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Tatsien Li
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Synchronization for Wave Equations with Locally Distributed Controls

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Preface

Synchronization, as a type of prevalent natural and social phenomena, was discovered by Huygens in 1665 and began to be studied from the mathematical point of view by Wiener and others since the 1950s. At present, it is still a progressive research field with broad application prospects.

Starting from our systematic research on coupled systems of wave equations in 2012, the research on synchronization was expanded from the finite dimensional dynamical system based on ordinary differential equations to the infinite dimensional dynamical system based on partial differential equations, and it was closely connected with the research on controllability in control theory. For this purpose, we introduced the concepts of exact synchronization and approximate synchronization. The relevant results about synchronization achieved only through boundary control were collected in the monograph *Boundary Synchronization for Hyperbolic Systems* published by Birkhäuser Publishing House in 2019. This book was revised and published in Chinese by Shanghai Science and Technology Publishing House in 2021.

Realizing synchronization through boundary control is only a feasible option. In this monograph, we will further examine the situation of achieving synchronization through internal control, or through the combined effect of boundary control and internal control. Through in-depth analysis, it can be found that due to the use of internal controls, more deep-going results on synchronization can be obtained. Not only do they make the corresponding synchronization theory more precise and complete, but they propose some new research topics, which endow this monograph with distinctive features and its own style.

Since the major part of this monograph was completed during the COVID-19 pandemic from 2019 to 2023, when academic visits and exchange activities could not be carried out according to the original plan, we resorted to on-line communications instead. Nevertheless, it is gratifying that we never slackened, but redoubled our efforts to complete the preparation work and writing of this book.

Fudan University and its School of Mathematical Sciences, the Institut de Recherche Mathématique Avancée of University of Strasbourg, and the National

Natural Science Foundation of China have all provided long-term support and assistance to the research work. Here, we would like to express our heartfelt gratitude to them all.

In addition, Rao Bopeng would like to extend his sincere congratulations to his daughter, Isabelle, whose doctoral graduation ceremony coincided with the completion of the book.

Our thanks should also go to Dr. Zu Chengxia, who participated in writing and compiling parts of this book while studying for her doctor's degree. She will also be responsible for translating the book into Chinese.

Shanghai, China
June 2023

Tatsien Li
Bopeng Rao

Contents

Chapter 1
Introduction

In the monograph [1], we have made an abundant study on the boundary synchronization for a coupled system of wave equations. The present work will be concentrated on the internal controllability and synchronization for the problem with Dirichlet boundary condition. The material in this book is mainly selected from authors' recent works [2–6], which present the state of the art on the theory of internal synchronization.

Here are the main contributions.

In Part I, we consider the controllability and synchronization of a coupled system of wave equations with Dirichlet boundary condition by internal controls locally distributed on a subdomain ω of the domain Ω.

Firstly, we show that Kalman's rank condition is not only necessary but also surprisingly sufficient for the approximate internal controllability without any geometrical conditions on the subdomain ω, either any algebraic conditions on the coupling matrix A. Moreover, unlike the case of boundary control, the controllability time is determined only by the geodesic diameter of Ω, independently of the number of equations in the system or the rank of the control matrix. This is fundamentally different from the approximate boundary controllability, in which Ω should be a star-shaped domain, A must be a cascade matrix and the controllability time is undeterminable.

Secondly, based on this discovery, we clarify that a series of important properties, such as the independence of approximately synchronizable state by groups with respect to applied controls, the linear independence of the components of the approximately synchronizable state by groups, and the possibility of the extensibility of approximate synchronization etc., are all the consequence of the minimality of Kalman's rank condition. In particular, we affirm that the approximate internal synchronization is always in the pinning sense. So far, we have given a complete answer to these fundamental questions, which have plagued us for a long time.

Finally, we investigate the dependence of the exactly synchronizable state with respect to applied controls. We reveal that the exactly synchronizable state by groups can be divided into two groups. The first group can be approximately driven to zero, while the second group is independent of applied controls, only this group can be determined by te initial data. By this way, we have clarified the situation

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and satisfactorily answered the corresponding questions. The result presents a great interest for the applications as well as for the synchronization theory itself.

The same problem with Neumann boundary condition can be similarly considered without any essential difficulty.

In Part II, we consider the controllability and synchronization by both internal controls and Dirichlet boundary controls. The main novelty consists of the correspondence between the two kinds of controls.

We have shown that when the controls are fairly distributed within the system, Kalman's rank condition is still not only necessary but also sufficient for the uniqueness of solution to the adjoint system with incomplete internal and boundary observations, therefore for the approximate controllability by mixed internal and boundary controls. It is not a simple collection of known results on internal controllability and boundary controllability, but rather the coordination of several composites in a complex system!

Similarly, under suitable coordination between the mixed controls, the full rank condition on the control matrix is not only necessary but also sufficient for the exact controllability.

The work in this part raises many interesting questions and opens up a new direction on this topic.

Many results of the monograph could be extended to other time reversible linear evolutionary systems for example to plate models, Maxwell's equations, elasticity systems. Moreover, the feedback stabilization will be deeply developed in the forthcoming works.

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Chapter 2
Algebraic Preliminaries

For the sake of reading, here we collect some useful algebraic results, some of them can be found in the monograph [1]. We suggest that the readers skip this chapter at the first lecture. Only when they meet some difficulties in the forthcoming chapters, they may go back to this chapter and find useful material in it.

We denote by A a matrix of order N, and by D a full column-rank matrix of order $N \times M$. All these matrices are of constant entries.

Recall the following fundamental property on the Kalman's matrix.

Lemma 2.1 ([2, *Lemma 2.5*]) Let $d \geqslant 0$ be an integer. Then the control matrix D satisfies Kalman's rank condition:

$$\operatorname{rank}(D, AD, \ldots, A^{N-1}D) = N - d \tag{2.1.1}$$

if and only if d is the dimension of the largest subspace which is invariant for A^T and contained in $\operatorname{Ker}(D^T)$. The largest subspace invariant for A^T and contained in $\operatorname{Ker}(D^T)$ is given by

$$V = \operatorname{Ker}(D, AD, \ldots, A^{N-1}D)^T. \tag{2.1.2}$$

Consider the case with $D = (D_1, D_2)$, where D_1 and D_2 are full column-rank matrices of order $N \times M_1$ and $N \times M_2$ respectively.

Lemma 2.2 Let V_1, V_2 and V denote the largest subspaces invariant for A^T and contained in $\operatorname{Ker}(D_1^T)$, $\operatorname{Ker}(D_2^T)$ and $\operatorname{Ker}(D^T)$, respectively. We have

$$V_1 \cap V_2 = V. \tag{2.1.3}$$

Proof Since $\operatorname{Ker}(D_1^T) \cap \operatorname{Ker}(D_2^T) = \operatorname{Ker}(D^T)$, and $V_1 \cap V_2$ is invariant for A^T and contained in $\operatorname{Ker}(D_1^T) \cap \operatorname{Ker}(D_2^T)$, we get $V_1 \cap V_2 \subseteq V$. Conversely, V is invariant for A^T and contained in $\operatorname{Ker}(D^T) \subseteq \operatorname{Ker}(D_1^T) \cap \operatorname{Ker}(D_2^T)$, then $V \subseteq V_1 \cap V_2$. □

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Definition 2.1 Two systems of vectors $\mathcal{E}_1, \ldots, \mathcal{E}_d$ and $e_1, \ldots, e_d$ of $\mathbb{R}^N$ are bi-orthonormal if

$$\mathcal{E}_k^T e_l = \delta_{kl}, \quad 1 \leqslant k, l \leqslant d, \tag{2.1.4}$$

where δ_{kl} is the Kronecker symbol. Accordingly, the corresponding subspaces $V = \mathrm{Span}\{\mathcal{E}_1, \ldots, \mathcal{E}_d\}$ and $W = \mathrm{Span}\{e_1, \ldots, e_d\}$ are bi-orthonormal.

The following simple algebraic tools will be frequently used in this monograph.

Lemma 2.3 ([3]) Two non trivial subspaces V and W are bi-orthonormal if and only if

$$\dim(V) = \dim(W) \quad \text{and} \quad V \cap W^{\perp} = \{0\} \tag{2.1.5}$$

or equivalently if and only if V is a supplement of $W^{\perp}$.

Lemma 2.4 ([4]) A subspace V of $\mathbb{R}^N$ is invariant for A, namely, $AV \subseteq V$ if and only if its orthogonal supplement $V^{\perp}$ is invariant for A^T, namely, $A^T V^{\perp} \subseteq V^{\perp}$.

Now we introduce the notion of synchronization. Let $p \geqslant 1$ be an integer and

$$0 = n_0 < n_1 < \ldots < n_p = N \tag{2.1.6}$$

be a partition with $n_r - n_{r-1} \geqslant 2$ for $1 \leqslant r \leqslant p$.

Let $U = (u^{(1)}, \ldots, u^{(N)})^T$ be a vector of $\mathbb{R}^N$. We arrange its components into p groups:

$$(u^{(1)}, \ldots, u^{(n_1)}), (u^{(n_1+1)}, \ldots, u^{(n_2)}), \ldots, (u^{(n_{p-1}+1)}, \ldots, u^{(n_p)}) \tag{2.1.7}$$

such that the following condition of synchronization by p-groups

$$\begin{cases} u^{(1)} = \ldots = u^{(n_1)}, \\ u^{(n_1+1)} = \ldots = u^{(n_2)}, \\ \ldots\ldots \\ u^{(n_{p-1}+1)} = \ldots = u^{(n_p)} \end{cases} \tag{2.1.8}$$

holds.

Let S_r be a full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \leqslant r \leqslant p. \tag{2.1.9}$$

We define the $(N - p) \times N$ matrix C_p of synchronization by p-groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{2.1.10}$$

Then (2.1.8) can be equivalently written as

$$C_p U = 0. \tag{2.1.11}$$

Moreover, we have

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \ldots, e_p\} \tag{2.1.12}$$

with

$$e_r = (0, \ldots, 0, \overset{(n_{r-1}+1)}{1}, \ldots, \overset{(n_r)}{1}, 0, \ldots, 0)^T, \quad 1 \leqslant r \leqslant p. \tag{2.1.13}$$

The followings properties on the matrix C_p will be frequently used.

Lemma 2.5 ([1, *Proposition 2.11*]) We have

$$\mathrm{rank}(C_p D) = \mathrm{rank}(D) \text{ if and only if } \mathrm{Ker}(C_p) \cap \mathrm{Im}(D) = \{0\}, \tag{2.1.14}$$

or equivalentely,

$$\mathrm{rank}(C_p D) = \mathrm{rank}(C_p) \text{ if and only if } \mathrm{Ker}(D^T) \cap \mathrm{Im}(C_p^T) = \{0\}. \tag{2.1.15}$$

Lemma 2.6 Assume that

$$\mathrm{rank}(C_p D) = \mathrm{rank}(D) = N - p. \tag{2.1.16}$$

Then $\mathrm{Ker}(D^T)$ and $\mathrm{Ker}(C_p)$ are bi-orthogonal, consequently, we have

$$\mathrm{Ker}(D^T) \bigoplus \mathrm{Im}(C_p^T) = \mathbb{R}^N. \tag{2.1.17}$$

Proof By Lemma 2.5 and noting $\mathrm{rank}(C_p) = N - p$, we have

$$\mathrm{Ker}(D^T) \cap \mathrm{Im}(C_p^T) = \{0\}.$$

Noting $\dim \mathrm{Ker}(D^T) = \dim \mathrm{Ker}(C_p) = p$, we conclude the proof by Lemma 2.3. □

Lemma 2.7 ([1, *Proposition 2.15*]) The following assertions are equivalent:
(a) A satisfies the condition of C_p-compatibility:

$$A\,\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p); \tag{2.1.18}$$

(b) there exists a unique matrix A_p of order $(N-p)$, such that

$$C_p A = A_p C_p, \tag{2.1.19}$$

where the reduced matrix A_p is given by

$$A_p = C_p A C_p^+ \tag{2.1.20}$$

with the Moore-Penrose inverse:

$$C_p^+ = C_p^T (C_p C_p^T)^{-1}. \tag{2.1.21}$$

Lemma 2.8 ([1, *Proposition 2.16*]) Assume that A satisfies the condition of C_p-compatibility (2.1.18). Let A_p be defined by (2.1.20) and $D_p = C_p D$. Then we have

$$\operatorname{rank}(D_p, A_p D_p, \ldots, A_p^{N-p-1} D_p) = \operatorname{rank} C_p(D, AD, \ldots, A^{N-1}D). \tag{2.1.22}$$

When A does not satisfy the condition of C_p-compatibility, we introduce the internal extension matrix $C_{\tilde{p}}^T$ of order $(N-\widetilde{p}) \times N$ with $\widetilde{p} \leqslant p$ given by

$$\operatorname{Im}(C_{\tilde{p}}^T) = \operatorname{Span}\{C_p^T, A^T C_p^T, \ldots, (A^T)^{N-1} C_p^T\}. \tag{2.1.23}$$

By Cayley-Hamilton's Theorem, $\operatorname{Im}(C_{\widetilde{p}}^T)$ is invariant for A^T. Then, by Lemma 2.4, $A \operatorname{Ker}(C_{\widetilde{p}}) \subseteq \operatorname{Ker}(C_{\widetilde{p}})$, namely, A satisfies the condition of $C_{\widetilde{p}}$-compatibility (2.1.18) with C_p replaced by $C_{\widetilde{p}}$. Moreover, we have

Lemma 2.9 Assume that

$$\operatorname{Im}(C_{\widetilde{p}}^T) \cap V = \{0\}, \tag{2.1.24}$$

$$\operatorname{rank}(D, AD, \ldots, A^{N-1}D) = N - p, \tag{2.1.25}$$

where $V = \operatorname{Ker}(D, AD, \ldots, A^{N-1}D)^T$ is the largest subspace invariant for A^T and contained in $\operatorname{Ker}(D^T)$; or assume that

$$\operatorname{rank}(D) = N - p, \tag{2.1.26}$$

$$\operatorname{rank}(C_{\widetilde{p}} D) = N - \widetilde{p}. \tag{2.1.27}$$

Then A satisfies the condition of C_p-compatibility (2.1.18).

Proof By Lemma 2.5, conditions (2.1.24) and (2.1.25) imply the non extensibility of $\operatorname{Im}(C_p^T)$:

$$N - p \geqslant \operatorname{rank} C_{\widetilde{p}}(D, AD, \ldots, A^{N-1}D) = \operatorname{rank}(C_{\widetilde{p}}) = N - \widetilde{p}. \tag{2.1.28}$$

Similarly, conditions (2.1.26) and (2.1.27) imply the non extensibility of $\operatorname{Im}(C_p^T)$:

$$N - p \geqslant \operatorname{rank}(C_{\widetilde{p}} D) = \operatorname{rank}(C_{\widetilde{p}}) = N - \widetilde{p}. \tag{2.1.29}$$

It follows that $A^T \operatorname{Im}(C_p^T) \subseteq \operatorname{Im}(C_p^T)$. By Lemma 2.4, A satisfies the condition of C_p-compatibility (2.1.18). □

Lemma 2.10 Assume that

$$\operatorname{rank} C_p(D, AD, \ldots, A^{N-1}D) = N - p, \tag{2.1.30}$$

$$\operatorname{rank}(D, AD, \ldots, A^{N-1}D) = N - p. \tag{2.1.31}$$

Then there exists a matrix Q_p of order $N \times (N - p)$, such that for any given $U \in \mathbb{R}^N$, we have

$$U = \sum_{r=1}^{p} \psi_r e_r + Q_p C_p U, \tag{2.1.32}$$

where $\operatorname{Ker}(C_p) = \operatorname{Span}\{e_1, \ldots, e_p\}$, $V = \operatorname{Span}\{\mathcal{E}_1, \ldots, \mathcal{E}_p\}$ is the largest subspace invariant for A^T and contained in $\operatorname{Ker}(D^T)$, and $\psi_r = \mathcal{E}_r^T U$ for $r = 1, \ldots, p$.

Proof Noting that $\dim \operatorname{Im}(C_p^T) = N - p$, by Lemma 2.5, condition (2.1.30) implies that $V \cap \operatorname{Im}(C_p^T) = \{0\}$. By Lemma 2.1, $\dim(V) = \dim \operatorname{Ker}(C_p) = p$. Applying Lemma 2.3, V and $\operatorname{Ker}(C_p)$ are bi-orthonormal, and $\operatorname{Im}(C_p^T)$ and $V^\perp$ are also bi-orthonormal. Then we can choose

$$\mathcal{E}_r^T e_s = \delta_{rs}, \quad 1 \leqslant r, s \leqslant p \tag{2.1.33}$$

and an $N \times (N - p)$ matrix Q_p by $\operatorname{Im}(Q_p) = V^\perp$, such that

$$C_p Q_p = I_{N-p}. \tag{2.1.34}$$

Moreover, $\operatorname{Ker}(C_p)$ is a supplement of $\operatorname{Im}(Q_p)$, then, for any given $U \in \mathbb{R}^N$, there exist $x_1, \ldots, x_p \in \mathbb{R}$ and $Y \in \mathbb{R}^{N-p}$, such that

$$U = \sum_{s=1}^{p} x_s e_s + Q_p Y. \tag{2.1.35}$$

Noting (2.1.34) and applying C_p to (2.1.35), we get $Y = C_p U$. Similarly, noting (2.1.33) and applying $\mathcal{E}_r^T$ to (2.1.35), we get $x_r = \psi_r$ for $r = 1, \ldots, p$. The proof is complete. □

When conditions (2.1.30) and (2.1.31) don't hold simultaneously, we have

$$\operatorname{rank}(D, AD, \ldots, A^{N-1}D) > \operatorname{rank} C_p(D, AD, \ldots, A^{N-1}D). \tag{2.1.36}$$

In order to apply Lemma 2.10, we will introduce the external extension matrix C_q.

For $1 \leqslant i \leqslant m$, let λ_i be the eigenvalues of A^T and denote by

$$\mathcal{E}_{i0} = 0, \quad A^T \mathcal{E}_{ij} = \lambda_i \mathcal{E}_{ij} + \mathcal{E}_{i,j-1}, \quad 1 \leqslant j \leqslant d_i \tag{2.1.37}$$

the corresponding Jordan chain (see [5, 6]). Let I denote the set of indices i such that

$$I = \{i : \quad \mathcal{E}_{i\overline{d}_i} \in \mathrm{Im}(C_p^T) \text{ with } 1 \leqslant \overline{d}_i \leqslant d_i\}. \tag{2.1.38}$$

The internal extension matrix of order $(N-q) \times N$ by

$$\mathrm{Im}(C_q^T) = \bigoplus_{i \in I} \mathrm{Span}\{\mathcal{E}_{i1}, \ldots, \mathcal{E}_{i\overline{d}_i}, \ldots, \mathcal{E}_{id_i}\} \tag{2.1.39}$$

with

$$\mathrm{Ker}(C_q) = \mathrm{Span}\{\epsilon_1, \ldots, \epsilon_q\} \tag{2.1.40}$$

and

$$q = N - \sum_{i \in I} d_i. \tag{2.1.41}$$

We first improve the number of rank in (2.1.30).

Lemma 2.11 Let A satisfy the condition of C_p-compatibility (2.1.18). Assume that (2.1.30) holds. Then we have

$$\mathrm{rank}\, C_q(D, AD, \ldots, A^{N-1}D) = N - q, \tag{2.1.42}$$

where C_q is defined by (2.1.39).

Proof Assume that

$$\mathrm{rank}(C_q(D, AD, \ldots, A^{N-1}D)) < N - q. \tag{2.1.43}$$

By Lemma 2.5, we have

$$\mathrm{Im}(C_q^T) \cap \mathrm{Ker}(D, AD, \ldots, A^{N-1}D)^T \neq \{0\}. \tag{2.1.44}$$

By Lemma 2.1, $V = \mathrm{Ker}(D, AD, \ldots, A^{N-1}D)^T$ is invariant for A^T and contained in $\mathrm{Ker}(D^T)$. Since $\mathrm{Im}(C_q^T)$ is invariant for A^T, then, A^T admits an eigenvector $E \in \mathrm{Im}(C_q^T) \cap V$. By the construction given by (2.1.39), $\mathrm{Im}(C_q^T)$ is the extension of $\mathrm{Im}(C_p^T)$ by adding root vectors of A^T, so $E \in \mathrm{Im}(C_p^T) \cap V$, namely,

$$\mathrm{Im}(C_p^T) \cap \mathrm{Ker}(D, AD, \ldots, A^{N-1}D)^T = \mathrm{Im}(C_p^T) \cap V \neq \{0\}. \tag{2.1.45}$$

By Lemma 2.5, we have

$$\operatorname{rank}(C_p(D, AD, \ldots, A^{N-1}D)) < N - p. \tag{2.1.46}$$

This contradicts (2.1.30). □

Condition (2.1.42) implies that

$$\operatorname{rank}(D, AD, \ldots, A^{N-1}D) \geqslant N - q. \tag{2.1.47}$$

In particular, the equality holds in (2.1.47) with the control matrix D_q of order $N \times (N-p)$ defined by

$$\operatorname{Ker}(D_q^T) = \bigoplus_{i \in I^c} \operatorname{Span}\{\mathcal{E}_{i1}, \ldots, \mathcal{E}_{id_i}\} \bigoplus \bigoplus_{i \in I} \operatorname{Span}\{\mathcal{E}_{i\bar{d}_i+1}, \ldots, \mathcal{E}_{id_i}\}, \tag{2.1.48}$$

where I^c denotes the supplement of I. More precisely, we have the following

Lemma 2.12 Let C_q and D_q be defined by (2.1.39) and (2.1.48), respectively. We have

$$A\operatorname{Ker}(C_q) \subseteq \operatorname{Ker}(C_q), \tag{2.1.49}$$

$$\operatorname{rank}(D_q, AD_q, \ldots, A^{N-1}D_q) = N - q, \tag{2.1.50}$$

$$\operatorname{rank} C_q(D_q, AD_q, \ldots, A^{N-1}D_q) = N - q, \tag{2.1.51}$$

$$\operatorname{rank} C_p(D_q, AD_q, \ldots, A^{N-1}D_q) = N - p. \tag{2.1.52}$$

Proof By (2.1.39), $\operatorname{Im}(C_q^T)$ is invariant for A^T, then by Lemma 2.4, $\operatorname{Ker}(C_q)$ is invariant for A.

By (2.1.48), we easily check that the subspace

$$\bigoplus_{i \in I^c} \operatorname{Span}(\mathcal{E}_{i1}, \ldots, \mathcal{E}_{id_i}) \tag{2.1.53}$$

is the largest subspace invariant for A^T and contained in $\operatorname{Ker}(D_q^T)$. By Lemma 2.1, we have

$$\operatorname{Ker}(D_q, AD_q, \ldots, A^{N-1}D_q)^T = \bigoplus_{i \in I^c} \operatorname{Span}(\mathcal{E}_{i1}, \ldots, \mathcal{E}_{id_i}). \tag{2.1.54}$$

Still by Lemma 2.1, we get (2.1.50).

Similarly, by (2.1.39) and (2.1.54), we have

$$\begin{aligned} &\operatorname{Ker}(D_q, AD_q, \ldots, A^{N-1}D_q)^T \cap \operatorname{Im}(C_q^T) \\ = &\bigoplus_{i \in I^c} \operatorname{Span}(\mathcal{E}_{i1}, \ldots, \mathcal{E}_{id_i}) \bigcap \bigoplus_{i \in I} \operatorname{Span}(\mathcal{E}_{i1}, \ldots, \mathcal{E}_{id_i}) = \{0\}. \end{aligned}$$

Noting $\operatorname{Im}(C_p^T) \subseteq \operatorname{Im}(C_q^T)$, we get