Lecture Notes in Networks and Systems 934

Radek Silhavy
Petr Silhavy *Editors*

Software Engineering Methods in Systems and Network Systems

Proceedings of 7th Computational Methods in Systems and Software 2023, Vol. 2

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Preface

Welcome to Volume 1 of the conference proceedings for the esteemed Computational Methods in Systems and Software 2023 (CoMeSySo). This volume, titled "Software Engineering Methods in Systems and Network Systems," encapsulates the innovative strides and groundbreaking research presented by experts, scholars, and professionals from around the globe.

In today's digital age, the role of software engineering in shaping the future of systems and network systems cannot be understated. The papers and articles contained within this volume delve deep into the methodologies, practices, and tools that are at the forefront of this dynamic field. From novel approaches to software development to the optimisation of network systems, the breadth and depth of topics covered here are a testament to the vibrant and evolving nature of software engineering.

The CoMeSySo conference has always been a melting pot of ideas, fostering collaborations, and discussions that push the boundaries of what's possible in computational methods. This year, we were privileged to witness a confluence of minds, all dedicated to advancing the state of the art in software engineering for systems and network systems.

We want to extend our heartfelt gratitude to all the authors, reviewers, and members of the organising committee. Their dedication, hard work, and passion have made this volume not just a collection of papers but a beacon for future research and development.

To our readers, we hope this volume serves as both an inspiration and a resource. Whether you are a seasoned professional, a budding researcher, or a curious enthusiast, the insights and knowledge shared within these pages will enrich your understanding and fuel your passion for software engineering.

Thank you for being a part of this journey. We look forward to the continued growth and evolution of the CoMeSySo community and to the innovations that the future holds.

Radek Silhavy
Petr Silhavy

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Computer-Aided Adaptive Technologies for Teaching Law Students

Natalia Vyacheslavovna Yalaeva[1](✉), Natalia Valerievna Sadykova[2], and Tatiana Viktorovna Kudinova[1]

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Abstract. Modern legal education is inseparably linked to the use of information technologies and e-courses, as they are an effective learning tool. Among the main advantages of using LMS Moodle is the possibility to develop individual learning trajectories for teaching English. The article deals with the notion of adaptive teaching and the possibilities that LMS Moodle provides for the implementation of adaptive learning. #COMESYSO1120.

Keywords: Adaptive Technologies · LMS · Moodle · Testing · Individual Trajectory

1 Introduction

The purpose of the study was to explore the possibility of using Moodle for adaptive learning for teaching lawyers. Since the teacher cannot ignore the fact that different students have different levels of training even within the same group, the authors understood the importance of the task to individualize the educational routes of students for the successful development of the discipline with the help of new computer technology appeared. Within the framework of adaptive learning with modern learning tools, it is possible to create an environment that incorporates individual characteristics of each learner. The idea of adaptive learning and especially e-learning was introduced by a number of scholars. According to S. Oxman and W. Wong, the main goal of adaptive teaching is to improve the quality of student learning, and we as teachers are striving to do the same. An essential feature of adaptive learning then becomes the ability to automate the process [1]. The adaptive e-learning system was developed by Bradác and Klimes [2] in order to improve language learning substantially, focusing on the English language, using learning management systems (LMS). They created an adaptive system for the decision making support that automatically creates learning options suitable for the needs of each student in a group.

Souki et al. [3] emphasized the learning pathways of the students' preference. The scholars provided the evidence of the system's potential in developing specific aspects of self-managed learning and student performance.

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Adaptive testing is generally possible in personalized learning. For instance, Balogh et al. [4] examined the way in which IT subjects can be personalized in terms of a constructivist approach to the learner and proposed a methodology for creating an e-course that responds to the hypotheses of educational personalization.

A study by Susanti et al. [5] empirically proved the advantages of computer adaptive tests over basic, linear tests.

Unfortunately research on computerized adaptive testing in training lawyers (teaching English for professional purposes), has not been implemented.

The widely used Learning Management System (LMS) Moodle offers a number of functionalities that can be used for adaptive teaching.

According to a number of researchers, *e.g.* Fatueva, S. A. [6], Chelyshkova M.B. [7], Chernova N.I., etc., adaptive learning is a prospective approach for improving the efficiency of the educational process. N.I. Chernova points out that adaptive teaching is "a means for organizing highly effective, person-oriented student activity, that ensures productive learning of disciplinary knowledge and competencies and creates opportunities for applying them in the professional activity" [8]. Adaptive education means that learning material and methods are presented to the student according to their individual needs and the level of knowledge, thus maximizing the effectiveness of the training process. Various studies, including those in psychology and pedagogy, show that adaptive teaching can improve learning outcomes and reduce the time required to achieve the desired goals.

Thus, the theme of the work seems relevant and necessary for the individualization of the educational process of law students within the framework of adaptive learning through computerized programmes.

2 Methods

The first stage of the research (2020–2022 academic year) included the experiment with 40 first-year Master's students of the Ural State Law University's extramural department aged 22–28. The second stage (2022–2023 academic year) will include the experiment with Master's students of the full-time department of the Russian Technological University MIREA.

Participants of the research were offered to undergo training in the educational electronic environment, which includes instruction in the learning management system (LMS) Moodle and its activities (Forum, Chat, Lesson, Assignment). The materials used in the LMS Moodle e-course were taken from the textbook "Legal English for Advanced and Postgraduate Students", written by Ural State Law University English language teachers [9].

To solve the tasks set forth, a range of interconnected methods was used to achieve the goal and test the hypothesis:

- analytical method to analyze the functionality of LMS Moodle and its potential for adaptive learning.
- comparative analysis and synthesis of scientific sources to identify the best practices of adaptive learning.

- empirical method to monitor future lawyers' training for professional activity in order to identify the levels of the said training using interviewing and questionnaire survey;
- mathematical statistics method to analyze the experimental research data and their interpretation.

Adaptive English teaching is a method that incorporates technology to customize and individualize the learning process for each student. This method provides the opportunity to design an individual educational program that is sensitive to the level of knowledge, interests and needs of each learner.

The methodology can be implemented using computer programs and online platforms that can analyze learner data such as progress, task responses, time spent on the course and so forth. The information collected about every student is used to create an individual educational trajectory.

2.1 An Overview of Moodle

Moodle is a learning management system (LMS) that provides tools for creating and managing online courses.

The system enables users to create courses containing various types of materials such as text documents, videos, audio, quizzes, assignments, and so on. Moodle courses consist of modules that represent individual elements of the course such as forums, quizzes, assignments, H5P interactive content and SCORM package, etc. The Moodle forums allow students and teachers to communicate with each other, discuss course topics and ask questions.

LMS Moodle provides an opportunity for teachers to create different types of test tasks: multiple choice, matching, fill in the blanks, etc.

The "Assignment" element in Moodle is used for student submissions, teacher assessment and feedback.

Furthermore, students are provided with group activities for project work. The teacher can grade and mark the submissions. Timetable of activities, conferences and notifications can be created and shared with all users of the system.

Moodle allows system administrators to manage courses, modules and other elements of the system and provides a high level of security for users and the data stored in the system. N.V.Sadykova et al. point out "the flexibility and multifunctioning of Moodle system allows adequate conducting distance educational process with incorporation of all the educational elements and methods" [10].

In the Russian Technological University (RTU) MIREA and in the Ural State Law University, Master's students study both full-time and part-time, thus using Moodle platform for effective learning.

The structure of the e-learning course that we designed for Master's students includes an entry test, 2 levels of training (pre-intermediate and intermediate), mastering vocabulary, reading texts, studying grammar, listening comprehension, completing training exercises (in the adaptive mode) and a control test (in the interactive mode).

Let us focus on the Moodle testing system in more detail. The testing system is only a small part of the larger Moodle program. The system has several test modes: interactive and adaptive. To create tests in interactive mode in Moodle, the following steps should be taken:

1. Enter the course where you want to create a test.
2. Click on the "Add material or activity" button and select "Test".
3. Enter the name of the quiz and a description.
4. Select the type of test (e.g. multiple choice, matching, etc.).
5. Add questions to the test.
6. Set up test parameters such as execution time, number of attempts, etc.
7. Save the quiz and publish it in the course.

Once the quiz is created by the instructor, students will be able to take the quiz interactively, receiving automatic feedback on their answers and a grade for completing the quiz. The instructor will be able to analyze the results of the quiz and assess students' performance.

2.2 Adaptive Mode in Moodle

Adaptive mode in Moodle is a feature that allows instructors to create personalized learning paths for their students based on their individual needs and performance. The Adaptive Mode feature in Moodle uses rule-based mechanisms that automatically adapt course content, assignments and grades according to each student's learning needs.

In order to use the adaptive mode in Moodle, the teacher needs to do the following:

- set learning objectives: formulate learning objectives for the course and make a list of skills and knowledge to be acquired by the learners;
- define assessment criteria: define the criteria to be used to assess students' work, for example, quizzes, assignments and tests;
- define student profiles: the profiles of the students should be analyzed in order to identify their learning preferences, abilities and weaknesses. This can be done by means of an entry test;
- develop adaptive requirements: based on the learning objectives, assessment criteria and student profiles, adaptive rules should be developed to determine what content, activities and assessments should be presented to students, taking into account their individual needs;
- monitor students' progress: students' progress should be monitored and adaptive rules should be adjusted as necessary to ensure progress towards the learning objectives;
- provide feedback: provide students with feedback on their progress and achievements, and offer additional support as required.

The following steps are to be taken in order to create a test in adaptive mode in the Moodle system:

1. Enter the course for which the test is to be created.
2. Click on the "Add Material or Action" button and select "Test".
3. Enter the name of the quiz and a description.
4. Select the type of test "Adaptive test".
5. Add questions to the test. In this type of test, you need to add questions from the test bank of different levels of difficulty and level of knowledge.

6. Set up test settings such as test execution time, number of attempts, and add comments. In the comments, the teacher usually gives further advice on the material to be studied, based on the textbook, the relevant section in the course, and the PowerPoint presentation.
7. Save the test and publish it in the course.

Now the test tasks are selected and customized by the teacher from the test task bank. When re-testing, questions on a given topic are changed to check the mastery of the studied material. In the future, we are planning to use artificial intelligence to select training tasks from the bank.

During the study, which lasted a year and a half, all of the above steps and research methods were tested.

3 Outcomes

The conducted research showed that LMS Moodle offers a number of functionalities that can be used for adaptive learning of lawyers. The researchers made the conclusion that the adaptive mode in Moodle is an effective tool for providing an individual approach to student learning. By implementing the recommendations described above and using adaptive methods to adjust the course content, lessons and assessments according to the individual needs of each student, the researchers came to the following results: the involvement, motivation and performance of students increased; the adaptive teaching method allowed to create personalized courses and assignments based on the level of students.

4 Discussions

Discussions Based on the results, we see an increase in success rate (by 32%) and quality of knowledge (by 26%) (see Fig. 1). The quality success rate was taken as the number of students with good and very good grades multiplied by 100% and divided by the total number of students. The indicator of absolute success was taken as the number of students with ‘good’, ‘very good’ and ‘satisfactory’ grades multiplied by 100% and divided by the total number of students.

The data was taken from the LMS Moodle USLU control test results processing. The control point is testing conducted at the end of each module. Progress tests are conducted twice a semester.

The survey conducted by the teachers of English was created with the help of Google forms. The link to the survey was placed in the LMS Moodle course “Legal English for Advanced and Postgraduate Students”. The results of the survey showed the following data, graphically presented in Fig. 2.

Students were asked to answer several questions about the benefits of the new computerized adaptive course and testing system.

The basic idea behind adaptive language instruction is that students should receive individualized assistance based on their unique needs and level of language proficiency.

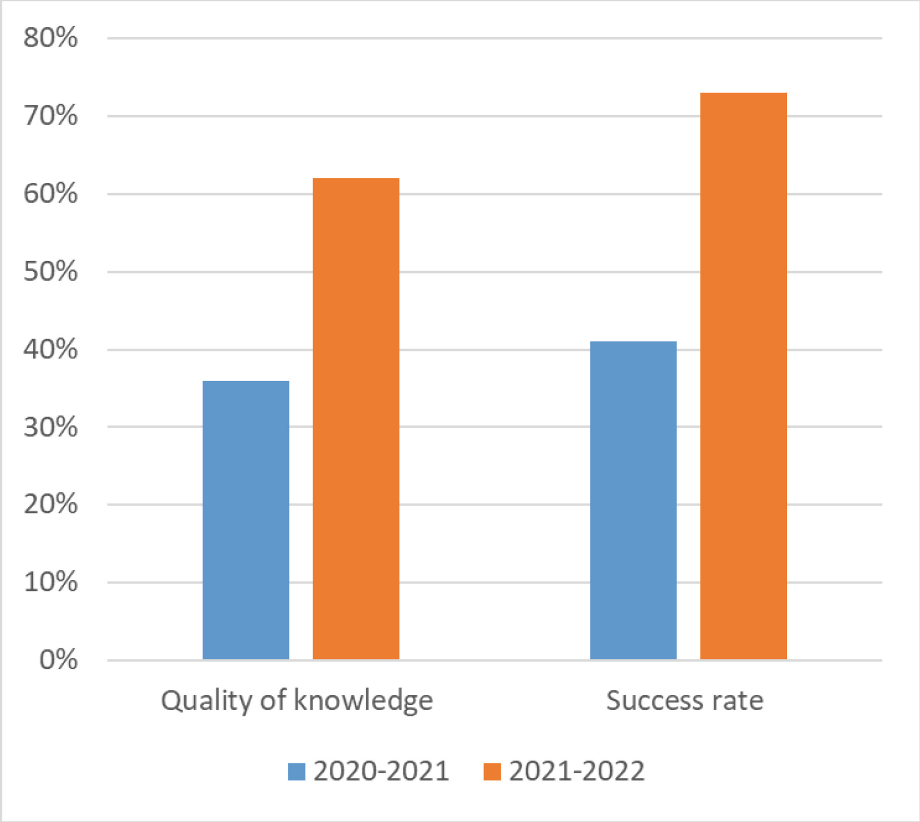

Fig. 1. Academic performance statistics in 2020–2021 and 2021–2022 academic years.

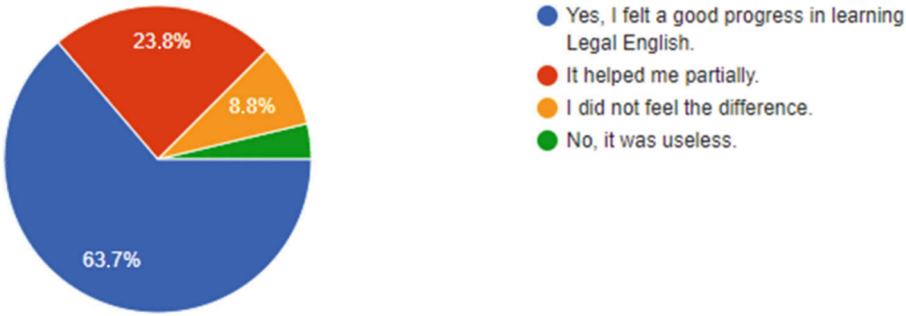

Fig. 2. The results of the survey conducted among the students who studied LMS Moodle course "Legal English for Advanced and Postgraduate Students" in 2020–22 academic years.

At the same time, materials and methods should be adapted to each individual student in order to ensure the best learning outcome.

The study is practically significant because it offers practical recommendations for using LMS Moodle for adaptive learning of lawyers.

The authors noted that while working with the course in Moodle students achieved a higher "individualization of learning, optimization of learning material, increased independence and level of motivation" [11].

According to other researchers of this topic, the purpose of teaching lawyers is "formation of a harmoniously developed personality, relying on information and communication capabilities of modern technologies, in particular, on the use of computer tests" [12]. We fully agree with the author of this statement. In the course of work, the authors used interactive technologies, including individual practical tasks, which significantly increased the number of correct answers during testing. The researchers identified a number of advantages of using Moodle adaptive learning. LMS Moodle allows you to work at any time in any place, get results in real time, track the time spent on tasks and progress of students, creating individual educational routes. It is also useful as a means of testing reducing the time spent on the test, individualizing tests to the students' needs, and fostering the progress of students by gradually making the tasks more and more complex.

The researchers identified a number of drawbacks of the system. First, it is necessary to have access to the Internet. Though, many students have various devices to have Internet access, for some of those who live in rural areas it is still a problem. The only decision here is to conduct tests on the territory of the campus during session time. Second, it is unknown whether the student performs tasks independently or uses someone's help as LMS Moodle system does not require to fulfil the tasks using cameras. We cannot solve this big problem at present. But we think that by increasing the motivation of students through working with adaptive learning system we will foster their desire to do all the tasks independently. Third, it should be noted that adaptive learning in Moodle requires significant time and effort at the stage of creating learning materials and system settings. In order for adaptive learning to work effectively, it is necessary to conduct a detailed analysis of students' needs, establish criteria for determining the level of knowledge and skills, and develop learning materials that will meet these needs. However, we hope that being engaged in this process we will be able to create new approaches to the said problem and find the way out of this situation by forming working groups of teachers, communicating with the colleagues from other universities and familiarizing ourselves with new computer technologies through special courses.

We are planning to continue our experiment and use neural networks to select training tasks from the bank. This kind of research is already underway at RTU MIREA.

5 Conclusion

Adaptive learning can significantly improve students' learning efficiency and increase their motivation to learn as it was achieved according to the results of the tests and questionnaire. Adaptive learning methods using Moodle are important for law students because they allow them to build individual learning trajectories. By adapting instructional methods to each student's individual needs, teachers can help students gain a deeper understanding of the material and develop critical thinking skills. In addition, adaptive teaching methods can be useful for students with disabilities. In general, the study confirms the prospects of using LMS Moodle for adaptive learning of law students.

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Application of a Local Additive Approximation Method for Evaluating the Efficiency of Maximum Likelihood Algorithms for the Joint Estimation of Regular and Discontinuous Information Process Parameters

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Abstract. An asymptotic method for evaluating the efficiency of the joint estimates of the observed information process parameter vector is considered, in view of these estimates are synthesized using the maximum likelihood method. It is assumed that the specified vector of parameters includes an arbitrary number of regular parameters and an arbitrary number of discontinuous parameters. Based on the additive-multiplicative representation of the moments of the decision determining statistics, the general asymptotic expressions for the probability densities and the first two moments of the estimates are found. It is shown that, under the conditions of a high a posteriori accuracy, the estimates of regular and discontinuous parameters are statistically independent. The application of the study results is illustrated by the example of determining the characteristics of the estimates of the time of arrival (the discontinuous parameter) and the central frequency (the continuous parameter) of a pulse signal with a Gaussian random substructure. #COMESYSO1120.

Keywords: Random process · Maximum likelihood estimate · Regular parameter · Discontinuous parameter · Local Markov approximation method · Local additive approximation method

1 Introduction

One of the most common methods for measuring (estimating) signal parameters against the noise background is the maximum likelihood (ML) method [1–5]. This method makes it possible to obtain simple but sufficiently efficient algorithms for estimating signal parameters and requires only a relatively small amount of a priori information. However, a reasonable choice of one or another estimation algorithm in various practical applications in the noise presence can be made only after the analysis of the following statistical characteristics of the estimates: estimate distribution function, bias, variance, dispersion, etc.

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The problem of obtaining exact analytical expressions for the characteristics of the signal parameter estimates remains a rather complex mathematical problem even in the simplest cases. Therefore, the examination of estimation algorithms is often carried out mainly by means of the computer statistical simulation, and only in the case of the fixed signal and noise parameters. Thus, it appears to be difficult to identify the general regularities occurring during the operation of estimation algorithms that are valid in a wide range of a priori conditions.

At present, there are several methods that allows obtaining asymptotically exact (as the signal-to-noise ratio (SNR) increases) expressions for the characteristics of maximum likelihood estimates (MLEs). The possibility of practical application of these methods for calculating the MLE characteristics depends on the analytical properties of the decision determining statistics of the estimation algorithm – the logarithm of the functional of the likelihood ratio (FLR).

Using the small parameter method [3], one can find asymptotically exact (with an increase in SNR) expressions for the characteristics of joint MLEs of an arbitrary finite number of signal parameters. However, in order to apply the method, it is required that the regularity conditions for the decision determining statistics of the MLE algorithm, that is the FLR logarithm, be satisfied as the function of all the estimated signal parameters. The signal parameters for which the regularity conditions are satisfied are called regular ones. In the case of the Gaussian FLR logarithm, the regularity conditions are reduced to the existence of continuous second derivatives for the first two moments of the FLR logarithm taken for the estimated parameters [3, 4].

If the derivatives of the moments of the FLR logarithm for the estimated parameters have discontinuities of the first kind at the point of the real values of these parameters, then the regularity conditions for this logarithm are violated and the small parameter method is inapplicable. Such signal parameters are called discontinuous or non-analytical [4, 6]. To calculate asymptotically exact (with an increase in SNR) expressions for the characteristics of MLE of a single discontinuous signal parameter, the local Markov approximation (LMA) method has been developed in [5, 6]. The characteristics of joint MLEs of several discontinuous parameters can be obtained using the local additive approximation (LAA) method together with the LMA method, as it has been done in [7]. The LAA method is applicable, if the moments of FLR logarithm, as the functions of the estimated signal parameters, allow a local additive-multiplicative representation [7]. With this representation, the moments of the FLR logarithm are expressed as the sum of a finite number of terms, each of which is a product of the functions of a single estimated parameter only. According to the LAA method, it is sufficient to implement the additive-multiplicative representation of the moments of the FLR logarithm in a small neighborhood of the point of real values of the estimated parameters.

In practice, signal processing is often carried out under the conditions of a partial violation of the regularity conditions of the decision determining statistics, when both discontinuous and regular parameters are unknown and subject to joint estimation. In [8], a method is proposed for calculating asymptotically exact (with an increase in SNR) expressions for the characteristics of joint MLEs of a single discontinuous and several regular parameters of quasi deterministic signals. In [9], this technique is generalized for a wider class of signals, including the Gaussian stochastic signals [10, 11]. Due to the results in [9], one can write asymptotically exact (with an increase in SNR) expressions for the characteristics of joint MLEs of an arbitrary finite number of discontinuous and regular parameters. To apply the results of [9], similarly to [7], it is necessary to have a local additive-multiplicative representation of the moments of the FLR logarithm for the estimated discontinuous signal parameters. In addition, similarly to [8], the local limits on the first and second derivatives of the moments of the FLR logarithm are imposed for the estimated regular signal parameters.

It is noteworthy that the local additive-multiplicative representation of the moments of FLR logarithm is possible for a wide class of signal parameters. For example, when processing a pulse with a Gaussian random substructure [10, 11], such a representation of the moments of the FLR logarithm is possible for the time of arrival, duration, moments of appearance and disappearance of the pulse, and also for the central frequency and the bandwidth of the spectral density of the random substructure. At the same time, these parameters in the general expressions for the moments of the logarithm of the FLR are usually not distinguished explicitly, but are included as the formal parameters of the modulating function (envelope) of the signal and its spectral density. Therefore, verification of the local limits [9] for the first and second derivatives of the moments of the FLR logarithm for these parameters generally turns out to be difficult. It is required to specify the modulating function and the spectral density of the signal, and thus a generality of the results obtained suffers.

In some cases, these difficulties can be overcome, if the characteristics of the joint estimates of discontinuous and regular parameters are calculated using the LAA method [7], taking into account the necessary generalizations for the case of regular parameters. Here it suffices to require a local additive-multiplicative representation of the moments of the FLR logarithm for two vector parameters – the vector of discontinuous parameters and the vector of regular parameters. Then the use of the LAA method [7] makes it possible, similarly to [9], to reduce the problem of finding the characteristics of joint MLEs of discontinuous and regular signal parameters to two simpler problems: to calculating 1) the characteristics of the joint MLEs of discontinuous parameters for the known regular parameters and 2) the characteristics of the joint MLEs of the regular parameters when the discontinuous parameters are known. To calculate the characteristics of the joint MLEs of the discontinuous signal parameters for a priori known regular parameters, the LAA and LMA methods can be applied as in [7]. To find the characteristics of the joint MLEs of the regular signal parameters for a priori known discontinuous parameters, one can use the small parameter method [3].

The applicability of the LAA method with a partial preservation of the regularity conditions for the FLR logarithm can be explained by the fact that this method does not refer to the regularity property of the decision determining statistics. The LAA method (subject to some modifications) is applicable both when the regularity conditions of the decision determining statistics for all or for some number of the jointly estimated parameters are met and when they are violated. It is only necessary that a local additive-multiplicative representation of the moments of the FLR logarithm of these parameters be allowed.

So, it is time to consider the application of the LAA method for the calculation of the asymptotic characteristics of the joint MLEs of an arbitrary (but finite) number of discontinuous and regular signal parameters.

2 Maximum Likelihood Estimates for Signal Parameters

2.1 Estimation Problem Statement

It is presupposed that, at the input of the processing device, the mixture

$$x(t) = s(t, \mathbf{l}_0, \mathbf{\Theta}_0) \otimes n(t), t \in [0, T] \tag{1}$$

arrives of the useful information signal $s(t, \mathbf{l}_0, \mathbf{\Theta}_0)$ and the noise $n(t)$ during the observation interval $[0, T]$. Here $\otimes$ generally means an arbitrary combination of signal and noise, that may, for example, be additive, multiplicative, etc.

The signal $s(t, \mathbf{l}_0, \mathbf{\Theta}_0)$ is characterized by the vectors of informative parameters $\mathbf{l}_0 = \| l_{01}, l_{02}, \ldots, l_{0p} \|$ and $\mathbf{\Theta}_0 = \| \Theta_{01}, \Theta_{02}, \ldots \Theta_{0r} \|$, containing $p \geq 1$ and $r \geq 1$ of the scalar parameters, respectively. The signal parameters l_{01}, l_{02}, …, l_{0p} are expected to be discontinuous, while the parameters Θ_{01}, Θ_{02}, …, Θ_{0r} – regular. It is presupposed that the discontinuous parameters l_{0i}, $i = \overline{1, p}$ of the received signal $s(t, \mathbf{l}_0, \mathbf{\Theta}_0)$ are unknown and take the values from the a priori definitional domain $\mathbf{l}_0 \in \Re_l$. The regular parameters $\Theta_{0j}, j = \overline{1, r}$ of the received signal are also unknown and take their values from the definitional domain $\mathbf{\Theta}_0 \in \Re_\Theta$.

Based on observed data $x(t)$ (1) and the available a priori information about the signal and noise, it is necessary to measure (estimate) the discontinuous $l_{0i}, i = \overline{1, p}$ and regular $\Theta_{0j}, j = \overline{1, r}$ parameters of the signal $s(t, \mathbf{l}_0, \mathbf{\Theta}_0)$.

2.2 Maximum Likelihood Estimation Algorithms

According to the ML method [1–5], in order to obtain the joint estimates (measurements) of the unknown parameters $(\mathbf{l}_0, \mathbf{\Theta}_0)$ of the signal $s(t, \mathbf{l}_0, \mathbf{\Theta}_0)$, one should, based on the observed data $x(t)$ (1), build a decision determining statistics – the logarithm $L(\mathbf{l}, \mathbf{\Theta}) = \ln \Lambda(\mathbf{l}, \mathbf{\Theta})$ of the FLR $\Lambda(\mathbf{l}, \mathbf{\Theta})$, as the function of the vectors $\mathbf{l} = \| l_1, l_2, \ldots, l_p \|$ and $\mathbf{\Theta} = \| \Theta_1, \Theta_2, \ldots \Theta_r \|$ of the current values of the discontinuous and regular signal parameters. The expression for the FLR logarithm

$$L(\mathbf{l}, \mathbf{\Theta}) \equiv L\big(l_1, l_2, \ldots, l_p, \Theta_1, \Theta_2, \ldots, \Theta_r\big) \tag{2}$$

in each particular case can be found based on the probabilistic (statistical) description of the received signal and noise. According to the definition [1–5], the FLR $\Lambda(\mathbf{l}, \mathbf{\Theta})$ is determined as the limit

$$\Lambda(\mathbf{l}, \mathbf{\Theta}) = \lim_{\Delta \to 0, N \to \infty} \big[W(\mathbf{X} | \mathbf{l}, \mathbf{\Theta}) \big/ W_0(\mathbf{X}) \big],$$

where $W(\mathbf{X} | \mathbf{l}, \mathbf{\Theta})$ is the conditional probability density of the sample $\mathbf{X} = \| x_1, x_2, \ldots, x_N \|, x_i = x(t_i)$ from the observed data $x(t)$ (1) at the time moments $t_i = i\Delta$

under the condition when the vectors of the unknown parameters of the received signal are equal to $\mathbf{l}$, $\mathbf{\Theta}$, and $W_0(\mathbf{X})$ is the probability density of sample $\mathbf{X}$ in the absence of a signal in the observed data. Here N is the number of the elements x_i of the sample $\mathbf{X}$, and $\Delta = T/N$ is the sampling time step, while the limit is calculated at the constant value of the observation interval duration $N\Delta = T$. According to the definition, the FLR $\Lambda(\mathbf{l}, \mathbf{\Theta})$ and the corresponding FLR logarithm $L(\mathbf{l}, \mathbf{\Theta}) = \ln \Lambda(\mathbf{l}, \mathbf{\Theta})$ characterize the probability density of the values $\mathbf{l}$, $\mathbf{\Theta}$ of the received signal parameters for the specified realization of the observed data $x(t)$ (1).

Next, $\Re = \Re_l \cup \Re_\Theta$ is introduced as the a priori range of the possible values of the estimated parameters $(\mathbf{l}_0, \mathbf{\Theta}_0)$ that is a sum of the a priori ranges $\Re_l$ and $\Re_\Theta$ of the possible values of the discontinuous $\mathbf{l}_0$ and regular $\mathbf{\Theta}_0$ parameters. Then the joint MLEs $l_{1m}, l_{2m}, \ldots, l_{pm}$ and $\Theta_{1m}, \Theta_{2m}, \ldots, \Theta_{rm}$ of the discontinuous $l_{01}, l_{02}, \ldots, l_{0p}$ and regular $\Theta_{01}, \Theta_{02}, \ldots, \Theta_{0r}$ signal parameters are calculated as the coordinates of the FLR logarithm (2) absolute maximum location within the a priori range $\Re$, namely:

$$\left(l_{1m}, l_{2m}, \ldots, l_{pm}, \Theta_{1m}, \Theta_{2m}, \ldots \Theta_{rm}\right) = \underset{(\mathbf{l},\mathbf{\Theta})\in\Re}{\arg\sup} L\left(l_1, l_2, \ldots, l_p, \Theta_1, \Theta_2, \ldots \Theta_r\right). \tag{3}$$

Similarly, the vectors $\mathbf{l}_m = \| l_{1m}, l_{2m}, \ldots, l_{pm} \|$ and $\mathbf{\Theta}_m = \| \Theta_{1m}, \Theta_{2m}, \ldots \Theta_{rm} \|$ of the joint MLEs of the parameters $\mathbf{l}_0$ and $\mathbf{\Theta}_0$ can be written as

$$(\mathbf{l}_m, \mathbf{\Theta}_m) = \underset{(\mathbf{l},\mathbf{\Theta})\in\Re}{\arg\sup} L(\mathbf{l}, \mathbf{\Theta}) \tag{4a}$$

or as

$$\mathbf{l}_m = \underset{\mathbf{l}\in\Re_l}{\arg\sup} L(\mathbf{l}, \mathbf{\Theta}_m),\ \mathbf{\Theta}_m = \underset{\mathbf{\Theta}\in\Re_\Theta}{\arg\sup} L(\mathbf{l}_m, \mathbf{\Theta}) \tag{4b}$$

If only the MLEs $\mathbf{l}_m$ or $\mathbf{\Theta}_m$ of the discontinuous or regular parameters is to be estimated and analyzed, then it would be better to represent the estimates as in [2, 3]:

$$\begin{aligned} \mathbf{l}_m &= \underset{\mathbf{l}\in\Re_l}{\arg\sup}\left[\sup_{\mathbf{\Theta}\in\Re_\Theta} L(\mathbf{l}, \mathbf{\Theta})\right] = \underset{\mathbf{l}\in\Re_l}{\arg\sup} L_{m1}(\mathbf{l}), \\ \mathbf{\Theta}_m &= \underset{\mathbf{\Theta}\in\Re_\Theta}{\arg\sup}\left[\sup_{\mathbf{l}\in\Re_l} L(\mathbf{l}, \mathbf{\Theta})\right] = \underset{\mathbf{\Theta}\in\Re_\Theta}{\arg\sup} L_{m2}(\mathbf{\Theta}), \end{aligned} \tag{5}$$

where the decision determining statistics $L_{m1}(\mathbf{l})$ and $L_{m2}(\mathbf{\Theta})$ of the discontinuous and regular signal parameter estimates $\mathbf{l}_m$ and $\mathbf{\Theta}_m$ appear to be the maximizations of the decision determining statistics $L(\mathbf{l}, \mathbf{\Theta})$ (2) by the discontinuous $\mathbf{l}$ and the regular $\mathbf{\Theta}$ parameters, respectively:

$$L_{m1}(\mathbf{l}) = \sup_{\mathbf{\Theta}\in\Re_\Theta} L(\mathbf{l}, \mathbf{\Theta}),\ L_{m2}(\mathbf{\Theta}) = \sup_{\mathbf{l}\in\Re_l} L(\mathbf{l}, \mathbf{\Theta}). \tag{6}$$

If one states that

$$\mathbf{l}_A(\mathbf{\Theta}) = \underset{\mathbf{l}\in\Re_l}{\arg\sup} L(\mathbf{l}, \mathbf{\Theta}),\ \mathbf{\Theta}_A(\mathbf{l}) = \underset{\mathbf{\Theta}\in\Re_\Theta}{\arg\sup} L(\mathbf{l}, \mathbf{\Theta}),$$

then the functionals $L_{m1}(\mathbf{l})$ and $L_{m2}(\mathbf{\Theta})$ can be defined as the section of the FLR logarithm $L(\mathbf{l}, \mathbf{\Theta})$ (2) by the surfaces $\mathbf{\Theta} = \mathbf{\Theta}_A(\mathbf{l})$ and $\mathbf{l} = \mathbf{l}_A(\mathbf{\Theta})$, respectively, i.e.

$$L_{m1}(\mathbf{l}) = L[\mathbf{l}, \mathbf{\Theta}_A(\mathbf{l})], L_{m2}(\mathbf{\Theta}) = L[\mathbf{l}_A(\mathbf{\Theta}), \mathbf{\Theta}] \tag{7}$$

If the regular parameters $\mathbf{\Theta}_0$ are a priori known, then one can write the joint MLEs $\mathbf{l}_{m0} = \| l_{1m0}, l_{2m0}, \ldots, l_{pm0} \|$ of the discontinuous parameters $\mathbf{l}_0$ as in [1–5]:

$$\mathbf{l}_{m0} = \arg\sup_{\mathbf{l} \in \Re_l} L_{01}(\mathbf{l}), L_{01}(\mathbf{l}) = L(\mathbf{l}, \mathbf{\Theta}_0). \tag{8}$$

The decision determining statistics $L_{01}(\mathbf{l})$ of the estimate algorithm (8) is the section of the FLR logarithm $L(\mathbf{l}, \mathbf{\Theta})$ (2) by the surface $\mathbf{\Theta} = \mathbf{\Theta}_0$. The asymptotically exact (with an increase in SNR) expressions for the characteristics of the joint MLEs $\mathbf{l}_{m0}$ (8) of the discontinuous parameters $\mathbf{l}_0$ in the case of the a priori known regular parameters $\mathbf{\Theta}_0$ can be found using LAA and LMA methods, as it is done in [7].

When the discontinuous parameters $\mathbf{l}_0$ are a priori known, the joint MLEs $\mathbf{\Theta}_{m0} = \| \Theta_{1m0}, \Theta_{2m0}, \ldots \Theta_{rm0} \|$ of the regular parameters $\mathbf{\Theta}_0$ can be defined as

$$\mathbf{\Theta}_{m0} = \arg\sup_{\mathbf{\Theta} \in \Re_\Theta} L_2(\mathbf{\Theta}), L_{02}(\mathbf{\Theta}) = L(\mathbf{l}_0, \mathbf{\Theta}) \tag{9}$$

The decision determining statistics $L_{02}(\mathbf{\Theta})$ of the estimate algorithm (9) is the section of the FLR logarithm $L(\mathbf{l}, \mathbf{\Theta})$ by the surface $\mathbf{l} = \mathbf{l}_0$. The asymptotically exact (with an increase in SNR) expressions for the characteristics of the joint MLEs $\mathbf{\Theta}_{m0}$ (9) of the regular parameters $\mathbf{\Theta}_0$ in the case of the a priori known discontinuous parameters $\mathbf{l}_0$ can be found using the small parameter method [3].

In the general case, $\mathbf{\Theta}_A(\mathbf{l}) \neq \mathbf{\Theta}_0$ and $\mathbf{l}_A(\mathbf{\Theta}) \neq \mathbf{l}_0$. And that is why the joint MLEs $\mathbf{l}_m$ and $\mathbf{\Theta}_m$ (3)–(5 of the discontinuous and regular signal parameters do not coincide with the corresponding MLEs $\mathbf{l}_{m0}$ (8) of the discontinuous parameters in the case of the known regular parameters as well as with the MLEs $\mathbf{\Theta}_{m0}$ (9) of the regular parameters in the case of the known discontinuous parameters. The small parameter and the LMA methods cannot be directly applied when determining the characteristics of the joint MLEs $(\mathbf{l}_m, \mathbf{\Theta}_m)$ (3)–(5) of the discontinuous and regular signal parameters.

The next task is to consider the LAA method application for the calculation of the asymptotically exact (with an increase in SNR) expressions for the characteristics of the joint MLEs (3)–(5) of the discontinuous and regular signal parameters. For this purpose, one should specify the local representations of the statistical characteristics of the FLR logarithm (2) resulting from the discontinuous and regular signal properties.

3 The Local Representations of the Estimate Algorithm Decision Determining Statistics Characteristics

3.1 The General Terms

The efficiency of the MLEs (3)–(5) is uniquely determined by the FLR logarithm (2) statistical characteristics. When analyzing MLEs, it is sufficient to represent the FLR logarithm as the sum $L(\mathbf{l}, \mathbf{\Theta}) = S(\mathbf{l}, \mathbf{\Theta}) + N(\mathbf{l}, \mathbf{\Theta})$, where $S(\mathbf{l}, \mathbf{\Theta}) = \langle L(\mathbf{l}, \mathbf{\Theta}) \rangle$ is the signal function (the deterministic component of the FLR logarithm), and $N(\mathbf{l}, \boldsymbol{\Theta}) = L(\mathbf{l}, \boldsymbol{\Theta}) - \langle L(\mathbf{l}, \boldsymbol{\Theta})\rangle$ is the noise function (the fluctuation component). Here and further $\langle\cdot\rangle$ means an averaging by the realizations of the observed data $x(t)$ (1) (or by the FLR logarithm (2) realizations) when the real values $\mathbf{l}_0$ and $\boldsymbol{\Theta}_0$ of the estimated signal parameters are fixed [3].

Following [1–5], one considers the FLR logarithm $L(\mathbf{l}, \boldsymbol{\Theta})$ as the Gaussian random field. Then, for the calculations of the characteristics of the MLEs, one suffices to the analysis of the first two FLR logarithm moments: namely, the signal function $S(\mathbf{l}, \boldsymbol{\Theta})$ and the correlation function $K(\mathbf{l}_1, \boldsymbol{\Theta}_1, \mathbf{l}_2, \boldsymbol{\Theta}_2) = \langle N(\mathbf{l}_1, \boldsymbol{\Theta}_1)N(\mathbf{l}_2, \boldsymbol{\Theta}_2)\rangle$ of the noise function $N(\mathbf{l}, \boldsymbol{\Theta})$. Here $\mathbf{l}_j = \| l_{j1}, l_{j2}, \ldots, l_{jp}\|$, $\boldsymbol{\Theta}_j = \| \Theta_{j1}, \Theta_{j2}, \ldots \Theta_{jr}\|$, $j = 1, 2$ are the notations for the vectors of the discontinuous and regular signal parameters.

In accordance with [2–5], one can state that the signal function $S(\mathbf{l}, \boldsymbol{\Theta})$ within the a priori range $\Re$ has a single maximum at the point $\mathbf{l} = \mathbf{l}_0$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0$ that is the internal point of the range $\Re$, while the realizations of the noise function $N(\mathbf{l}, \boldsymbol{\Theta})$ are the continuous ones with the probability of 1. Then the output voltage SNR for the MLE algorithm (3)–(5) can be written as in [2, 3]:

$$z = S(\mathbf{l}_0, \boldsymbol{\Theta}_0)\Big/\sqrt{\langle N^2(\mathbf{l}_0, \boldsymbol{\Theta}_0)\rangle} = A_S/\sigma_N, \tag{10}$$

where $A_S = S(\mathbf{l}_0, \boldsymbol{\Theta}_0) > 0$ is the value of the signal function absolute maximum and $\sigma_N^2 = \langle N^2(\mathbf{l}_0, \boldsymbol{\Theta}_0)\rangle$ is the noise function dispersion at $\mathbf{l} = \mathbf{l}_0$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0$. To find the asymptotic (with an increase in SNR) expressions for the MLE (3)-(5) characteristics, one presupposes that the SNR z (10) is so high that a high a posteriori estimate precision is achieved [3–5]. In this case, the joint MLEs $\mathbf{l}_m$, $\boldsymbol{\Theta}_m$ (3)–(5) are located within the small neighborhood of the point $\mathbf{l} = \mathbf{l}_0$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0$ of the signal function maximum and while $z \to \infty$ the estimates $\mathbf{l}_m$, $\boldsymbol{\Theta}_m$ undergo mean-square convergence towards the values $\mathbf{l}_0$, $\boldsymbol{\Theta}_0$ [3–5]. Then, for the MLE characteristics calculation, accounting for the behavior of the signal function $S(\mathbf{l}, \boldsymbol{\Theta})$ and the correlation function $K(\mathbf{l}_1, \boldsymbol{\Theta}_1, \mathbf{l}_2, \boldsymbol{\Theta}_2)$ in the small neighborhood of the point $\mathbf{l} = \mathbf{l}_0$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0$ with the increasing SNR z resulting in the decrease of this neighborhood value.

Specification of the local (in the small neighborhood of the point $(\mathbf{l}_0, \boldsymbol{\Theta}_0)$) representations of the first two moments of the FLR logarithm allows applying the LAA method for the calculation of the characteristics of the joint estimates (3)–(5) of the discontinuous and regular signal parameters.

3.2 An Additive-Multiplicative Representation of the FLR Logarithm Moments by the Vectors of the Discontinuous and Regular Signal Parameters

Now one focuses on the small neighborhood $\Re_\delta$ of the point $(\mathbf{l}_0, \boldsymbol{\Theta}_0)$ within the range of values $\Re$ of the parameters $\mathbf{l}$, $\boldsymbol{\Theta}$. Let $\delta_{li} = \max|l_i - l_{0i}|$, $l_i \in \Re_\delta$ is the value of the maximum deviation of the points belonging to the neighborhood $\Re_\delta$ from the point $(\mathbf{l}_0, \boldsymbol{\Theta}_0)$ by the coordinate l_i corresponding to the discontinuous parameter l_{0i}, while $\delta_{\Theta i} = \max|\Theta_i - \Theta_{0i}|$ is the value of the same deviation but by the coordinate Θ_i corresponding to the regular parameter Θ_{0i}. Then $\delta_l = \max(\delta_{l1}, \delta_{l2}, \ldots, \delta_{lp})$ and $\delta_\Theta = \max(\delta_{\Theta 1}, \delta_{\Theta 2}, \ldots, \delta_{\Theta r})$ are the maximum deviations of the points within the range $\Re_\delta$

from the point $(\mathbf{l}_0, \mathbf{\Theta}_0)$ by the discontinuous and regular parameters, respectively, while $\delta = \max(\delta_l, \delta_\Theta)$ is this very deviation but by the all the estimated signal parameters.

It is presupposed that the signal function $S(\mathbf{l}, \mathbf{\Theta})$ and the correlation function $K(\mathbf{l}_1, \mathbf{\Theta}_1, \mathbf{l}_2, \mathbf{\Theta}_2)$ allow the asymptotic (at $\delta \to 0$) additive-multiplicative representations in the neighborhood $\Re_\delta$ of the point $(\mathbf{l}_0, \mathbf{\Theta}_0)$:

$$S(\mathbf{l}, \mathbf{\Theta}) = \sum_{k=1}^{v} A_{Sk} S_{Uk}(\mathbf{l})\, S_{Rk}(\mathbf{\Theta}) + o\left(\delta^2\right),$$
$$K(\mathbf{l}_1, \mathbf{\Theta}_1, \mathbf{l}_2, \mathbf{\Theta}_2) = \sum_{k=1}^{u} \sigma_{Nk}^2 K_{Uk}(\mathbf{l}_1, \mathbf{l}_2)\, K_{Rk}(\mathbf{\Theta}_1, \mathbf{\Theta}_2) + o\left(\delta^2\right), \tag{11}$$

where $S_{Rk}(\mathbf{\Theta})$ and $K_{Rk}(\mathbf{\Theta}_1, \mathbf{\Theta}_2)$ are the functions depending on the regular parameters only, while $S_{Uk}(\mathbf{l})$ and $K_{Uk}(\mathbf{l}_1, \mathbf{l}_2)$ are the functions depending on the discontinuous parameters only, $v \geq 1, u \geq 1$ denoting an arbitrary but a finite number of the summands in the sums and $o\left(\delta^2\right)$ denotes the higher-order infinitesimal terms compared with δ^2.

The parameters $\mathbf{\Theta}$ $(\mathbf{\Theta}_i)$ are the regular ones, so that the functions $S_{Rk}(\mathbf{\Theta})$, $K_{Rk}(\mathbf{\Theta}_1, \mathbf{\Theta}_2)$ are continuous and continually differentiated within the considered neighborhood of the point $(\mathbf{l}_0, \mathbf{\Theta}_0)$ doubly, at least [2–5]. Then the functions $S_{Rk}(\mathbf{\Theta})$ and $K_{Rk}(\mathbf{\Theta}_1, \mathbf{\Theta}_2)$ can be normalized so that the conditions

$$S_{Rk}(\mathbf{\Theta}_0) = 1,\ K_{Rk}(\mathbf{\Theta}_0, \mathbf{\Theta}_0) = 1 \tag{12}$$

hold.

The parameters $\mathbf{l}$ $(\mathbf{l}_i)$ are the discontinuous ones, so that the functions $S_{Uk}(\mathbf{l})$ and $K_{Uk}(\mathbf{l}_1, \mathbf{l}_2)$ are continuous, but their first derivatives may have the discontinuities of the first kind at the points $\mathbf{l} = \mathbf{l}_0$ and $\mathbf{l}_1 = \mathbf{l}_2 = \mathbf{l}_0$, respectively [2–5]. Like in (12), the functions $S_{Uk}(\mathbf{l})$ and $K_{Uk}(\mathbf{l}_1, \mathbf{l}_2)$ are normalized so that

$$S_{Uk}(\mathbf{l}_0) = 1, K_{Uk}(\mathbf{l}_0, \mathbf{l}_0) = 1. \tag{13}$$

From (11)-(13), it follows that the constants A_{Sk} meet the requirements

$$\sum_{k=1}^{v} A_{Sk} + o\left(\delta^2\right) = A_S, \tag{14}$$

where $A_S = S(\mathbf{l}_0, \mathbf{\Theta}_0)$ is the value of the maximum of the signal function $S(\mathbf{l}, \mathbf{\Theta})$. At the same time,

$$\sum_{k=1}^{v} A_{Sk} S_{Uk}(\mathbf{l}) + o\left(\delta^2\right) = A_S S_U(\mathbf{l}), \sum_{k=1}^{v} A_{Sk} S_{Rk}(\mathbf{\Theta}) + o\left(\delta^2\right) = A_S S_R(\mathbf{\Theta}), \tag{15}$$

where the functions