{x} \tag{3}$$

a continuous and coercive bilinear form, and a linear and continuous functional, respectively. Introducing a *high-fidelity* space $V_h \subset V$ of dimension $\mathcal{N}$. The *high-fidelity* solution to problem (2) obtained by Galerkin-FE method reads: for any $\mu \in \mathbb{P}$, find $u_h(\mu) \in V_h$ such that

$$a(u_h(\mu), v_h; \mu) = f(v_h) \qquad \forall v_h \in V_h. \tag{4}$$

When dealing with advection-dominated, i.e., for $\frac{|\boldsymbol{b}|}{\mu} \ll 1$, solution to (4) yields numerical oscillations unless a suitable stabilization technique is introduced. Therefore, in this case SUPG [7] is applied. The stabilized formulation of (4) reads:

$$a_{stab}(u_h(\mu), v_h; \mu) = f_{stab}(v_h) \qquad \forall v_h \in V_h, \tag{5}$$

where $a_{stab}(.,.;\mu)$ and $f_{stab}(.)$ are bilinear and linear forms including the stabilization terms defined as:

$$\begin{aligned} a_{stab}(u_h(\mu), v_h; \mu) &= a(u_h(\mu), v_h; \mu) + s(u_h(\mu), v_h; \mu), \\ f_{stab}(.) &= f(v_h) + f_s(v_h), \end{aligned} \tag{6}$$

being

$$\begin{aligned} s(u_h(\mu), v_h; \mu) &= \sum_{K\in\tau_h} (L(\mu)u_h, \delta_K L_{SS} v_h)_{L^2(K)}, \\ f_s(v_h) &= \sum_{K\in\tau_h} (f, \delta_K L_{SS} v_h)_{L^2(K)}, \end{aligned} \tag{7}$$

where $L_{SS}u = \boldsymbol{b} \cdot \nabla u$ is skew symmetric part of operator L and $\delta_K > 0$ a suitable stabilization coefficient. Algebraic formulation of (6) can be written as:

$$A_{stab}(\mu)\boldsymbol{u}_h(\mu) = F_{stab}, \tag{8}$$

where

$$A_{stab}(\mu) = A(\mu) + S(\mu), \qquad F_{stab} = F + F_s, \tag{9}$$

being $\boldsymbol{u}_h(\mu) \in \mathbb{R}^{\mathcal{N}}$ the vectors whose components are the degrees of freedom of $u_h(\mu)$ and for $i, j = 1, \ldots, \mathcal{N}$

$$\begin{aligned} (A(\mu))_{ij} = a(\phi_j, \phi_i; \mu), \qquad (S(\mu))_{ij} = s(\phi_j, \phi_i; \mu), \\ (F)_i = f(\phi_i), \qquad (F_s)_i = f_s(\phi_i), \end{aligned} \tag{10}$$

where $\{\phi\}_{i=1}^{\mathcal{N}}$ denote the set of (Lagrangian) basis functions on V_h.

Now introducing a low dimensional subspace V_N of dimension N, where $N \ll \mathcal{N}$ and V_N is built from a set of *high-fidelity* solutions (snapshots) computed for properly selected parameter values [14, 19], i.e.,

$$V_N = \text{span}\{u_h(\mu^n)|1 \leq n \leq N\} \subset V_h, \tag{11}$$

The RB is obtained by Galerkin-projection onto V_N and reads as follows: for any $\mu \in \mathbb{P}$ find $u_N(\mu) \in V_N$ such that

$$a(u_N(\mu), v_N; \mu) = f(v_N) \qquad \forall v_N \in V_N. \tag{12}$$

For advection-dominated case *offline-only stabilization* [17] is not stable and shows spurious oscillations. Therefore, in order to overcome these oscillations, we look for the following two possibilities.

2.1 Offline-Online Stabilization

Performing a Galerkin projection of the stabilized problem (5) onto V_N using a stabilized RB formulation [2, 17] yields stable RB approximation and it reads: for any $\mu \in \mathbb{P}$ find $u_N(\mu) \in V_N$ such that

$$a_{stab}(u_N(\mu), v_N; \mu) = f_{stab}(v_N) \qquad \forall v_N \in V_N. \tag{13}$$

Algebraically, the RB approximation for SUPG case is the solution of following system:

$$A_N^{stab}(\mu)\boldsymbol{u}_N(\mu) = F_N^{stab}, \tag{14}$$

where

$$A_N^{stab}(\mu) = A_N(\mu) + A_N^{SUPG}(\mu), \qquad F_N^{stab} = F_N + F_N^{SUPG}. \tag{15}$$

These RB matrices are obtained as:

$$\begin{aligned} A_N(\mu) = Z^T A_h(\mu) Z, \qquad A_N^{SUPG}(\mu) = Z^T A_h^{SUPG}(\mu) Z, \\ F_N = Z^T F_h, \qquad F_N^{SUPG} = Z^T F_h^{SUPG}, \end{aligned} \tag{16}$$

where $Z \in \mathbb{R}^{\mathcal{N} \times N}$ is the basis matrix, such that $Z = [\xi_1 | \cdots | \xi_N]$.

2.2 Post-processing Based on Rectification

After solving the problem (12), a further post-processing based on a rectification method [8–10, 13] is applied to improve the accuracy of solution. In other words, this rectification method is used to correct the consistency error of RB approximation

$$u_N(\boldsymbol{\mu}) = \sum_{k=1}^{N} \alpha_k(\boldsymbol{\mu}) \xi_k,$$

i.e., the fact that

$$u_N(\boldsymbol{\mu}^i) \neq u_h(\boldsymbol{\mu}^i) \quad \forall \boldsymbol{\mu}^i \in S_N = \{\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^N\}.$$

In order to cure this issue, an alternative linear combination of the reduced basis functions has been chosen.

We start by computing the RB Galerkin approximations for all values $\boldsymbol{\mu} = \boldsymbol{\mu}^i$; $i = 1, \ldots, N$ which gives the coefficients $u_N(\boldsymbol{\mu}^i) = \sum_{k=1}^{N} \alpha_k(\boldsymbol{\mu}^i)\xi_k$. We define the matrix R_N with coefficients α_k^i, i.e.,

$$R_N = \begin{pmatrix} \alpha_1(\boldsymbol{\mu}^1) & \ldots & \alpha_1(\boldsymbol{\mu}^N) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_N(\boldsymbol{\mu}^1) & \ldots & \alpha_N(\boldsymbol{\mu}^N) \end{pmatrix}. \tag{17}$$

We also express the N snapshots over the reduced basis which gives the coefficients $u_h(\boldsymbol{\mu}^i) = \sum_{j=1}^{N} \beta_j(\boldsymbol{\mu}^i)\xi_j$, from which we define the matrix R of coefficients β_j^i, i.e.,

$$R = \begin{pmatrix} \beta_1(\boldsymbol{\mu}^1) & \dots & \beta_1(\boldsymbol{\mu}^N) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \beta_N(\boldsymbol{\mu}^1) & \dots & \beta_N(\boldsymbol{\mu}^N) \end{pmatrix}. \tag{18}$$

We set $J = RR_N^{-1}$ done in the offline stage and the matrix is stored.

Finally, the rectified solution $u_N^r(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathbb{P}$ is computed online by using the new coefficients $\boldsymbol{\alpha}_{new} = J\boldsymbol{\alpha}$, i.e.,

$$u_N^r(\boldsymbol{\mu}) = \sum_{j=1}^{N} \alpha_{new,j}(\boldsymbol{\mu})\xi_j. \tag{19}$$

2.3 Numerical Results and Discussion

Generally, combining the SUPG method with rectification method, one can discuss the following options to do the numerical tests:

- *offline-online stabilization* with/without rectification;
- *offline-only stabilization* with/without rectification.

The first option above is consistent for any case [2]. Therefore we focus here on second option, because we know that *offline-only stabilization* is not consistent [2] and we are interested here to correct the consistency by using rectification method. We provide some numerical results of problem (1) using the two solution methodologies described in Sects. 2.1 and 2.2.

Figure 1 plots the RB solutions obtained by *offline-online stabilization*, whereas Fig. 2 plots the RB solution using online rectification and without any online stabilization.

Figure 3 plots the error between FE and RB solutions obtained for various stabilization options. In all cases the offline stage is stabilized with SUPG-stabilization method but online stage is obtained for different options. We point out that the online rectification option was not reported by Maday et al. [16]. From these results we see that if we perform a post-processing (online rectification) on *offline-only stabilization*, we are able to improve the error upto 3 order of magnitude when compared to *offline-only stabilization*.

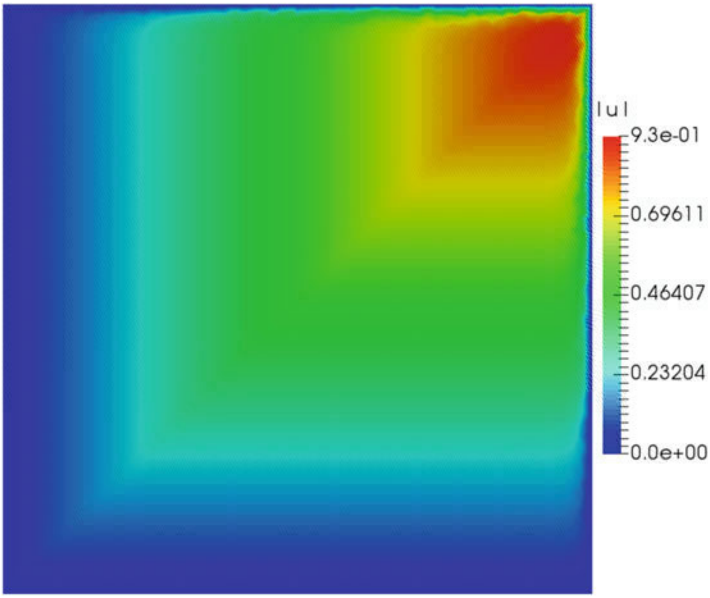


Fig. 1 RB solutions at $\boldsymbol{\mu} = 10^{-6}$ obtained by online stabilization

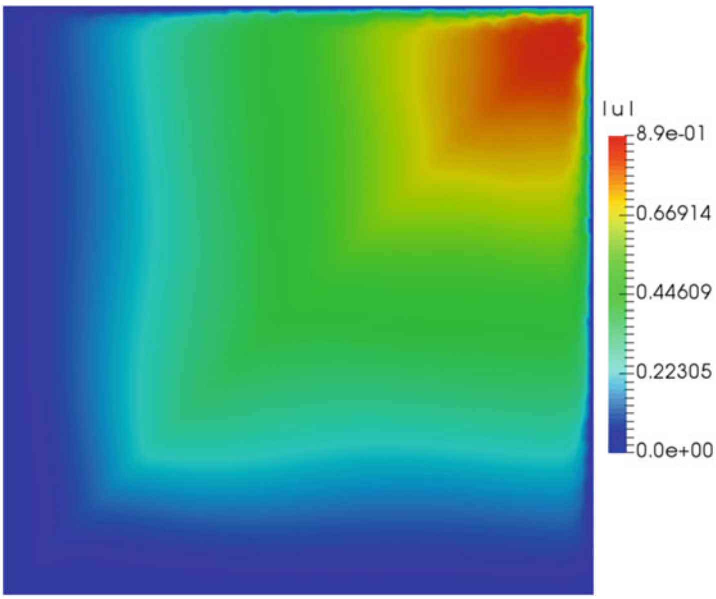


Fig. 2 RB solutions at $\boldsymbol{\mu} = 10^{-6}$ obtained by online rectification

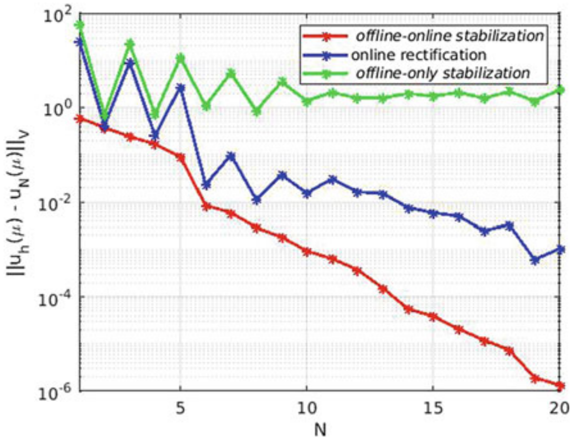


Fig. 3 Error comparison between different stabilization options for $N = 20$ and $\boldsymbol{\mu} = 10^{-6}$

3 Rectification Method for Stokes Problem

In this section we propose the online rectification method for Stokes problem introduced in Sect. 2.2. In order to proceed with the rectification process, first we recall the steady Stokes problem in parametrized domain. The steady Stokes problem in a two-dimensional parametrized domain $\Omega_0(\boldsymbol{\mu}) \subset \mathbb{R}^2$ read as: find $(\boldsymbol{u}_o, p_o) \in \boldsymbol{V} \times Q$ such that

$$\begin{cases} -\nu\Delta\boldsymbol{u}_o + \nabla p_o = \boldsymbol{f} & \text{in } \Omega_o(\boldsymbol{\mu}), \\ \text{div}\boldsymbol{u}_{\text{o}} = 0 & \text{in } \Omega_o(\boldsymbol{\mu}), \\ \boldsymbol{u}_o = \boldsymbol{g} & \text{on } \partial\Omega_o, \end{cases} \tag{20}$$

where $\boldsymbol{V} = [H_0^1(\Omega)]^2$ and $L_0^2(\Omega)$ are functional spaces, $\boldsymbol{u}_o$ is the unknown velocity and p_o is the unknown pressure, $\boldsymbol{f}$ is a given forcing function and ν is the viscosity of fluid, $\boldsymbol{\mu} \in \mathbb{P}$ (parameter domain) denotes a parameter which may be physical or geometrical. For the sake of simplicity we take $\boldsymbol{f} = 0$. The boundary $\partial\Omega_o$ is divided into two parts in such a way that $\partial\Omega_o = \Gamma_{D_0} \cup \Gamma_{D_g}$, where Γ_{D_g} is the Dirichlet boundary with non-homogeneous data and Γ_{D_0} denotes the Dirichlet boundary with zero data. For further detail of weak formulation we refer to [2]. We directly write the stabilized FE formulation

$$\begin{cases} \text{Find } \boldsymbol{u}_h(\boldsymbol{\mu}) \in \boldsymbol{V}_h, \ p_h(\boldsymbol{\mu}) \in Q_h : \\ a(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h(\boldsymbol{\mu}); \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) + s_h^v(\boldsymbol{v}_h; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h(\boldsymbol{\mu}), q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) + s_h^q(q_h; \boldsymbol{\mu}) & \forall\, q_h \in Q_h, \end{cases} \tag{21}$$

where $s_h^v(\boldsymbol{v}_h; \boldsymbol{\mu})$ and $s_h^q(q_h; \boldsymbol{\mu})$ are the stabilization terms (residual based) with possible choices given by [2]. We stabilize the offline stage to get stable basis functions and then, we project on RB without taking into consideration the stabilization terms for online solve. Therefore, the RB formulation is given by

$$\begin{cases} \text{Find } (\boldsymbol{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in \boldsymbol{V}_N \times Q_N : \\ a(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N(\boldsymbol{\mu})) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \boldsymbol{V}_N, \\ b(\boldsymbol{u}_N(\boldsymbol{\mu}), q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N. \end{cases} \tag{22}$$

where the RB spaces $\boldsymbol{V}_N$ and Q_N for velocity and pressure, respectively are defined as:

$$\boldsymbol{V}_N = \text{span}\left\{\xi_n^u = \boldsymbol{u}_h(\boldsymbol{\mu}^n), 1 \leq n \leq N_u\right\}, \tag{23}$$

and

$$Q_N = \text{span}\left\{\xi_n^p = p_h(\boldsymbol{\mu}^n), 1 \leq n \leq N_p\right\}, \tag{24}$$

where N_u and N_p are the dimensions of RB velocity space $\boldsymbol{V}_N$ and RB pressure space Q_N, respectively. $\{\xi_n^u\}_{n=1}^{N_u}$ and $\{\xi_n^p\}_{n=1}^{N_p}$ are mutually orthonormal basis functions for RB velocity and pressure, respectively obtained by applying the Gram-Schmidt orthogonalization process [14].

We recall that solving the stabilized FE formulation (21) in the offline stage and non-stabilized formulation (22) in the online stage is called *offline-only stabilization*. This option has been discussed in our previous work [1, 2] but we saw that in all cases this choice is not consistent and we were not able to get a stable RB solution.

In this section we try to recover the consistency of RB solution obtained by *offline-only stabilization* using the idea of post-processing based on rectification method [8, 13, 16]. We know that in case of *offline-only stabilization*, the solutions from which RB is constructed are actually not the solutions of the problem (22) for $\boldsymbol{\mu} = \boldsymbol{\mu}^i$, i.e.,

$$\boldsymbol{u}_N(\boldsymbol{\mu}^i) \neq \boldsymbol{u}_h(\boldsymbol{\mu}^i), \qquad p_N(\boldsymbol{\mu}^i) \neq p_h(\boldsymbol{\mu}^i), \quad \forall \boldsymbol{\mu}^i \in S_N = \{\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^N\}.$$

In other words, we are interested in correcting the consistency error of the RB approximation for velocity and pressure, respectively:

$$\boldsymbol{u}_N(\boldsymbol{\mu}) = \sum_{k=1}^{N_u} \alpha_k^u(\boldsymbol{\mu}) \xi_k^u \quad \text{and} \quad p_N(\boldsymbol{\mu}) = \sum_{k=1}^{N_p} \alpha_k^p(\boldsymbol{\mu}) \xi_k^p, \tag{25}$$

where $\{\xi_k^u\}_{k=1}^{N_u}$ and $\{\xi_k^p\}_{k=1}^{N_p}$ are mutually orthonormal basis functions for RB velocity and pressure, respectively while $\alpha_k^u(\boldsymbol{\mu})$ and $\alpha_k^p(\boldsymbol{\mu})$ denotes the coefficients of the reduced basis approximation for velocity and pressure, respectively. The method of rectification basically replaces these reduced basis coefficients with alternate ones.

In order to calculate the alternate coefficients, first we express the N snapshots for velocity and pressure, respectively over the RB as:

$$\boldsymbol{u}_h(\boldsymbol{\mu}^i) = \sum_{k=1}^{N} \beta_k^u(\boldsymbol{\mu}^i) \xi_k^u \qquad \text{and} \quad p_h(\boldsymbol{\mu}^i) = \sum_{k=1}^{N} \beta_k^p(\boldsymbol{\mu}^i) \xi_k^p \tag{26}$$

from which we obtain the matrices R^u (for velocity) and R^p (for pressure) with columns equal to the coordinates of $\boldsymbol{u}_h(\boldsymbol{\mu}^i)$ and $p_h(\boldsymbol{\mu}^i)$ in the reduced basis ξ_k^u and ξ_k^p, respectively, i.e. the coefficient matrices

$$R^u = \begin{pmatrix} \beta_1^u(\boldsymbol{\mu}^1) & \ldots & \beta_1^u(\boldsymbol{\mu}^N) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \beta_N^u(\boldsymbol{\mu}^1) & \ldots & \beta_N^u(\boldsymbol{\mu}^N) \end{pmatrix}, \qquad R^p = \begin{pmatrix} \beta_1^p(\boldsymbol{\mu}^1) & \ldots & \beta_1^p(\boldsymbol{\mu}^N) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \beta_N^p(\boldsymbol{\mu}^1) & \ldots & \beta_N^p(\boldsymbol{\mu}^N) \end{pmatrix}. \tag{27}$$

We compute the *offline-only* approximation of (22) for $\boldsymbol{\mu} = \boldsymbol{\mu}^i; i = 1, \ldots, N.$, i.e.,

$$\boldsymbol{u}_N(\boldsymbol{\mu}^i) = \sum_{k=1}^{N} \alpha_k^u(\boldsymbol{\mu}^i)\xi_k^u \quad \text{and} \quad p_N(\boldsymbol{\mu}^i) = \sum_{k=1}^{N} \alpha_k^p(\boldsymbol{\mu}^i)\xi_k^p, \tag{28}$$

which gives us the coefficient matrices R_N^u (for velocity) and R_N^p (for pressure) with entries α_k^u and α_k^p, respectively, i.e.,

$$R_N^u = \begin{pmatrix} \alpha_1^u(\boldsymbol{\mu}^1) & \ldots & \alpha_1^u(\boldsymbol{\mu}^N) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_N^u(\boldsymbol{\mu}^1) & \ldots & \alpha_N^u(\boldsymbol{\mu}^N) \end{pmatrix}, \qquad R_N^p = \begin{pmatrix} \alpha_1^p(\boldsymbol{\mu}^1) & \ldots & \alpha_1^p(\boldsymbol{\mu}^N) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_N^p(\boldsymbol{\mu}^1) & \ldots & \alpha_N^p(\boldsymbol{\mu}^N) \end{pmatrix}. \tag{29}$$

Finally, we set $J^u = R^u(R_N^u)^{-1}$ and $J^p = R^p(R_N^p)^{-1}$. The computation of J^u and J^p is done once in the offline stage and matrices are stored.

In the online stage, we compute the rectified solutions $\boldsymbol{u}_N^r(\boldsymbol{\mu})$ and $p_N^r(\boldsymbol{\mu})$ to problem (22) for any $\boldsymbol{\mu} \in \mathbb{P}$ as

$$\boldsymbol{u}_N^r(\boldsymbol{\mu}) = \sum_{k=1}^{N} \bar{\alpha}_k^u(\boldsymbol{\mu})\xi_k^u \quad \text{and} \quad p_N^r(\boldsymbol{\mu}) = \sum_{k=1}^{N} \bar{\alpha}_k^p(\boldsymbol{\mu})\xi_k^p, \tag{30}$$

where $\bar{\boldsymbol{\alpha}}^u = J^u\boldsymbol{\alpha}^{\boldsymbol{u}}$ and $\bar{\boldsymbol{\alpha}}^p = J^p\boldsymbol{\alpha}^{\boldsymbol{p}}$ are the coordinates for velocity and pressure, respectively. Now, combining three approaches; the supremizer enrichment [21], the *offline-online stabilization* [2] and the rectification approach, one can have the following possible options in the online stage:

- *offline-online stabilization* with/without supremizer with/without rectification
- *offline-only stabilization* with/without supremizer with/without rectification.

In this work we are only interested in the following options:

- *offline-only stabilization* with supremizer with rectification
- *offline-only stabilization* without supremizer with rectification.

3.1 Numerical Results and Discussion

In this section we present some numerical solutions for the new stabilization strategy presented in Sect. 3. We consider the following two test cases with increasing complexity as we move from test case one to test case two.

3.1.1 Cavity Test Case

As a first example we consider the parametrized cavity domain shown in Fig. 4. Figure 5 shows the RB velocity obtained by online stabilization (left) and online rectification (right). Similarly Fig. 6 shows the RB solutions for pressure obtained by online stabilization (left) and online rectification (right). We recall that in both cases, the offline stage is stabilized. From these plots, we see that the solutions obtained by two different stabilization approaches are same.

Figures 7 and 8 illustrates the absolute error between FE and RB solutions for velocity and pressure, respectively, using different stabilization options. We see that in case of velocity, the error for rectification method is 10^{-6} which is almost zero. A similar behavior is observed in case of relative error, that we do not show here. However *offline-online stabilization* method is still better. In case of pressure, the rectification method is able to reduce the error down to 10^{-5}, apart from some peaks at different values of N. These peaks are due to the poor conditioning of the matrix R_N^p, which, in this case is controlled by the enrichment of RB velocity space with supremizer solutions and the error is decreased to 10^{-7}.

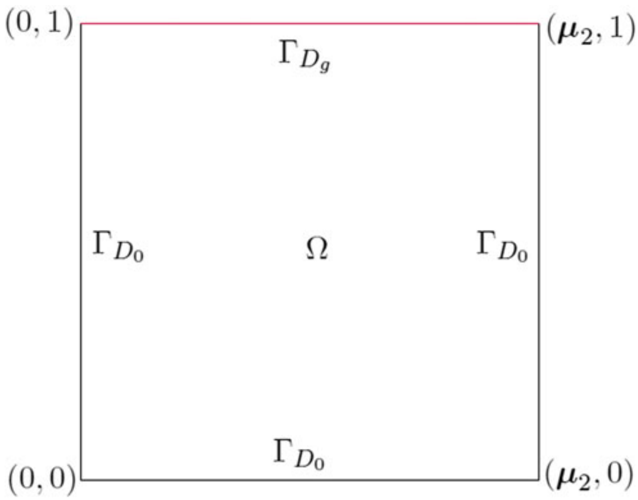


Fig. 4 Parametrized domain

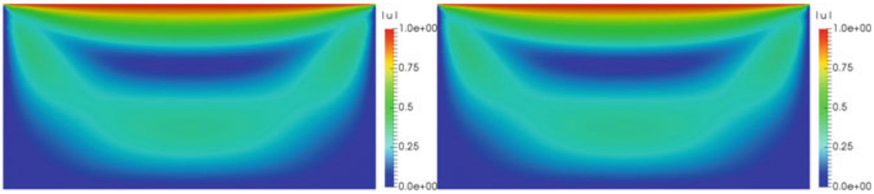


Fig. 5 RB velocity: online stabilization (left) and online rectification (right)

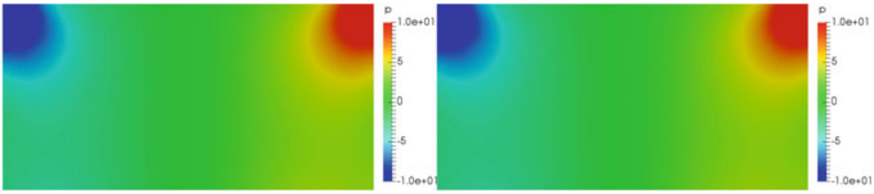


Fig. 6 RB pressure: online stabilization (left) and online rectification (right)

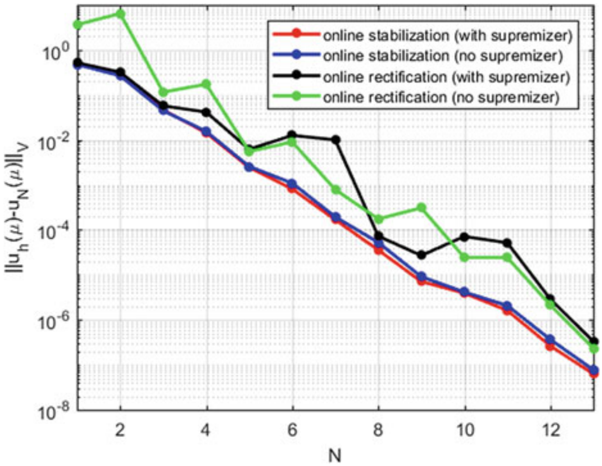


Fig. 7 Stokes cavity problem: error between FE and RB velocity for different possible options with $N_u = 13$

3.1.2 T-Bypass Test

In order to see the validity of rectification method in more challenging problems, for instance, in this example we consider the problem with many parameters. We take the example of "T-bypass" configuration from Rozza and Veroy [21]. Parametrized domain is shown in Fig. 9 with vector of parameters $\boldsymbol{\mu} = [t, D, L, S, H, \theta]$ labeled. The parameter ranges in the offline stage are $t = D = L = S = H \in [0.5, 1.5]$ and

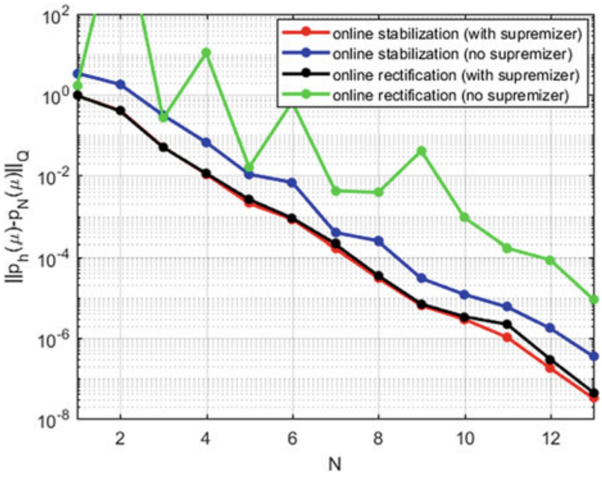
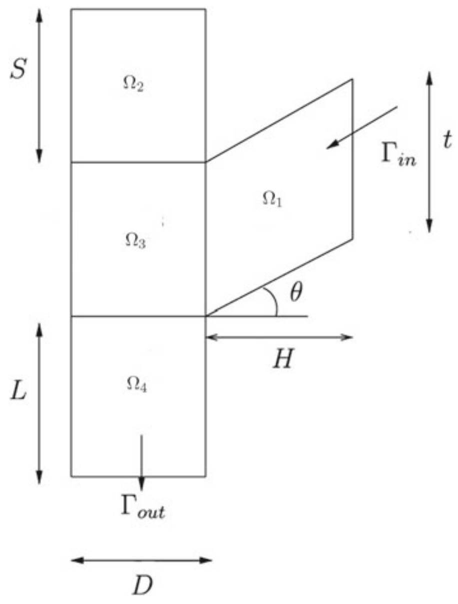


Fig. 8 Stokes cavity problem: error between FE and RB pressure for different possible options with $N_p = 13$

Fig. 9 Parametrized domain for T-bypass example



$\theta \in [0, \pi/6]$. The online parameter values are $t = D = L = S = H = 1.0$ and $\theta = \pi/6$.

In Figs. 10 and 11, we show the absolute error between FE and RB solutions for velocity and pressure, respectively for different stabilization options. From these results we see that in the online rectification there are some peaks at different values

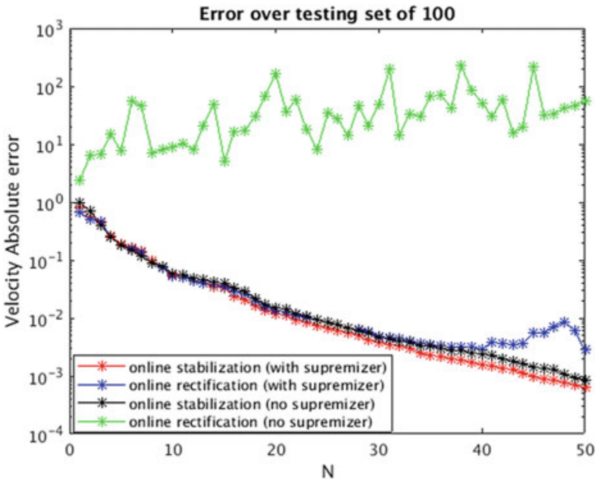


Fig. 10 T-bypass example: error between FE and RB velocity for different possible options with $N_u = 50$

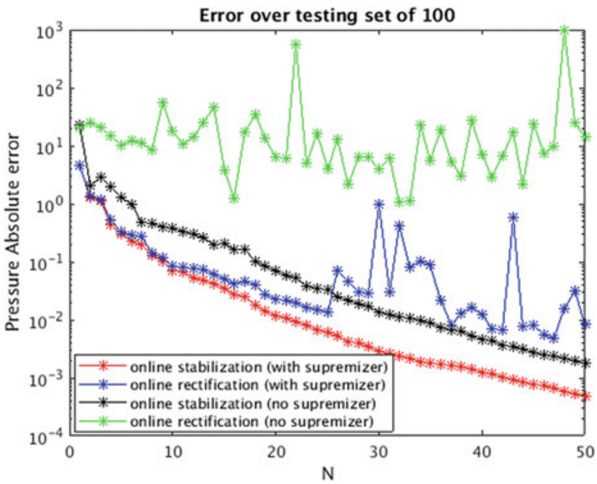


Fig. 11 T-bypass example: error between FE and RB velocity for different possible options with $N_p = 50$

of N. Some of these peaks are controlled by the enrichment of supremizer [3] in RB velocity spaces. But in case of pressure these peaks are not completely controlled by supremizer enrichment. In such cases one can use the POD orthonormalization [9] which can help to reduce the condition number of rectification matrix R_N^p.

4 Concluding Remarks

In this paper we have introduced the rectification method for parametrized Stokes problem. We have reviewed the paper by Maday et al. [16] for the advection-diffusion problem from which we extended the idea of post processing to get the rectified solution of reduced parametric viscous problem. More specifically the rectification method is used to improve the *offline-only stabilization* option. The main outcomes of this paper based on numerical experiments are as follows:

- we point out that in case of advection-dominated problem, even if we do not consider the vanishing viscosity (done in [16]), we are able to get a stable RB solution with the post processing (rectification) only, see for instance Fig. 3 (blue line);
- in case of Stokes problem we are able to get a stable RB solution for velocity and pressure while doing the rectification on *offline-only stabilized* RB solution;
- we have also compared rectification method with *offline-online stabilization* approach and conclude that *offline-online stabilization* is best way to stabilize;
- supremizers improves the pressure approximation and do not effect the velocity. However in more complex problem (T-shape), the role of supremizer for both velocity and pressure is more important.

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References

1. Ali, S., Ballarin, F., Rozza, G.: A reduced basis stabilization for the unsteady Stokes and Navier-Stokes equations. Submitted (2023)
2. Ali, S., Ballarin, F., Rozza, G.: Stabilized reduced basis methods for parametrized steady Stokes and Navier-Stokes equations. Comp. Math. Appl. **80**(11), 2399–2416 (2020)
3. Ballarin, F., Manzoni, A., Quarteroni, A., Rozza, G.: Supremizer stabilization of POD-Galerkin approximation of parametrized steady incompressible Navier-Stokes equations. Int. J. Num. Meth. Eng. **102**(5), 1136–1161 (2015)
4. Ballarin, F., Sartori, A., Rozza, G.: RBniCS - reduced order modelling in FEniCS. http://mathlab.sissa.it/rbnics (2016)
5. Barrault, M., Maday, Y., Nguyen, N.C., Patera, A.T.: An ‘empirical interpolation’ method: application to efficient reduced-basis discretization of partial differential equations. Comp. Ren. Math. **339**(9), 667–672 (2004)

6. Binev, P., Cohen, A., Dahmen, W., DeVore, R., Petrova, G., Wojtaszczyk, P.: Convergence rates for Greedy algorithms in reduced basis methods. SIAM J. Math. Anal. **43**(3), 1457–1472 (2011)
7. Brooks, A., Hughes, T.J.R.: Streamline Upwind/Petrov-Galerkin methods for advection dominated flows. In: Third International Conference on Finite Element Methods in Fluid Flow, vol. 2 (1980)
8. Chakir, R., Maday, Y.: A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE. In: 9e Colloque national en calcul des structures (2009)
9. Chakir, R., Hammond, K.: A non-intrusive reduced basis method for elastoplasticity problems in geotechnics. J. Comp. Appl. Math. **337**, 1–17 (2019)
10. Chakir, R., Maday, Y., Parnaudeau, P.: A non-intrusive reduced basis approach for parametrized heat transfer problems. J. Comput. Phys. **376**, 617–633 (2019)
11. Cohen, A., DeVore, R.: Kolmogorov widths under holomorphic mappings. IMA J. Num. Anal. **36**(1), 1–12 (2015)
12. Edmonds, J.: Matroids and the greedy algorithm. Math. Program. **1**, 127–136 (1971)
13. Herrero, H., Maday, Y., Pla, F.: RB (Reduced basis) for RB (Rayleigh-Bénard). Comp. Meth. App. Mech. Eng. **261**, 132–141 (2013)
14. Hesthaven, J.S., Rozza, G., Stamm, B.: Certified Reduced Basis Methods for Parametrized Partial Differential Equations. Springer Briefs in Mathematics (2015)
15. Logg, A., Mardal, K.A., Wells, G.: Automated Solution of Differential Equations by the Finite Element Method. Springer, Berlin (2012)
16. Maday, Y., Manzoni, A., Quarteroni, A.: An online intrinsic stabilization strategy for the reduced basis approximation of parametrized advection-dominated problems. Comp. Ren. Math. **354**(12), 1188–1194 (2016)
17. Pacciarini, P., Rozza, G.: Stabilized reduced basis method for parametrized advection-diffusion PDEs. Comp. Meth. App. Mech. Eng. **274**, 1–18 (2014)
18. Pacciarini, P., Rozza, G.: Stabilized reduced basis method for parametrized scalar advection-diffusion problems at higher Péclet number: roles of the boundary layers and inner fronts. In: 11th World Congress on Computational Mechanics, WCCM 2014, 5th European Conference on Computational Mechanics, ECCM 2014 and 6th European Conference on Computational Fluid Dynamics, ECFD 2014, pp. 5614–5624 (2014)
19. Quarteroni, A., Manzoni, A., Negri, F.: Reduced Basis Methods for Partial Differential Equations: An Introduction. Springer International Publishing (2015)
20. Rheinboldt, W.C.: On the theory and error estimation of the reduced basis method for multi-parameter problems. Nonlinear Anal. Theor. Meth. Appl. **21**(11), 849–858 (1993)
21. Rozza, G., Veroy, K.: On the stability of the reduced basis method for Stokes equations in parametrized domains. Comp. Meth. Appl. Mech. Eng. **196**, 1244–1260 (2007)