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Multiscale Analysis of Viscous Flows in Thin Tube Structures

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Preface

Mathematical models of viscous flows in thin domains have multiple applications. Such domains have one or several dimensions which are much smaller than other ones. In particular, tube structures are some unions of thin cylinders or rectangles (pipes or channels). This geometry simulates a network of blood vessels in biological applications or pipelines and cooling systems in technical applications.

Full dimension numerical computations of flows in thin domains require huge computer resources, for example, for a network of blood vessels. To reduce these resources and accelerate computations, we use asymptotic analysis where the small parameter is the ratio of thickness of pipes or channels to their length. This analysis leads to the construction of asymptotic expansions justified by error estimates. It is also implemented in some special numerical methods combining the description with reduced dimension and full dimension zooms for small zones of singular behavior of the solution.

Basically, we consider the Newtonian rheology for the fluid motion corresponding to the stationary and nonstationary Navier–Stokes or Stokes equations. These equations introduced two centuries ago are still in the spotlight of contemporary mathematics. In particular, one of the main challenges ("millennium problem") concerns the question of the global existence and uniqueness of a solution of the nonstationary Navier–Stokes equations.

The nonstationary Navier–Stokes equations have the form:

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, \\ \hfill \operatorname{div} \mathbf{v} = 0 \ , \end{cases} \tag{1}$$

where the unknown functions are $\mathbf{v}$ (the n-dimensional velocity vector field) and p, the pressure (more exactly, this unknown scalar function stands for the pressure divided by the density of the fluid); these unknown functions depend on the space variable $x \in \mathbb{R}^n$ and the time $t \in [0, +\infty)$. The positive coefficient ν is the kinematic viscosity. It is related to the dynamic viscosity μ as follows: $\nu = \mu/\rho$, where ρ is the density of the fluid. This system of partial differential equations requires one initial condition for $\mathbf{v}(x, 0)$. If these equations are stated in a bounded domain, then they should be completed by a boundary condition. Classical boundary condition is the given velocity $\mathbf{v}$ on the boundary (the Dirichlet type condition); however, recently other boundary conditions involving pressure were studied. We will consider also the stationary version of the Navier–Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, \\ \qquad\qquad\qquad\qquad \operatorname{div}\mathbf{v} = 0, \end{cases} \tag{2}$$

as well as the linearized versions: the nonstationary Stokes equations

$$\begin{cases} \dfrac{\partial\mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + \nabla p = 0, \\ \qquad\qquad \operatorname{div}\mathbf{v} = 0, \end{cases} \tag{3}$$

and the stationary Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = 0, \\ \qquad \operatorname{div}\mathbf{v} = 0. \end{cases} \tag{4}$$

The Navier–Stokes equations are often used for the description of the blood flow, although sometimes for the hemodynamical applications non-Newtonian rheologies are used. The stationary and nonstationary Navier–Stokes equations are considered in thin tube structures. The main results of the book are formulated in the form of theorems. The complete asymptotic expansions of the solutions are constructed. The estimates for the difference of the exact solution and its J-th asymptotic approximation is proved. The method of asymptotic partial decomposition of the domain (MAPDD) is formulated and justified for the stationary and nonstationary cases. It gives the asymptotically exact interface conditions of coupling of the 1D and 3D models of the flow. Namely, the geometry of the blood circulation system is presented as a big union B_ε of thin cylindrical "vessels," and the small parameter ε is the ratio of the radius to the height of the cylinders. The MAPDD simplifies the solution of the Stokes and Navier–Stokes equations in such structures by combining the description of different dimensions in one model of hybrid dimension. It provides the one-dimensional reduction in the main part of the thin structure, at some distance from the junction area, and it keeps the $3D$ (or $2D$) description within the junction area. The asymptotic analysis justifies the appropriate interface conditions between these $3D$ and $1D$ descriptions. In particular, for the steady Stokes or Navier–Stokes equations, the above $1D$ models are reduced to some Poiseuille type flows. For the nonstationary Navier–Stokes equations, the $1D$ models are reduced to some nonsteady Poiseuille type flows. This approach effectively reduces the computational burden on solvers dealing with flows within intricately shaped domains, thereby significantly expediting computations.

The structure of the monograph is as follows. It starts by the introductory Chap. 1 presenting the methods used in the book applied to the simplified settings. This chapter helps to catch the main ideas of the book with the examples of "toy problems," i.e., drastically simplified realistic models. In Chap. 1, we define the main geometrical object of the book: thin tube structure. The further chapters systematically introduce all necessary notions and recall fundamental theorems used in the book. Thus, Chap. 2 (Preliminaries) recalls the well-known inequalities in normed spaces, as well as some facts from functional analysis on linear operators in Banach and Hilbert spaces, Sobolev and Hölder spaces. A spotlight of Chap. 2 is the study of functional spaces for unbounded domains with cylindrical outlets at infinity (such domains are used for the construction of the boundary layer correctors), the analysis of the divergence equation, and the spaces of solenoidal vector-valued functions. This chapter also contains the formulations and proofs of the estimates of solutions of the divergence equation for tube structures taking into account the dependence of constants on the small parameter. Chapter 3 studies the so-called Poiseuille flows which are stationary or non-stationary flows in an infinite cylindrical tube with the given flux in the case when the pressure is a linear function of the longitudinal variable x_n, independent of the transversal variables $(x_1, \ldots, x_{n-1})$; the tangential velocity of the flow vanishes while the normal velocity is independent of x_n. We prove the existence and uniqueness of such solution and the a priori estimates, i.e., the estimates of the norm of the solution via the norms of the data. We also introduce the linear operators relating the pressure slope of the Poiseuille flow to the flux. In Chap. 4, we consider the general theory of the Stokes equations in bounded and unbounded domains. For the tube structures, the existence and uniqueness theorems are proved, a priori estimates of the solution are obtained. For the problems in unbounded domains with cylindrical outlets to infinity, we prove the existence and uniqueness theorems as well as the theorems on the stabilization (exponential decay) of the solution in the outlets and for time tending to infinity. These theorems are used further for the construction of boundary layers. Chapters 5 and 6 are the spotlights of the book. In Chap. 5, we consider the stationary Stokes and Navier–Stokes equations in thin tube structures B_ε. We start with the Dirichlet conditions on the boundary of the tube structure; we construct the complete asymptotic expansion of the solution, prove the error estimates for the asymptotic approximations, and formulate and justify the method of asymptotic partial decomposition of the domain. In the second part of the chapter, we consider the Stokes equations with the given pressure at the inlets and outlets and no-slip boundary condition on the lateral surface of the tubes. We prove the existence and uniqueness theorem for this boundary value problem and construct and justify the asymptotic expansion and the MAPDD approximations. Finally, we consider the Navier–Stokes equation with the given Bernoulli pressure on the inlets and outlets and construct and justify the asymptotic expansion and the MAPDD. Chapter 6 introduces the Dirichlet boundary value problem for the nonstationary Navier–Stokes equations stated in B_ε with the initial condition $\mathbf{v}(x, 0) = 0$. The existence and uniqueness of the solution for $\varepsilon << 1$ is proved. The asymptotic expansion is constructed. It contains the boundary layers "in space" and "in time." The error estimates are proved for the asymptotic approximations and for the time-dependent MAPDD approximations. Chapter 7 is devoted to the time-periodic setting of the Navier–Stokes equations. As in the previous chapters, we prove the existence and uniqueness theorems and construct and justify the asymptotic and the MAPDD approximations.

The theoretical analysis developed in Chaps. 5 and 6 is confirmed by several numerical experiments showing that in reality the limitations of the applicability of the proposed methods are more flexible than the theoretically predicted conditions. Several numerical experiments for the flows in thin tube structures are presented. Applying the finite element method (FEM)-based codes, we provide the direct numerical computations for the full dimension and reduced dimension (MAPDD) models and compare the solutions. We evaluate the difference between these solutions and point out the limitations of the asymptotic theory and estimate the size of the boundary layer zones where the boundary layer correctors' contribution is essential. The last chapter is a brief bibliographical review on the adjacent topics.

The book is accessible for a wide range of readers: specialists in engineering, applied mathematicians working in fluid mechanics as well as in applications to biophysics and medicine, and master and PhD students in mathematics and mechanics.

The "users" of methods who are not interested in reading the proofs may pass directly to the description of algorithms.

In the book we use generally triplet numbering of formulas and propositions (number of chapter, number of section, number of formula or proposition within the section).

Vilnius, Lithuania Grigory Panasenko
September, 2023 Konstantin Pileckas

Contents

Chapter 1
Introduction

1.1 Asymptotic Methods for Elliptic Equations in Thin Domains with Dirichlet Boundary Conditions

This section presents an overview of the methodology used to construct asymptotic expansions for solutions of elliptic equations and stationary Stokes and Navier–Stokes equations within thin tube structures, which serve as models for blood vessel networks. These structures are composed of finite unions of thin cylinders. We begin by introducing the concept of thin tube structures and formulating a model problem. It is important to note that the construction of asymptotic expansions for solutions of Stokes or Navier–Stokes equations, subject to no-slip boundary conditions, is more technically intricate compared to the case of the Laplace equation. Nevertheless, the fundamental ideas underlying the process remain similar. As a result, we focus on addressing Dirichlet's problem for the Laplace equation in the context of thin tube structures.

For our simplified model problems, we develop the asymptotic expansion of the solution and validate it by computing the residual. Furthermore, we establish the existence, uniqueness, and a priori estimation of the solution. Additionally, we evaluate the discrepancy between the exact solution and its asymptotic approximation. Throughout this work, we adopt conventional notations and draw upon well-established theorems of functional analysis. While we provide a brief introduction to these notations and theorems in this introductory section, a more comprehensive review will be presented in Chap. 2. In that chapter, we will also introduce functional spaces and formulate the primary theorems of functional analysis that will be utilized in subsequent chapters.

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1.1.1 *Dimensional Reduction for the Poisson Equation in a Thin Rectangle*

We begin by considering the simplest formulation of the boundary value problem for partial differential equations within a thin domain. Specifically, we examine the Dirichlet problem for the Laplacian in a thin rectangle denoted as $G_\varepsilon = (0, 1) \times (0, \varepsilon)$:

$$\begin{cases} -\Delta u_\varepsilon = f(x_1), \; x \in G_\varepsilon \;, \\ \quad u_\varepsilon = 0, \qquad x \in \partial G_\varepsilon, \end{cases} \tag{1.1.1}$$

where $f \in C^\infty([0, 1])$.

1.1.1.1 Construction of High-Order Asymptotic Approximations for the Case $f \in C_0^\infty([0, 1])$

We aim to construct an approximate solution for small values of ε. The process of constructing asymptotic approximations does not require any specialized mathematical prerequisites, so we present it directly. However, to establish the existence of a unique solution for the boundary value problem (1.1.1) and to evaluate the error of the constructed approximation, we will need some results from functional analysis. These necessary facts are briefly summarized below, with more extensive information available in Chap. 2.

Let us proceed with constructing the asymptotic expansion for the solution of (1.1.1) as the small parameter ε tends to zero. We assume that $f \in C_0^\infty([0, 1])$. Here, $C_0^\infty([0, 1])$ represents the space of infinitely differentiable functions defined on the interval $[0, 1]$, which vanish in some neighborhood of the interval's ends.

We seek an asymptotic solution in the form:

$$u_\varepsilon^{(J)} = \sum_{l=0}^{J} \varepsilon^{l+2} u_l \left(x_1, \frac{x_2}{\varepsilon} \right), \tag{1.1.2}$$

where $u_l \in C^2([0, 1] \times [0, 1])$ and J is an even integer with $J \geq 0$. By substituting $u_\varepsilon^{(J)}$ into (1.1.1), we obtain the following expression:

$$
\begin{aligned}
-\Delta u_\varepsilon^{(J)} &= -\frac{\partial^2}{\partial x_1^2} u_\varepsilon^{(J)} - \frac{\partial^2}{\partial x_2^2} u_\varepsilon^{(J)} \\
&= \left\{ -\sum_{l=0}^{J} \varepsilon^{l+2} \frac{\partial^2 u_l}{\partial x_1^2}(x_1, \xi_2) - \sum_{l=0}^{J} \varepsilon^{l+2} \varepsilon^{-2} \frac{\partial^2 u_l}{\partial \xi_2^2}(x_1, \xi_2) \right\} \Bigg|_{\xi_2 = \frac{x_2}{\varepsilon}} \\
&= \left\{ -\sum_{l=0}^{J} \varepsilon^{l} \frac{\partial^2 u_l}{\partial \xi_2^2}(x_1, \xi_2) - \sum_{l'=2}^{J+2} \varepsilon^{l'} \frac{\partial^2 u_{l'-2}}{\partial x_1^2}(x_1, \xi_2) \right\} \Bigg|_{\xi_2 = \frac{x_2}{\varepsilon}, l' = l+2} \\
&= -\sum_{l=0}^{J} \varepsilon^{l} \left(\frac{\partial^2 u_l}{\partial \xi_2^2} + \frac{\partial^2 u_{l-2}}{\partial x_1^2} \right) - \varepsilon^{J+1} \frac{\partial^2 u_{J-1}}{\partial x_1^2} - \varepsilon^{J+2} \frac{\partial^2 u_J}{\partial x_1^2},
\end{aligned}
$$

where $u_l = 0$ if $l < 0$. In this derivation, we utilized the evident formula for changing the subscript in the sum:

$$
\sum_{l=0}^{N} a_l = \sum_{l'=M}^{N+M} a_{l'-M} = \sum_{l=M}^{N+M} a_{l-M},
$$

where $l' = l + M$ and $l = l' - M$.

Equating this expansion to the right-hand side $f(x_1)$ leads to a series of problems for u_l:

$$
\begin{cases}
-\dfrac{\partial^2}{\partial \xi_2^2} u_l(x_1, \xi_2) = \dfrac{\partial^2}{\partial x_1^2} u_{l-2}(x_1, \xi_2) + f(x_1)\delta_{l0}, & \xi_2 \in (0, 1), \\
u(x_1, \xi_2) = 0, & \xi_2 = 0, \quad \xi_2 = 1,
\end{cases}
\tag{1.1.3}
$$

where δ_{ij} is the Kronecker delta. For $l = 0$, the solution is given by:

$$
u_0(x_1, x_2) = \frac{1}{2} x_2 (x_2 - 1)(-f(x_1)).
$$

One can directly verify by induction that $u_l = 0$ for odd values of l. By calculating the residual on the right-hand side, we obtain the following expression:

$$
\begin{cases}
-\Delta u_\varepsilon^{(J)} = f(x_1) + r_\varepsilon(x), & x \in G_\varepsilon, \\
u_\varepsilon^{(J)} = 0, & x \in \partial G_\varepsilon,
\end{cases}
\tag{1.1.4}
$$

where

$$
r_\varepsilon = \varepsilon^{J+2} \frac{\partial^2 u_J}{\partial x_1^2} = O(\varepsilon^{J+2}).
\tag{1.1.5}
$$

The vanishing of $u_\varepsilon^{(J)}$ on ∂G_ε follows from the condition $f \in C_0^\infty([0,1])$. This can be proven by induction using (1.1.3).

Thus, we have constructed the J-th order approximation of the solution. In order to prove its proximity to the exact solution, we require some prerequisites from functional analysis.

1.1.2 Justification of the Asymptotic Expansion

To establish the validity of the error estimate and prove the existence of the solution, we need to revisit certain definitions. In our analysis, we utilize the spaces $L^2(G)$ and $W^{1,2}(G)$, where G represents a domain in $\mathbb{R}^n$. The space $L^2(G)$ is a Hilbert space of real-valued functions u defined on G, which possess a finite Lebesgue integral $\int_G u^2(x)dx$. It is equipped with an inner product $(u, w)_{L^2(G)} = \int_G u(x)w(x)dx$ and a norm $\|u\|_{L^2(G)} = \sqrt{(u,u)_{L^2(G)}}$. We write $u \in C_0^\infty(G)$ if u is an infinitely differentiable function that vanishes outside some subdomain $G' \subset\subset G$ and $u \in C^\infty(G)$ if u is infinitely differentiable function in G extebdable to $\mathbb{R}^n$. Moreover, if u belongs to $L^2(G)$ and there exists a function $u_i \in L^2(G)$ satisfying the relation

$$\int\limits_G u(x)\frac{\partial v}{\partial x_i}dx = -\int\limits_G u_i(x)v(x)dx$$

for all functions $v \in C_0^\infty(G)$, then u_i is referred to as a weak partial derivative and denoted by $\dfrac{\partial u}{\partial x_i}$.

The Sobolev space $W^{1,2}(G)$ is the space of functions in $L^2(G)$ having all weak partial derivatives $\dfrac{\partial u}{\partial x_i} \in L^2(G)$, where $i = 1, \dots, n$. It is endowed with an inner product $(u, w)_{W^{1,2}(G)} = \int_G \big(u(x)w(x) + \nabla u \cdot \nabla w\big)dx$ and a norm $\|u\|_{W^{1,2}(G)} = \sqrt{(u,u)_{W^{1,2}(G)}}$. The spaces $L^2(G)$ and $W^{1,2}(G)$ can also be defined as the closures of the space $C^\infty(G)$ with respect to the norms $\|u\|_{L^2(G)}$ and $\|u\|_{W^{1,2}(G)}$, respectively. Additionally, we introduce the space $\mathring{W}^{1,2}(G)$, which is the closure of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{W^{1,2}(G)}$.

We define a weak solution of problem (1.1.1) as a function $u_\varepsilon \in \mathring{W}^{1,2}(G_\varepsilon)$ that satisfies the following identity for any test function $v \in \mathring{W}^{1,2}(G_\varepsilon)$:

$$\int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla v dx = \int\limits_{G_\varepsilon} f(x_1)v(x)dx. \tag{1.1.6}$$

To prove the existence of the solution, we employ Poincaré inequality, which holds for any function $w \in \mathring{W}^{1,2}(G_\varepsilon)$ and is given by:

$$\|w\|_{L^2(G_\varepsilon)} \leq C\varepsilon \|\nabla w\|_{L^2(G_\varepsilon)}$$

where the constant C is independent of ε.

Let H be a Hilbert space of functions in $\overset{\circ}{W}{}^{1,2}(G_\varepsilon)$, endowed with a new inner product

$$(w, v)_H = \int\limits_{G_\varepsilon} \nabla w(x) \cdot \nabla v(x) dx.$$

Then, (1.1.6) can be written as

$$(u_\varepsilon, v)_H = (f, v)_{L^2(G_\varepsilon)}. \tag{1.1.7}$$

Considering the linear functional $\Phi : H \to \mathbb{R}^1$ defined by $\Phi(v) = (f, v)_{L^2(G_\varepsilon)}$, we note that due to Poincaré inequality, it is a continuous functional on H. Therefore, we can apply the Riesz representation theorem, which states that there exists a unique element $u_\varepsilon \in H$ such that for all $v \in H$, $\Phi(v) = (u_\varepsilon, v)_H$. This element $u_\varepsilon \in H$ is the unique weak solution of problem (1.1.1). By taking $v = u_\varepsilon$ in (1.1.7) and applying the Cauchy-Schwarz-Bunyakovsky inequality $|(u, v)_{L^2(G_\varepsilon)}| \leq \|u\|_{L^2(G_\varepsilon)} \|v\|_{L^2(G_\varepsilon)}$, followed by the Poincaré inequality, we obtain the a priori estimate

$$\|u_\varepsilon\|_{W^{1,2}(G_\varepsilon)} \leq C\varepsilon \|f\|_{L^2(G_\varepsilon)}. \tag{1.1.8}$$

It is important to note that the a priori estimate (1.1.8) is valid not only for functions f dependent on one variable x_1 but also for all $f \in L^2(G_\varepsilon)$.

This estimate serves as the basis for justifying the asymptotic expansion (1.1.2). By subtracting problem (1.1.1) from (1.1.4), we find that the difference $w = u_\varepsilon - u_\varepsilon^{(J)}$ satisfies the following problem:

$$\begin{cases} -\Delta w = r_\varepsilon, \ x \in G_\varepsilon, \\ \quad w = 0, \ \ x \in \partial G_\varepsilon. \end{cases} \tag{1.1.9}$$

Applying estimate (1.1.8) and taking into account (1.1.5) and the definition of the $L^2(G_\varepsilon)$-norm, we derive the estimate:

$$\|u_\varepsilon - u_\varepsilon^{(J)}\|_{W^{1,2}(G_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes}G_\varepsilon}\right). \tag{1.1.10}$$

Furthermore, this estimate can be improved. For even J, replacing $u_\varepsilon^{(J)}$ by $u_\varepsilon^{(J+2)}$, we have:

$$\|u_\varepsilon - u_\varepsilon^{(J+2)}\|_{W^{1,2}(G_\varepsilon)} = O\left(\varepsilon^{J+4}\sqrt{\text{mes}G_\varepsilon}\right). \tag{1.1.11}$$

On the other hand, we find:

$$\|u_\varepsilon^{(J)} - u_\varepsilon^{(J+2)}\|_{W^{1,2}(G_\varepsilon)} = O\left(\varepsilon^{J+3}\sqrt{\text{mes}G_\varepsilon}\right). \tag{1.1.12}$$

Combining the estimates (1.1.11) and (1.1.12) and using the triangle inequality, we obtain:

$$\|u_\varepsilon - u_\varepsilon^{(J)}\|_{W^{1,2}(G_\varepsilon)} = O\left(\varepsilon^{J+3}\sqrt{\text{mes}G_\varepsilon}\right) . \tag{1.1.13}$$

This concludes the justification of the asymptotic expansion (1.1.2) and the error estimate for the problem (1.1.1).

1.1.3 Construction of High-Order Asymptotic Approximations for $f \in C^\infty([0, 1])$

In this section, we consider the scenario where the function $f(x_1)$ is infinitely differentiable on the interval [0, 1]. Additionally, let us assume that $f(x_1) = 0$ in some neighborhood of $x_1 = 1$, but $f(0) \neq 0$. In such cases, the boundary condition at $\gamma_0 = \partial G_\varepsilon \cap \{x_1 = 0\}$ is not satisfied. To rectify this, we introduce the concept of a boundary layer corrector denoted as $u_\varepsilon^{[BL,J]}$ given by the asymptotic expansion:

$$u_\varepsilon^{[BL,J]} = \sum_{l=0}^{J} \varepsilon^{l+2} u_l^{[BL]}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) , \tag{1.1.14}$$

where $u_l^{[BL]}|_{x_1=0}$ compensates the trace $u_l|_{x_1=0}$:

$$u_l^{[BL]}|_{x_1=0} = -u_l|_{x_1=0} \; .$$

Thus, we encounter a chain of problems for $u_l^{[BL]}$ in the dilated variables ξ belonging to the half-strip $\Pi = \mathbb{R}_+ \times (0, 1)$ (see Fig. 1.1), expressed as:

$$\begin{cases} -\Delta_\xi u_l^{[BL]} = 0, & \xi \in \Pi, \\ u_l^{[BL]} = 0, & \xi_2 = 0, \quad \xi_2 = 1, \\ u_l^{[BL]}(0, \xi_2) = -u_l(0, \xi_2). \end{cases} \tag{1.1.15}$$

To demonstrate that the solutions of (1.1.15) decay exponentially as $\xi_1 \to +\infty$, we employ the Fourier series technique. Specifically, we assume:

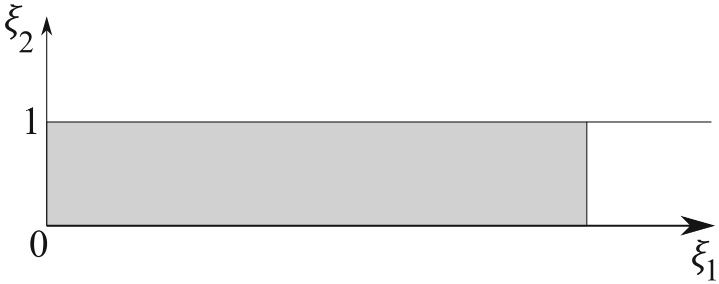

Fig. 1.1 Domain of definition of boundary layer problem: half-strip

$$-u_l(0,\xi_2) = \sum_{n=1}^{\infty} b_n \sin(\xi_2 \pi n),$$

and then seek $u_l^{[BL]}(\xi_1, \xi_2)$ in the form of:

$$u_l^{[BL]}(\xi_1, \xi_2) = \sum_{n=1}^{\infty} B_n(\xi_1) \sin(\xi_2 \pi n) \ ,$$

which leads to the following ordinary differential equations for the coefficients B_n:

$$\begin{cases} -B_n'' + (\pi n)^2 B_n \ = \ 0, \quad \xi_1 \in \mathbb{R}_+, \\ B_n(0) \ = \ b_n, \\ B_n(\xi_1) \to 0 \quad \text{as} \ \ \xi_1 \to +\infty. \end{cases}$$

The solution to these equations is given by $B_n = b_n e^{-\pi n \xi_1}$, resulting in the final expression for $u_l^{[BL]}$ as follows:

$$u_l^{[BL]}(\xi_1, \xi_2) = \sum_{n=1}^{\infty} b_n e^{-\pi n \xi_1} \sin(\xi_2 \pi n) \ . \qquad (1.1.16)$$

Hence, we have established the theorem of exponential stabilization of the solution at infinity, also known as the theorem of Phrägmen–Lindelöf type:

$$\exists C_1 > 0 : \ \ \left|u_l^{[BL]}(\xi)\right|, \left|\nabla u_l^{[BL]}(\xi)\right| \le C_1 e^{-\pi \xi_1} \ .$$

To obtain an error estimate, we introduce the "corrected" asymptotic solution as:

$$u_\varepsilon^{(a,J)} = u_\varepsilon^{(J)}\left(x_1, \frac{x_2}{\varepsilon}\right) + u_\varepsilon^{[BL,J]}\left(\frac{x}{\varepsilon}\right)\eta(3x_1),$$

where $\eta \in C^2(\mathbb{R})$ is a cut-off function satisfying:

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \leq 1\,, \\ 0 & \text{if } |t| \geq 2. \end{cases}$$

Calculating the new residual

$$R_\varepsilon = r_\varepsilon - \left(\Delta u_\varepsilon^{[BL,J]} \eta(3x_1) + 2\frac{\partial u_\varepsilon^{[BL,J]}}{\partial x_1} \eta'(3x_1) + u_\varepsilon^{[BL,J]} \eta''(3x_1) \right),$$

and taking into account that the support of the terms with η belongs to the zone where the boundary layers are exponentially small, we obtain the following estimate in the $L^2(G_\varepsilon)$-norm:

$$||R_\varepsilon||_{L^2(G_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes } G_\varepsilon}\right).$$

As in the previous case, we obtain an error estimate:

$$\left\| u_\varepsilon - \left(u_\varepsilon^{(J)} + u_\varepsilon^{[BL,J]}\eta\right) \right\|_{W^{1,2}(G_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes} G_\varepsilon}\right),$$

which can be improved for even J as:

$$\left\| u_\varepsilon - \left(u_\varepsilon^{(J)} + u_\varepsilon^{[BL,J]}\eta\right) \right\|_{W^{1,2}(G_\varepsilon)} = O\left(\varepsilon^{J+3}\sqrt{\text{mes} G_\varepsilon}\right).$$

Additionally, we find a positive number δ such that for all x satisfying $\text{dist}(x, \gamma_0) \leq \delta$, the boundary layer corrector $u_\varepsilon^{[BL,J]}(\frac{x}{\varepsilon})$ has a "tail" smaller than ε^J (see Fig. 1.2). We use the estimate

$$\left|u_\varepsilon^{[BL,J]}(x)\right| \leq C_1 e^{-\pi x_1/\varepsilon},$$

to conclude that it is smaller than ε^J for $x_1 \geq \delta$, leading to the following formula for δ:

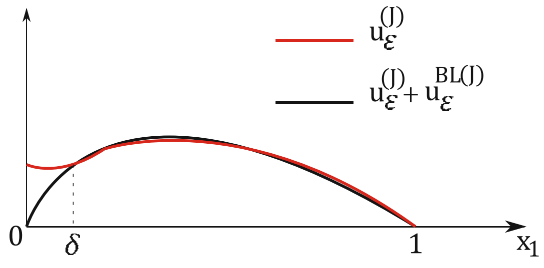

Fig. 1.2 Comparison of asymptotics with and without boundary layer

$$\delta = \text{const}\, J\varepsilon |\ln \varepsilon|\ .$$

Finally, it is important to note that the same approach can be applied when f does not vanish neither in 0 nor in 1.

1.2 Dirichlet's Problem for the Laplacian in a Thin Tube Structure

In this section, we investigate a "toy" problem related to the Laplacian in a thin tube structure. The domain consists of two regions: $B_\varepsilon = B_\varepsilon^1 \cup B_\varepsilon^2$, where $B_\varepsilon^1 = (-1, 1) \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, and $B_\varepsilon^2 = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) \times (0, 1)$.

The problem under consideration is given by:

$$\begin{cases} -\Delta u_\varepsilon = 1,\ x \in B_\varepsilon, \\ \quad u_\varepsilon = 0,\ x \in \partial B_\varepsilon. \end{cases} \tag{1.2.1}$$

This setting imitates a viscous flow in a T-shaped structure (see Fig. 1.3). While the real-life setting corresponds to the Navier–Stokes or Stokes equations, the main ideas can be effectively explained in this simplified setup.

The asymptotic expansion construction proceeds as follows:

Step 1. Define the "Poiseuille"-type solutions:

$$\begin{aligned} U_\varepsilon^1(x) &= -\frac{1}{2}\left(x_2^2 - \frac{\varepsilon^2}{4}\right) \quad \text{in } B_\varepsilon^1, \\ U_\varepsilon^2(x) &= -\frac{1}{2}\left(x_1^2 - \frac{\varepsilon^2}{4}\right) \quad \text{in } B_\varepsilon^2. \end{aligned} \tag{1.2.2}$$

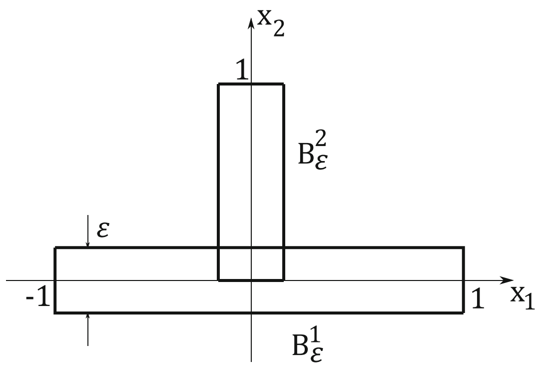

Fig. 1.3 T-shaped structure

Step 2. Multiply U_ε^1 and U_ε^2 by the cut-off functions:

$$u_\varepsilon^{12}(x) = U_\varepsilon^1(x)\left(1-\eta\left(\frac{x_1}{3\varepsilon}\right)\right) + U_\varepsilon^2(x)\left(1-\eta\left(\frac{x_2}{3\varepsilon}\right)\right) . \tag{1.2.3}$$

Substituting this expression into the equation, we obtain:

$$-\Delta u_\varepsilon^{12} = 1 + F\left(\frac{x}{\varepsilon}\right) ,$$

where the function $F(\xi)$ is defined as follows:

$$F(\xi) = \begin{cases} 1-\eta\left(\frac{\xi_1}{3}\right) - \frac{1}{18}\left(\xi_2^2-\frac{1}{4}\right)\eta''\left(\frac{\xi_1}{3}\right) + 1- \\ -\eta\left(\frac{\xi_2}{3}\right) - \frac{1}{18}\left(\xi_1^2-\frac{1}{4}\right)\eta''\left(\frac{\xi_2}{3}\right) - 1, & \xi \in (-6,6)^2 \cap \Pi, \\ 0, & \xi \in \Pi \setminus (-6,6)^2. \end{cases} \tag{1.2.4}$$

Here, $\Pi = \mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right) \bigcup \left(\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}_+$.

Step 3. To compensate the residual F, we introduce $u^{BL}(\xi)$, the solution to the boundary layer problem:

$$\begin{cases} -\Delta_\xi U^{BL}(\xi) = -F(\xi), \ \xi \in \Pi, \\ \quad U^{BL}(\xi) = 0, \qquad \xi \in \partial\Pi. \end{cases}$$

It is evident that there exists a unique solution in $\mathring{W}^{1,2}(\Pi)$ and $|u^{BL}|, |\nabla u^{BL}| \leq C_1 e^{-\pi|\xi|}$, where $C_1 > 0$.

Step 4. Construct boundary layer correctors as before to compensate the traces of u_ε^{12} on:

$$(a) \quad \gamma^{(-1,0)} = \left\{x_1 = -1, x_2 \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)\right\} :$$

find $U^{[BL(-1,0)]}(\xi)$ defined on the semi-strip $\Pi^{(-1,0)} = \mathbb{R}_+ \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and satisfying

$$\begin{cases} -\Delta_\xi U^{[BL(-1,0)]} = 0, & \xi \in \Pi^{(-1,0)}, \\ U^{[BL(-1,0)]}(\xi) = 0, & \xi \in \partial\Pi^{(1,0)} \setminus \{\xi_1 = 0\}, \\ U^{[BL(-1,0)]}(\xi) = \frac{1}{2}(\xi_2^2 - \frac{1}{4}), \ \xi_1 = 0; \end{cases}$$

on

$$(b) \quad \gamma^{(1,0)} = \left\{ x_1 = 1, x_2 \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right\} :$$

find $U^{[BL(1,0)]}(\xi)$ defined on a semi-strip $\Pi^{(1,0)} = \mathbb{R}_- \times \left(-\frac{1}{2}, \frac{1}{2} \right)$ satisfying

$$\begin{cases} -\Delta_\xi U^{[BL(1,0)]} = 0, & \xi \in \Pi^{(-1,0)}, \\ U^{[BL(1,0)]}(\xi) = 0, & \xi \in \partial\Pi^{(1,0)} \backslash \{\xi_1 = 0\}, \\ U^{[BL(1,0)]}(\xi) = \dfrac{1}{2}(\xi_2^2 - \frac{1}{4}), \ \xi_1 = 0; \end{cases}$$

on

$$(c) \quad \gamma^{(0,1)} = \left\{ x_1 \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right), x_2 = 1 \right\},$$

find $U^{[BL(0,1)]}(\xi)$ defined on a semi-strip $\Pi^{(0,1)} = \left(-\frac{1}{2}, \frac{1}{2} \right) \times \mathbb{R}_-$ satisfying

$$\begin{cases} -\Delta_\xi U^{[BL(0,1)]} & = 0, \qquad \xi \in \Pi^{(0,1)}, \\ U^{[BL(0,1)]}(\xi) & = 0, \qquad \xi \in \partial\Pi^{(0,1)} \backslash \{\xi_2 = 0\}, \\ U^{[BL(0,1)]}(\xi) & = \dfrac{1}{2}(\xi_1^2 - \frac{1}{4}), \ \xi_2 = 0. \end{cases}$$

The final asymptotic approximation is given by:

$$\begin{aligned} u_\varepsilon^a(x) = {} & U_\varepsilon^1(x) \left(1 - \eta\left(\frac{x_1}{3\varepsilon} \right) \right) + U_\varepsilon^2(x) \left(1 - \eta\left(\frac{x_2}{3\varepsilon} \right) \right) \\ & + U^{BL}\left(\frac{x}{\varepsilon} \right) \eta(3x_1)\eta(3x_2) \\ & + U^{BL(-1,0)}\left(\frac{x_1+1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \eta(3(x_1+1)) \\ & + U^{BL(1,0)}\left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \eta(3(x_1-1)) \\ & + U^{BL(0,1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) \eta(3(x_2-1)). \end{aligned}$$

All boundary layers decay exponentially, as proven by the Phrägmen–Lindelöf theorems, which are established by Fourier analysis for all semi-strips in the boundary layer problems.

By calculating the residual, we observe that it is of order $O(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon})$ in L^2-norm, and applying the a priori estimate, we obtain:

$$\|u_\varepsilon^{(a,J)} - u_\varepsilon\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon}).$$

1.3 Method of Asymptotic Partial Decomposition of Domain for a T-Shaped Domain

In this section, we present the method of asymptotic partial decomposition of domain, specifically designed for analyzing a T-shaped domain denoted as B_ε. The main objective is to decompose B_ε into several subdomains to efficiently solve the governing equation with appropriate boundary conditions. The decomposition is expressed in the form:

$$B_\varepsilon = B^1_{\varepsilon\delta} \cup B^2_{\varepsilon\delta} \cup B^{(0,0)}_{\varepsilon\delta} \cup B^{(-1,0)}_{\varepsilon\delta} \cup B^{(1,0)}_{\varepsilon\delta} \cup B^{(0,1)}_{\varepsilon\delta},$$

where the subdomains are defined as follows:

$$\begin{gathered}
B^1_{\varepsilon\delta} = (-1+\delta, -\delta) \times \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right) \cup (\delta, 1-\delta) \times \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right),\\
B^2_{\varepsilon\delta} = \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right) \times (\delta, 1-\delta),\\
B^{(0,0)}_{\varepsilon\delta} = (-\delta, \delta) \times \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right) \cup \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right) \times (0, \delta),\\
B^{(-1,0)}_{\varepsilon\delta} = (-1, -1+\delta) \times \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right),\\
B^{(1,0)}_{\varepsilon\delta} = (1-\delta, 1) \times \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right),\\
B^{(0,1)}_{\varepsilon\delta} = \left(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right) \times (1-\delta, 1).
\end{gathered}$$

See Fig. 1.4 for an illustration of the asymptotic domain decomposition of the T-shaped structure.

Consider the approximation u_d, which is a solution to the equation

$$-\Delta u_d = 1$$

in the various subdomains with specific boundary conditions:

(a) In $B^{(0,0)}_{\varepsilon\delta}$, the boundary conditions are:

$$\begin{aligned}
u_d &= 0, \quad x \in \partial B_\varepsilon \cap \partial B^{(0,0)}_{\varepsilon\delta},\\
u_d &= U^1_\varepsilon, \quad x \in \partial B^{(0,0)}_{\varepsilon\delta} \cap \{x_1 = \pm\delta\},\\
u_d &= U^2_\varepsilon, \quad x \in \partial B^{(0,0)}_{\varepsilon\delta} \cap \{x_2 = \delta\}.
\end{aligned}$$

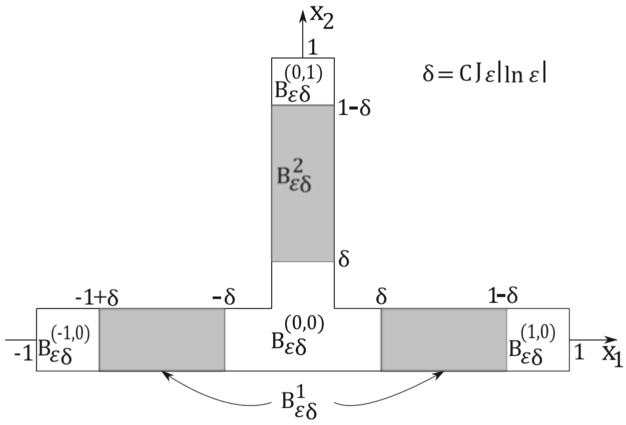

Fig. 1.4 Asymptotic domain decomposition of T-shaped structure

(b) In $B_{\varepsilon\delta}^{(\pm 1,0)}$, the boundary conditions are:

$$u_d = 0, \quad x \in \partial B_\varepsilon \cap \partial B_{\varepsilon\delta}^{(+1,0)},$$
$$u_d = U_\varepsilon^1, \quad x \in \partial B_{\varepsilon\delta}^{(\pm 1,0)} \cap \{x_1 = \pm(1-\delta)\}.$$

(c) In $B_{\varepsilon\delta}^{(0,1)}$, the boundary condition is:

$$u_d = 0, \quad x \in \partial B_\varepsilon \cap \partial B_{\varepsilon\delta}^{(0,1)},$$
$$u_d = U_\varepsilon^2, \quad x \in \{x_2 = 1-\delta\}.$$

Furthermore, we define

$$u_d = U_\varepsilon^1 \ \text{ in } \ B_{\varepsilon\delta}^1, \quad u_d = U_\varepsilon^2 \ \text{ in } \ B_{\varepsilon\delta}^2.$$

The main result of this method is an error estimate given by the following theorem:

Theorem 1.3.1 *Let $\delta = CJ\varepsilon|\ln\varepsilon|$, where C is a constant independent of ε. Then the error between the exact solution u_ε and the approximation u_d in the $W^{1,2}(B_\varepsilon)$ norm is bounded as:*

$$\|u_\varepsilon - u_d\|_{W^{1,2}(B_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon}\right), \tag{1.3.1}$$

where J is a positive integer and mes B_ε *denotes the measure of the domain B_ε.*

This method effectively reduces the computational resources needed. The problem is broken down into independent subproblems in $B_{\varepsilon\delta}^{(0,0)}$, $B_{\varepsilon\delta}^{(\pm 1,0)}$, and $B_{\varepsilon\delta}^{(0,1)}$, which can be solved in parallel.

The proof of inequality (1.3.1) is predicated on the estimation of the difference $u_\varepsilon^a - u_d$, achieved by selecting an appropriate value for δ. This choice ensures that the boundary layer functions $U^{[BL]}\left(\frac{x}{\varepsilon}\right)$, $U^{[BL(-1,0)]}\left(\frac{x_1+1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$, $U^{[BL(1,0)]}\left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$, and $U^{BL(0,1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon}\right)$, along with their first- and second-order derivatives, have magnitudes smaller than ε^{J+4} when evaluated at points x situated at a distance of $\delta/3$ from the corresponding nodes or vertices ((0, 0), (−1, 0), (1, 0), and (0, 1), respectively). As previously demonstrated, this value of δ is determined as $CJ\varepsilon|\ln\varepsilon|$, where C is a constant independent of ε. With this selection of δ, let us proceed to modify the factors η in the definition formula of $u_\varepsilon^{(a,J)}$ as follows:

$$\begin{aligned}\tilde{u}_\varepsilon^{(a,J)}(x) &= U_\varepsilon^1(x)\left(1-\eta\left(\frac{x_1}{3\varepsilon}\right)\right) + U_\varepsilon^2(x)\left(1-\eta\left(\frac{x_2}{3\varepsilon}\right)\right)\\ &+ U^{BL}\left(\frac{x}{\varepsilon}\right)\eta\left(\frac{3x_1}{\delta}\right)\eta\left(\frac{3x_2}{\delta}\right)\\ &+ U^{BL(-1,0)}\left(\frac{x_1+1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\eta\left(\frac{3(x_1+1)}{\delta}\right)\\ &+ U^{BL(1,0)}\left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\eta\left(\frac{3(x_1-1)}{\delta}\right)\\ &+ U^{BL(0,1)}\left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon}\right)\eta\left(\frac{3(x_2-1)}{\delta}\right).\end{aligned}$$

Utilizing the aforementioned choice of δ, we can now compute the residual r_ε in the right-hand side of the Laplace equation:

$$-\Delta\tilde{u}_\varepsilon^{(a,J)} = 1 + r_\varepsilon \tag{1.3.2}$$

We will then prove that this residual term satisfies the following estimate:

$$\|r_\varepsilon\|_{L^2(B_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon}\right).$$

It is important to note that $\tilde{u}_\varepsilon^{(a,J)}$ precisely satisfies Dirichlet's boundary condition on ∂B_ε.

Next, we subtract Equation (1.3.2) from $-\Delta u_d = 1$, leading to the following equation for the difference $w = \tilde{u}_\varepsilon^{(a,J)} - u_d$:

$$-\Delta w = r_\varepsilon.$$

Additionally, the condition $w = 0$ holds on the boundaries of small subdomains $B_{\varepsilon\delta}^{(0,0)}$, $B_{\varepsilon\delta}^{(-1,0)}$, $B_{\varepsilon\delta}^{(1,0)}$, and $B_{\varepsilon\delta}^{(0,1)}$. By applying the a priori estimate for the Laplace equation (similar to (1.1.8)), we are able to demonstrate that:

$$\|\tilde{u}_\varepsilon^{(a,J)} - u_d\|_{W^{1,2}(B_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon}\right).$$

In light of the chosen value of δ, we can now evaluate the difference between the original asymptotic expansion $u_\varepsilon^{(a,J)}$ and the modified asymptotic expansion $\tilde{u}_\varepsilon^{(a,J)}$. This yields the estimate:

$$\|\tilde{u}_\varepsilon^{(a,J)} - u_\varepsilon^{(a,J)}\|_{W^{1,2}(B_\varepsilon)} = O\left(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon}\right).$$

Finally, taking into account the obtained estimates of order $O\left(\varepsilon^{J+2}\sqrt{\text{mes } B_\varepsilon}\right)$ for the differences $\tilde{u}_\varepsilon^{(a,J)} - u_d$, $\tilde{u}_\varepsilon^{(a,J)} - u_\varepsilon^{(a,J)}$, and $u_\varepsilon^{(a,J)} - u_\varepsilon$ in the norm $W^{1,2}(B_\varepsilon)$, we finish the proof of the theorem.

This approach can be extended to encompass the case of the Stokes and Navier–Stokes equations in thin tube structures, facilitating the modeling of blood flow in a network of vessels.

1.4 Method of Asymptotic Partial Decomposition of Domain for Flows in a Tube Structure (Applications in Hemodynamics)

Motivated by the modeling of blood flow in a network of blood vessels, we investigate the Stokes equation in a tube structure. These domains consist of connected finite unions of thin finite cylinders (in the 2D case, thin rectangles). Each tube structure can be represented schematically by its graph, where the thickness of tubes approaches zero, reducing the tubes to segments.

1.4.1 Tube Structure and Graphs

Definition 1.4.1 Consider N different points $O_1, O_2, \ldots, O_N$ in $\mathbb{R}^n$, where $n = 2, 3$, and M closed segments $e_1, e_2, \ldots, e_M$ connecting pairs of these points (i.e., $e_j = \overline{O_{i_j} O_{k_j}}$ where $i_j, k_j \in \{1, \ldots, N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. A point O_i is a node if it is the common end of at least two edges, and it is a vertex if it is the end of only one edge. Two edges e_j and e_i can only intersect at their common nodes. The set of vertices is assumed to be non-empty.

A graph $\mathcal{B} = \bigcup_{j=1}^{M} e_j$ is defined as the union of edges, and it is assumed to be a connected set (see Fig. 1.5).

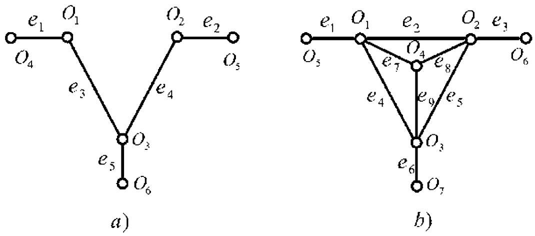

Fig. 1.5 Graphs of tube structures

Let e be an edge, denoted as $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first coordinate system has its origin at O_i, and the axis $O_i x_n^{(e)}$ aligns with the direction of the ray $[O_i O_j)$. On the other hand, the second coordinate system has its origin at O_j, with the axis $O_j \tilde{x}_n^{(e)}$ directed opposite to the ray $[O_j O_i)$.

We can choose either coordinate system, and for both cases, the local variable is denoted as x^e. It will be pointed out which end serves as the origin of the coordinate system.

Let us associate with each edge e_j a bounded domain $\sigma_j \subset \mathbb{R}^{d-1}$ with a C^2-smooth boundary $\partial\sigma_j$, where $j = 1, \ldots, M$. For each edge $e_j = e$ and its corresponding domain $\sigma_j = \sigma^{(e)}$, we introduce the cylinder $\Pi_\varepsilon^{(e)}$ as follows:

$$\Pi_\varepsilon^{(e)} = \left\{ x^{(e)} \in \mathbb{R}^n : x_n^{(e)} \in (0, |e|), \frac{x^{(e)'}}{\varepsilon} \in \sigma^{(e)} \right\},$$

where $x^{(e)'} = (x_1^{(e)}, \ldots, x_{n-1}^{(e)})$, $|e|$ represents the length of the edge e, and $\varepsilon > 0$ is a small parameter. Notably, the edges e_j, Cartesian coordinates of nodes and vertices O_j, as well as the domains σ_j, remain independent of ε.

Let $O_1, \ldots, O_{N_1}$ be nodes and $O_{N_1+1}, \ldots, O_N$ be vertices. Additionally, let $\omega^1, \ldots, \omega^N$ be bounded domains in $\mathbb{R}^n$, independent of ε, and possessing Lipschitz boundaries $\partial\omega^j$. We define the nodal domains $\omega_\varepsilon^j = \{x \in \mathbb{R}^n : \frac{x - O_j}{\varepsilon} \in \omega^j\}$.

Definition 1.4.2 A *tube structure* refers to the following domain:

$$B_\varepsilon = \left(\bigcup_{j=1}^{M} \Pi_\varepsilon^{(e_j)} \right) \bigcup \left(\bigcup_{j=1}^{N} \omega_\varepsilon^j \right). \tag{1.4.1}$$

We assume that this domain is connected and its boundary ∂B_ε is C^2-regular.

Define the surfaces $\gamma_\varepsilon^{N_1+1}, \ldots, \gamma_\varepsilon^N$ as the intersections of the entire boundary ∂B_ε with the boundaries of $\omega_\varepsilon^{N_1+1}, \ldots, \omega_\varepsilon^N$, respectively. These surfaces are referred to as the inflow/outflow parts of the boundary, as in fluid flow models, they represent the fluid inlets and outlets within the domain B_ε.