Einar N. Strømmen

Structural Dynamics

Second Edition

Springer

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Trondheim, Norway

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In loving memory of
Alf & Signe

Preface to the Second Edition

The study of structural dynamics is best achieved if it is based on comprehensive knowledge of the theory of mechanics of materials, a subject on which there are several textbooks available: among others, my recently published book "Structural Mechanics" on Springer Verlag, 2019 (ref. [46]). In that book, it was possible to devote space for more general developments of stress-strain relationships as well as differential equations and virtual work relations than that which was possible in the first edition of "Structural dynamics", a book which naturally was primarily devoted to dynamics rather than basic structural mechanics. Hence, it seemed natural to update "Structural dynamics", and issue a second edition where it has been possible to make references to my recent book on structural mechanics, i.e., to leave out much of the basic structural mechanics theory that for the sake of completeness had to be included in the first edition. For the same reason, it has been possible to include more comprehensive and applicable solutions, e.g., with respect to the effects of cross-sectional asymmetry and time-invariant stress resultants. Regarding damping, additional information has been included about material and structural damping properties. The theory of the suspension bridge as well as the theory of the tuned mass damper have been allocated to separate new chapters. MATLAB routines behind key examples may be made available on request.

Trondheim, Norway — Einar N. Strømmen
December 2023

Preface to the First Edition

This textbook is intended for studies in the theory of structural dynamics, with focus on civil engineering structures that may be described by line-like beam or beam-column type of systems, or by a system of rectangular plates. Throughout this book, the mathematical presentation contains a classical analytical description as well as a description in a discrete finite element format, covering the mathematical development from basic assumptions to the final equations ready for practical dynamic response predictions. Solutions are presented in time domain as well as in frequency domain. It has been my intention to start off at a basic level and step by step bring the reader up to a level where the necessary safety considerations to wind or horizontal ground motion-induced dynamic design problems can be performed, i.e., to a level where dynamic displacements and corresponding cross-sectional forces can actually be calculated. However, this is not a textbook in wind or earthquake engineering, and hence, relevant load descriptions are only included in so far as it has been necessary for the performance of illustrative examples. For more comprehensive descriptions of wind and earthquake-induced dynamic load and load effects, the reader should consult the literature, e.g., refs. [15] and [16]. Less attention has been given to other load cases, e.g., to any kind of shock or impact loading. Also, a comprehensive description of structural damping properties is beyond the scope of this book, but again, for the sake of completeness, a chapter covering the most important theories behind structural damping has been included. The special theory of the tuned mass damper has been given a comprehensive treatment, as this is a theory not fully covered elsewhere. For the same reason, a chapter on the problem of moving loads on beams has been included.

The reading of this book will require some knowledge of structural mechanics, i.e., the basic theory of elasticity. Also, readers unfamiliar with the theory of stochastic processes and time domain simulations should commence their studies by reading Appendices A and B, or another suitable textbook.

Anne Gaarden has prepared the drawings. Thanks to her and all others who have contributed to the writing of this book.

Trondheim, Norway Einar N. Strømmen
September 2012

Notation

Matrices and Vectors

Matrices are in general bold upper-case Latin or Greek letters, e.g., $\boldsymbol{K}$ or $\boldsymbol{\Phi}$.
Vectors are in general bold lower-case Latin or Greek letters, e.g., $\boldsymbol{q}$ or $\boldsymbol{\varphi}$.
$diag[\cdot]$ is a diagonal matrix whose content is written within the brackets.
$det(\cdot)$ is the determinant of the matrix within the brackets.
$tr(\cdot)$ is the trace of a matrix.
$\|\boldsymbol{x}\|$ is the norm of vector $\boldsymbol{x}$, i.e., $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$.

Imaginary Quantities

i is the imaginary unit (i.e., $i = \sqrt{-1}$). $Re(\cdot)$ and $Im(\cdot)$ are the real and imaginary parts of the variable within the brackets, respectively.

Superscripts and Bars Above Symbols

Super-script T indicates the transposed of a vector or a matrix.
Super-script * indicates the complex conjugate of a quantity.
Dots above symbols (e.g., $\dot{\boldsymbol{r}}$, $\ddot{\boldsymbol{r}}$) indicates the time derivatives, i.e., d/dt, d^2/dt^2.
Prime on a variable (e.g., C_L' or ϕ') indicates the first derivative, e.g., $\phi' = d\phi/dx$, two primes are the second derivative (e.g., $\phi'' = d^2\phi/dx^2$), and so on.
Bar ($-$) above a variable (e.g., $\bar{r}$) indicates its time-invariant average value.
Tilde (~) above a symbol (e.g., $\tilde{M}_n$) indicates a modal quantity.
Hat ($\wedge$) above a symbol (e.g., $\widehat{H}_\eta$) indicates a normalised quantity.

The Use of Indices and Superscript

Index x, y or z refers to structural axis. i and j are general indices on variables. n and m are mode shapes or element numbers. p and k are node numbers.

Abbreviations

CC and SC are short for the centre of cross-sectional neutral axis and the shear centre.
tot is short for total. max, min are short for maximum and minimum.
$\int_L$ or $\int_A$ means integration over the entire length or the area of the system.

Latin Letters

A, A_j	Area, cross-sectional area, coefficient associated with variable j
$A_1^* - A_6^*$	Aerodynamic derivatives associated with the motion in torsion
$\boldsymbol{A}$, $\boldsymbol{A}_m$, $\boldsymbol{A}_n$	Connectivity matrix (associated with element m or n)
a, a_i	Distance, coefficient, amplitude
a_x, a_y, a_z	Vector components associated with motion in x, y, z directions
B	Cross-sectional width
b, b_i	Coefficient, bandwidth parameter
b_c	Distance between cable planes is a suspension bridge
$\boldsymbol{b}_q$	Buffeting dynamic load coefficient matrix at cross-sectional level
C, $\boldsymbol{C}$	Damping coefficient, matrix containing damping coefficients
C_{ae}, $\boldsymbol{C}_{ae}$	Aerodynamic damping, aerodynamic damping matrix
c, c_i	Coefficient, damping coefficient at cross-sectional level
$\boldsymbol{c}$, $\boldsymbol{c}_{ae}$	Damping matrix at element level, aerodynamic damping matrix
Co, $\boldsymbol{Co}$	Co-spectral density, co-spectral density matrix
$\boldsymbol{Cov}_j$	Covariance matrix associated with variable j
D, d	Cross-sectional depth, coefficient
$\boldsymbol{d}$, d_i, d_k	Element displacement vector, element end displacement component
E, E_i	Modulus of elasticity, impedance function ($i = 1, 2, \ldots$)
$\boldsymbol{E}_0$	Matrix containing cross-sectional elastic constants
e, e_c	Exponential number (≈ 2.718281828), cable sag
F, $\boldsymbol{F}$, $\boldsymbol{F}_n$	Force, force vector, element force vector
f, f_n	Frequency [Hz], eigenfrequency associated with mode n
$f(\cdot)$	Function of variable within brackets
G	Modulus of elasticity in shear
$\boldsymbol{G}_0$	Matrix containing cross-sectional time invariant stress resultants
$g(\cdot)$, g	Function of variable within brackets, gravity constant
$H(t)$, $\bar{H}$	Horizontal cable force component, its mean value

$H_1^* - H_6^*$	Aerodynamic derivatives associated with the across-wind motion
H_n, $\boldsymbol{H}_r$	Frequency response function, frequency response matrix
$\widetilde{H}_\eta$, $\widetilde{\boldsymbol{H}}_\eta$	Modal frequency response functions, matrix containing $\widetilde{H}_{\eta_n}$
h_c, h_m	Length of suspension bridge hangers, hanger length at mid-span
h_r	Vertical distance between shear centre and hanger attachment
h_0	Height (above girder) of suspension bridge tower
I_t, I_w	St Venant torsion and warping constants
I_y, I_z	Moment of inertia with respect to bending about y or z axis
I_j	Turbulence intensity of flow components $j = u, v$ or w
$\boldsymbol{I}$	Identity matrix
i	The imaginary unit (i.e., $i = \sqrt{-1}$)
J, $\boldsymbol{J}$	Joint acceptance function, joint acceptance matrix
j	Index variable
K, $\boldsymbol{K}$	Stiffness, stiffness matrix
K_{ae}, $\boldsymbol{K}_{ae}$	Aerodynamic stiffness, aerodynamic stiffness matrix
k	Index variable, node, or sample number
k_p	Peak factor
$\boldsymbol{k}$, $\boldsymbol{k}_{ae}$	Stiffness matrix at element level, aerodynamic stiffness matrix
L, $L(t, \boldsymbol{r}, \dot{\boldsymbol{r}})$	Length (of structural system), Lagrange function
${}^m L_n$	Integral length scales ($m = x, y$ or z, $n = u, v$ or w)
ℓ_e	Effective length
M, M_g, $\boldsymbol{M}_g$	Mass, concentrated mass, mass matrix containing M_g
M_n	Cross-sectional bending moment ($n = y$ or z)
M_x, M_θ	Cross-sectional torsion moment, external torsion moment
m	Index variable
m_x, m_y, m_z	Mass per unit length associated with motion in x, y, z directions
$\boldsymbol{M}$	Mass matrix
$\widetilde{m}_n$	Modally equivalent and evenly distributed mass ($n = x$, y or z)
$\boldsymbol{m}_0$, $\boldsymbol{m}$	Mass matrix at a cross-sectional level, mass matrix at element level
N, N_r	Number, number of elements, number of degrees of freedom
N, N_x, N_y	Normal force, normal force in x or y directions
n	Index variable
P, P_F, P_q	External load energy
$P_1^* - P_6^*$	Aerodynamic derivatives associated with the along-wind motion
p, $p()$	Index variable, node or sample number, probability of occurrence
q, q_n, $\boldsymbol{q}$	Distributed load, load vector at cross-sectional level, $n = y, z$ or θ
R, $\boldsymbol{R}$	External load, reaction force, external load vector at system level
$\widetilde{R}$, $\widetilde{\boldsymbol{R}}$	Modal load, Modal load vector
r, r_i, $\boldsymbol{r}$	Displacement or rotation, displacement vector, $i = 1, 2, \ldots$
$r_{el}(x)$, $\boldsymbol{r}_{el}$	Element cross-sectional displacement, displacement vector
St	Strouhal number
S, $\boldsymbol{S}$, $\boldsymbol{S}_j$	Auto or cross-spectral density, cross-spectral density matrix
s	Cross-sectional surface coordinate, relative time variable
T_M, T_m	Motion energy of system body masses
t, T	Time, total length of time window

U	Instantaneous wind velocity in the main flow direction
U, U_M, U_m	Strain energy stored in the material fibres of the system
u	Infinitesimal element displacement in x direction, fluctuating along-wind component
V, V_R	Volume, mean wind velocity, resonance mean wind velocity
V_y, V_z	Shear forces
v	Infinitesimal element displacement in y direction, fluctuating across-wind horizontal component
w	Infinitesimal element displacement in z direction, fluctuating across-wind vertical component
X, Y, Z	Cartesian structural global axis
x, y, z	Cartesian structural element cross-sectional main neutral axis (with origo in the shear centre, x in span-wise direction and z vertical)
x_r	Chosen span-wise position for response calculation

Greek Letters

α, β	Coefficient, phase angle
$\boldsymbol{\beta}(x_p)$	Matrix containing mode shape values $(1 \cdots 6)$ at node position x_p
γ, γ_z, γ_θ	Shear strain, shear strain associated with shear force or torsion
δ, $\delta()$	Virtual displacement operator
∂	Derivative operator
ε, $\boldsymbol{\varepsilon}$, ε_j	Strain, strain vector, strain component ($j = x, y$ or z)
ζ, $\boldsymbol{\zeta}$	Damping ratio, damping ratio matrix
η, $\boldsymbol{\eta}$	Generalised coordinate, vector containing N_{mod} η components
θ	Cross-sectional rotation (about shear centre)
κ	Coefficient
ν	Poisson ratio
λ, λ_n	Wavelength, normalised eigenvalue
μ	Coefficient, friction coefficient
Π	Total energy
ϑ	Coefficient
ρ, ρ_j	Density, density of air, density of component associated with j
σ, σ^2	Standard deviation, variance
σ_n, τ_{mn}	Normal stress, shear stress, m and $n = x, y$ or z
τ	Time lag, dummy time variable
ϕ_y, ϕ_z, ϕ_θ	Mode shape components in y, z and θ directions
$\varphi(x, y)$	Plate mode shape functions
$\boldsymbol{\Phi}(x)$	3 by N_{mod} matrix containing all mode shapes $\boldsymbol{\varphi}_n$
$\hat{\boldsymbol{\Phi}}$, $\hat{\hat{\boldsymbol{\Phi}}}$	Matrices containing first and second-order derivatives of $\boldsymbol{\Phi}$
$\boldsymbol{\Phi}_r(x_r)$	3 by N_{mod} matrix containing the content of $\boldsymbol{\Phi}$ at $x = x_r$
$\boldsymbol{\varphi}_n$	Mode shape number n

ψ, $\boldsymbol{\psi}$, $\boldsymbol{\psi}_n$	Chosen approximate shape function, shape function matrix
$\hat{\boldsymbol{\psi}}$, $\hat{\hat{\boldsymbol{\psi}}}$	Contains first and second-order derivatives of $\boldsymbol{\psi}$
Ω	Coefficient
ω	Circular frequency (rad/s), sector coordinate
ω_n	Eigenfrequency associated with mode shape n
$\omega_n(V)$	Resonance frequency assoc. with mode n at mean wind velocity V
ω_0	Sector coordinate, $\omega_0 = \int_A r ds$

Symbols with Both Latin and Greek Letters

Δf, $\Delta\omega$	Frequency segment
Δt	Time step
Δs	Spatial separation ($s = x,\ y$ or z)
$\delta\Pi$	Change of energy

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Chapter 1
Basic Theory

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1.1 Introduction

This textbook focuses on the prediction of dynamic response of slender line-like or flat plate type of civil engineering structures. It is a general assumption that structural behaviour is linear elastic and that any non-linear part of the relationship between load and structural displacements may be disregarded. It is taken for granted that the load direction throughout the entire span of the structure is perpendicular to the axis in the direction of its span, as shown in Fig. 1.2a.

As shown in Figs. 1.1 and 1.2, a line-like beam type of structural element is described in Cartesian coordinate system (x, y, z), with its origo at the shear centre (SC) of the cross section, x in direction of its span and with y and z parallel to main neutral structural axes (i.e. the neutral axes with respect to cross sectional bending according to Hook's law and Navier's hypothesis), whose origo CC is defined by:

$$\int_A \begin{bmatrix} y_c \\ z_c \end{bmatrix} dA = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} y_c \\ z_c \end{bmatrix} = \begin{bmatrix} y - e_y \\ z - e_z \end{bmatrix} \tag{1.1}$$

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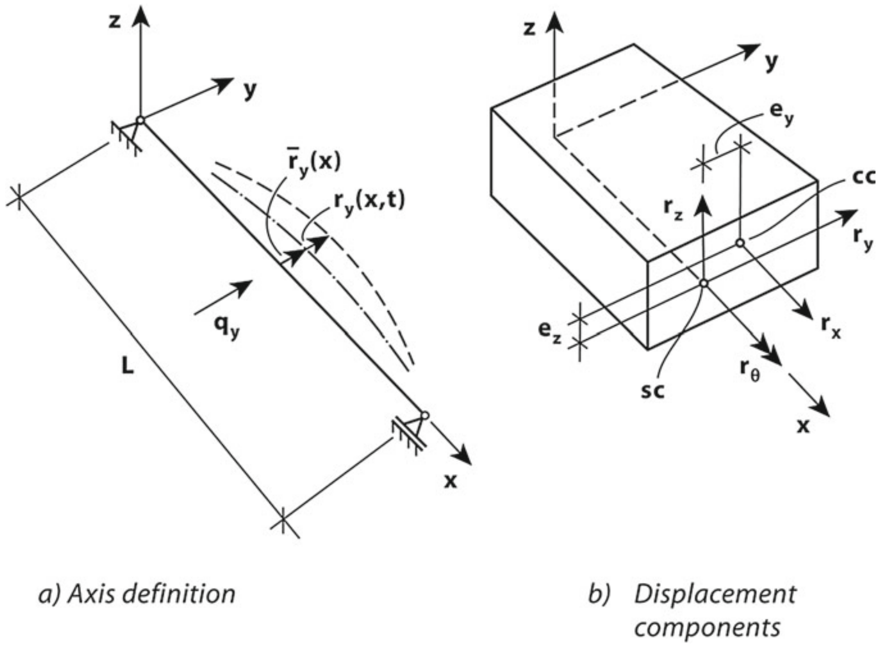

Fig. 1.1 Structural axes and displacement components

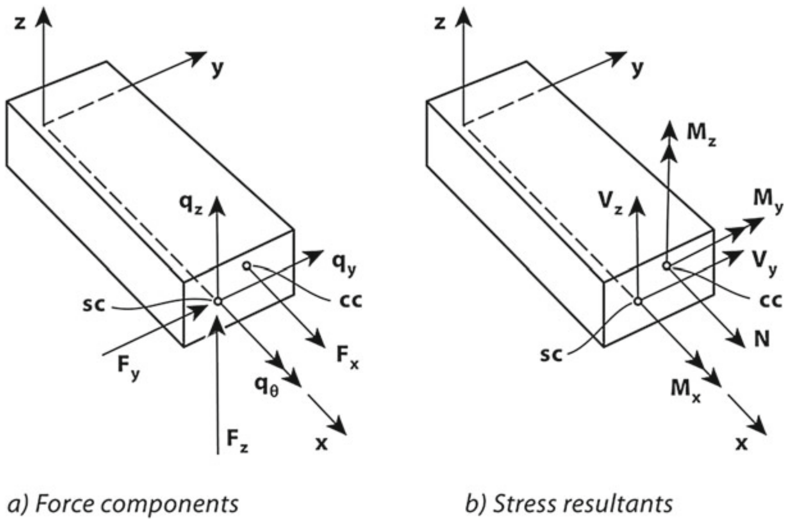

Fig. 1.2 Basic axis and vector definitions

Table 1.1 Approximate properties of materials [48]

Material	Density, $\rho 10^3$ kg/m^3	Elastic modulus, E 10^9 N/m^2	Shear modulus, G 10^9 N/m^2	Poisson ratio, υ
Aluminium	2.7	70	27	0.34
Brass	8.5	95	36	0.33
Copper	8.9	125	46	0.35
Concrete[a]	2.4	~30	12.5	0.2
Glass	2.5	50–90	20–30	0.2–0.3
Gold	19.3	80	28	0.423
Iron	7.8	200	77	0.3
Lead	11.3	17	6	0.43
Magnesium	1.74	43	17	0.29
Nickel	8.9	205	77	0.3
Silver	10.5	80	29	0.37
Steel	7.8	210	77	0.31
Tin	7.3	4.4	1.6	0.39
Wood, along grain	0.3–0.6	7–15	3–5	≈ 0.4

[a] In compression

If material density does not change over the area of the cross section, then main neutral axis (CC) will coincide with the mass centre, as defined by:

$$\int_A \rho \begin{bmatrix} y_c \\ z_c \end{bmatrix} dA = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.2}$$

Basic properties of some engineering materials are shown in Table 1.1. Units throughout the book adhere to ISO definitions:

- Displacement, rotation: meter (*m*), radians (*rad*)
- time: second (*s*)
- mass: kilogram (*kg*)
- force: Newton ($N = kg \cdot m/s^2$, [1, 2]).

The mean value (static part) of any load is assumed constant such that structural response can be predicted as the sum of a mean value and a fluctuating part, as indicated for the y-direction in Fig. 1.1a. Response displacements r_y, r_z, r_θ and load components F_y, F_z, q_y, q_z and q_θ are referred to shear centre (SC). Response displacement r_x and load component F_x are referred to origo of main neutral axis. Similarly, cross sectional stress resultants V_y, V_z, M_x are referred to shear centre, while bending moments M_y and M_z, as well as axial stress resultant N, are referred to origin of main neutral axis. All forces, moments, stresses, and stress resultants are vectors in (x, y, z) coordinates. If the load (concentrated or evenly distributed) is stationary, it may always be split into mean time invariant and a fluctuating part:

$$\boldsymbol{F}_{tot} = \overline{\boldsymbol{F}} + \boldsymbol{F}(t) = \begin{bmatrix} \overline{F}_x \\ \overline{F}_y \\ \overline{F}_z \end{bmatrix} + \begin{bmatrix} F_x(t) \\ F_y(t) \\ F_z(t) \end{bmatrix}, \quad \mathbf{q}_{tot} = \overline{\boldsymbol{q}} + \boldsymbol{q} = \begin{bmatrix} \overline{q}_y(x) \\ \overline{q}_z(x) \\ \overline{q}_\theta(x) \end{bmatrix} + \begin{bmatrix} q_y(x,t) \\ q_z(x,t) \\ q_\theta(x,t) \end{bmatrix} \quad (1.3)$$

Structural response displacements as well as corresponding cross-sectional stress resultants will then also be stationary, comprising mean and fluctuating parts:

$$\overline{\boldsymbol{r}} + \boldsymbol{r} = \begin{bmatrix} \overline{r}_x(x) \\ \overline{r}_z(x) \\ \overline{r}_\theta(x) \end{bmatrix} + \begin{bmatrix} r_y(x,t) \\ r_z(x,t) \\ r_\theta(x,t) \end{bmatrix} \quad (1.4)$$

and, as shown below, they may independently be obtained by satisfying the relevant static and dynamic equilibrium requirements of the system.

1.2 d'Alambert's Principle of Instantaneous Equilibrium

In statics the equilibrium conditions of a system subject to a set of time-invariant forces F_i (with unit N) and moments M_j (with unit Nm), is defined by:

$$\sum_i \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \sum_j \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.5)$$

Given a body with volume V, translatory mass $M = \int_V \rho dV$, and rotational mass $M_{\theta_n} = \int_V \rho a_n^2 dV$ about axis $n = x, y$ or z, where a_n is distance from mass centre to volume element dV. If the body is in rectilinear acceleration $\ddot{r}_n(t)$ or rotational acceleration $\ddot{r}_{\theta_n}(t)$; then Newton's second law states:

$$\sum_i \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}_i = M \begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix} \quad \text{and} \quad \sum_j \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}_j = \begin{bmatrix} M_{\theta_x} \ddot{r}_{\theta_x} \\ M_{\theta_y} \ddot{r}_{\theta_y} \\ M_{\theta_z} \ddot{r}_{\theta_z} \end{bmatrix} \quad (1.6)$$

where $\ddot{r}_n$ and $\ddot{r}_{\theta_n}(n = x, y, z)$ have units m/s^2 and rad/s^2. Static equilibrium, as given in Eq. 1.5, follows from the condition that the system is at rest or at a constant translatory and rotational velocity, i.e., that:

$$\begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \ddot{r}_{\theta_x} \\ \ddot{r}_{\theta_y} \\ \ddot{r}_{\theta_z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.7)$$

In dynamics (where Eq. 1.7 is not fulfilled) equilibrium considerations will have to include the motion of the system. This is done by adopting the principle of d'Alambert (first published by Lagrange [3]) that equilibrium of a system in motion can be established by considering the system at an arbitrary instantaneous (frozen) position in space and time, where accelerations can be interpreted as inertia forces in accordance with Newton's second law, i.e.: as forces $M\begin{bmatrix}\ddot{r}_x\\ \ddot{r}_y\\ \ddot{r}_z\end{bmatrix}$ and moments $\begin{bmatrix}M_{\theta_x}\ddot{r}_{\theta_x}\\ M_{\theta_y}\ddot{r}_{\theta_y}\\ M_{\theta_z}\ddot{r}_{\theta_z}\end{bmatrix}$ that resist acceleration.

Discrete Systems

A simple example is illustrated in Fig. 1.3. A mass M is suspended from a linear elastic spring with stiffness K. At left hand side the system is shown in its unloaded position.

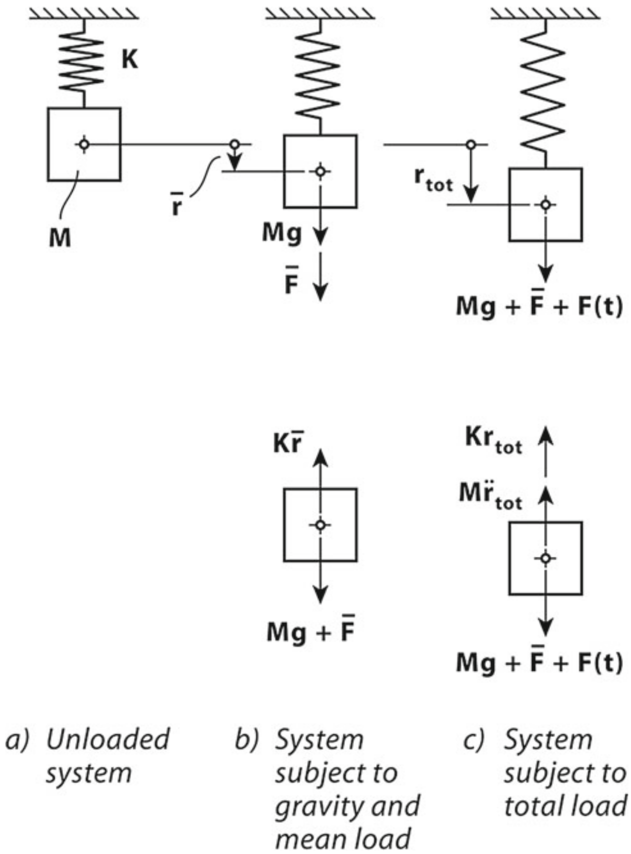

Fig. 1.3 Simple spring-mass system

Let the system first be subject to gravity Mg (where g is gravity acceleration constant) and a time invariant (static) force $\overline{F}$. The system is then at rest in a static position, it has been displaced a distance $\overline{r}$. As shown in Fig. 1.3b, the equilibrium requirement is that $K\overline{r} = Mg + \overline{F}$, from which $\overline{r}$ may be calculated if all other quantities are known. Let the system be subject to an additional dynamic force $F(t)$. Then, at arbitrary position $r_{tot} = \overline{r} + r(t)$, where $r(t)$ is the dynamic displacement, the equilibrium condition is defined by the sum of external forces $Mg + \overline{F} + F(t)$ being equal to elastic spring force Kr_{tot}, plus a motion resistance (inertia) force $M\ddot{r}_{tot}$ in accordance with Newton's second law, i.e.:

$$M\ddot{r}_{tot} + Kr_{tot} = Mg + \overline{F} + F(t) \tag{1.8}$$

Introducing $r_{tot} = \overline{r} + r(t)$, then:

$$M\ddot{r} + K(\overline{r} + r) = Mg + \overline{F} + F(t) \tag{1.9}$$

and, since the static equilibrium condition is defined by $K\overline{r} = Mg + \overline{F}$, it is seen that Eq. 1.9 may be reduced into a purely dynamic equilibrium condition:

$$M\ddot{r} + Kr = F(t) \tag{1.10}$$

Thus, it may be concluded that the equilibrium condition for such a linear elastic system may be split into two: a static time invariant equilibrium condition where only static loads are included ($K\overline{r} = Mg + \overline{F}$), and a dynamic equilibrium condition, where only dynamic loads are included [$M\ddot{r} + Kr = F(t)$], where forces due to instantaneous acceleration of the system is represented by inertia forces in opposite direction of the motion. For the system in Fig. 1.3, let us first assume that $F(t) = 0$, but that the mass has been set into an oscillating motion by imposing an initial displacement $r(0)$ and corresponding velocity $\dot{r}(0)$. Then Eq. 1.10 is reduced into $M\ddot{r} + Kr = 0$, and it is seen that its solution must be such that r is congruent to $\ddot{r}$, which is obtained by:

$$r(t) = b\,sin(\omega_n t) + c\,cos(\omega_n t) \tag{1.11}$$

where b and c are coefficients to match the position and velocity conditions at $t = 0$, from which it is seen that $r(0) = c$ and $\dot{r}(0) = \omega_n b$. The frequency of motion ω_n may then be obtained by introducing Eq. 1.11 into Eq. 1.10, from which:

$$\left(K - \omega_n^2 M\right) r(t) = 0 \tag{1.12}$$

It is seen that non-trivial solution $r(t) \neq 0$ can only be obtained if $K - \omega_n^2 M = 0$.

Thus, the frequency of free unloaded and oscillatory motion is given by $\omega_n = \sqrt{K/M}$. The motion is harmonic because it contains only a single and stationary frequency. This is what we call the eigenfrequency of the system. It has the unit

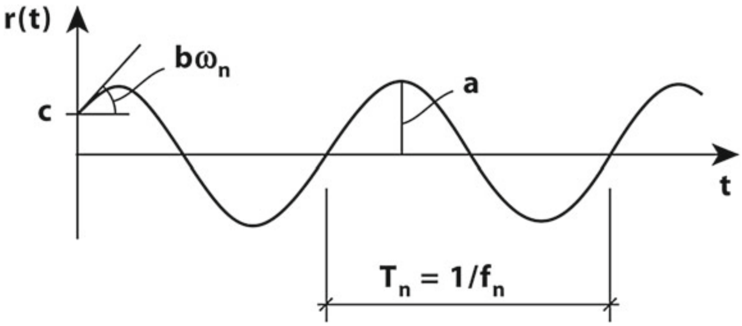

Fig. 1.4 Unloaded and undamped motion of single degree of freedom system

rad/s. In some cases, it may be convenient to convert it into $f_n = \omega_n/(2\pi)$ with unit $Hz = 1/s$, while yet another option is to introduce the period of the motion $T_n = 1/f_n$. A plot of a typical version of $r(t)$ is shown in Fig. 1.4.

As we shall see later, the only reason why there is a phase (time lag) between load and response in dynamics is the presence of damping, and therefore, it is a convenient mathematical simplification to convert the version of $r(t) = b\,sin(\omega_n t) + c\,cos(\omega_n t)$ in Eq. 1.11 into a complex format:

$$r(t) = Re\left(ae^{i\omega t}\right) \tag{1.13}$$

by using the trigonometric property of two arbitrary angles α_1 and α_2 : $sin\,\alpha_1 \cdot sin\,\alpha_2 + cos\,\alpha_1 \cdot cos\,\alpha_2 = cos(\alpha_1 - \alpha_2)$. Recalling that $c = r(0)$ and $b = \dot{r}(0)/\omega_n$, it is seen that $r(t)$ in Eq. 1.11 may then be written:

$$r(t) = a \cdot cos(\omega t - \beta) \text{ where : } \begin{cases} a = \sqrt{b^2 + c^2} = \sqrt{[r(0)]^2 + \left[\frac{\dot{r}(0)}{\omega_n}\right]^2} \\ tan\,\beta = b/c = \dot{r}(0)/[\omega_n r(0)] \end{cases} \tag{1.14}$$

Response $r(t)$ may be expressed in a complex format by defining: $a = c - ib = |a|e^{-i\beta} = (a^*a)^{1/2}e^{-i\beta}$, where i is the complex unit ($i = \sqrt{-1}$), i.e.:

$$\begin{aligned} r(t) &= Re\left(ae^{i\omega t}\right) = Re\left[(c - ib)e^{i\omega t}\right] = Re\left(|a|e^{-i\beta}e^{i\omega t}\right) \\ &= |a|Re\left[e^{i(\omega t - \beta)}\right] = |a|\,cos(\omega t - \beta) \end{aligned} \tag{1.15}$$

Example 1.1: Two Parallel Springs

A single mass with two parallel springs is shown on the left-hand side in Fig. 1.5. Next, it has been given an arbitrary harmonic motion $r(t) = Re\left(ae^{i\omega t}\right)$. Equilibrium will require (see right hand side free body diagram): $M\ddot{r} + (K_1 + K_2)r = 0$. Introducing $r(t) = Re\left(ae^{i\omega t}\right)$, then:

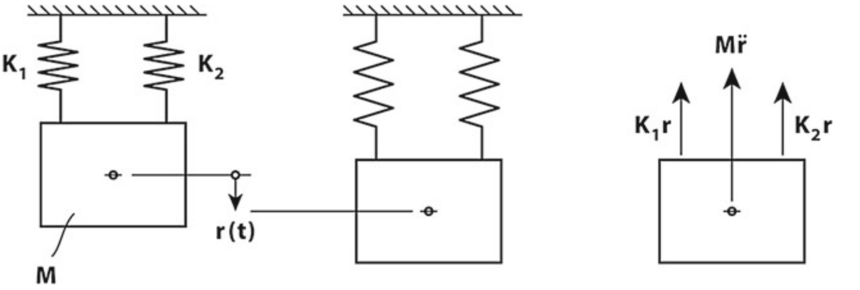

Fig. 1.5 Single mass with two parallel springs

$$-\omega^2 M + (K_1 + K_2) = 0 \Rightarrow \omega_n = \sqrt{(K_1 + K_2)/M}$$

from which it may be concluded that stiffness contributions in parallel are additive.

Example 1.2: Two Springs in Sequence

A single mass with two springs in sequence is shown on the left-hand side of Fig. 1.6. Next, it has been given arbitrary harmonic displacement $r_2(t) = Re\left(a_2 e^{i\omega t}\right)$. During this motion the connection between the two springs has undergone a harmonic displacement $r_1(t) = Re\left(a_1 e^{i\omega t}\right)$. Resisting force in upper spring is $F_{k_1} = K_1 r_1$, while resisting force in lower spring is $F_{k_2} = K_2(r_2 - r_1)$. The force throughout the sequence of springs must be unchanged. I.e., $F_{k_1} = F_{k_2}$. Thus:

$$K_1 r_1 = K_2(r_2 - r_1) \Rightarrow r_1 = \frac{K_2}{K_1 + K_2} r_2 \Rightarrow F_{k_1} = F_{k_2} = F_k = K_1 r_1$$
$$= \frac{K_1 K_2}{K_1 + K_2} r_2$$

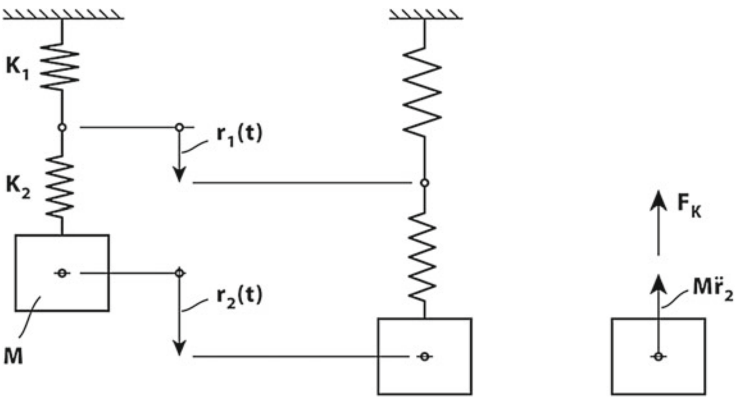

Fig. 1.6 Single mass with two springs in sequence

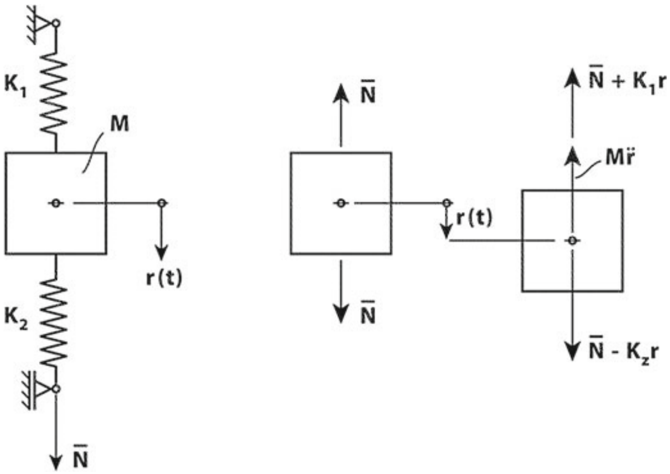

Fig. 1.7 Single mass with springs on either side

Equilibrium of the mass (see right hand side of Fig. 1.6) will then require: $M\ddot{r}_2 + [K_1K_2/(K_1 + K_2)]r_2 = 0$, which, with $r_2(t) = Re\left(a_2e^{i\omega t}\right)$, the following is obtained:

$$-\omega^2M + \frac{K_1K_2}{K_1 + K_2} = 0 \quad \text{and thus: } \omega_n = \sqrt{\frac{K_{tot}}{M}} \; where \; K_{tot} = \left(\frac{1}{K_1} + \frac{1}{K_2}\right)^{-1}$$

It may be concluded that stiffness contributions in sequence are inversely additive.

Example 1.3: Springs on Either Side of a Single Mass

A mass with springs on either side is shown at left-hand side of Fig. 1.7. The springs have been pre-stretched by a constant time invariant normal force $\overline{N}$, such that prior to any displacement the system is in a state of equilibrium. It is assumed that displacements are never smaller than that which will cause the springs to slacken. The mass has been set into harmonic motion $r(t) = Re\left(ae^{i\omega t}\right)$. At far right is shown free body diagram of forces acting on the mass. Equilibrium will require:

$$M\ddot{r} + \left(\overline{N} + K_1r\right) - \left(\overline{N} - K_2r\right) = 0.$$

Introducing $r(t) = Re\left(ae^{i\omega t}\right)$, the following is obtained:

$$-\omega^2M + K_1 + K_2 = 0 \quad and \text{ thus: } \quad \omega_n = \sqrt{(K_1 + K_2)/M}\omega_n = \sqrt{(K_1 + K_2)/M}$$

from which it may be concluded that stiffness contributions are additive.

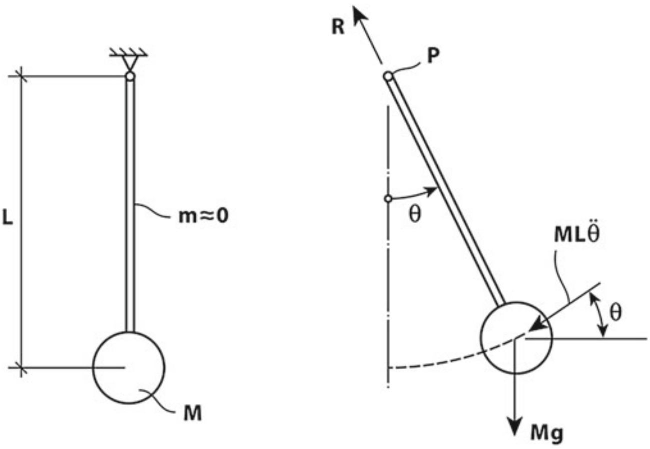

Fig. 1.8 Small displacement pendulum

Example 1.4: The Pendulum

The case of a simple pendulum is shown in Fig. 1.8. For simplicity, the mass of the rod is assumed negligible. At arbitrary rotation $\theta(t) = Re\left(a_\theta e^{i\omega t}\right)$, a free body diagram of the system is shown to the right. In this situation the mass M is subject to gravity force Mg, tangential acceleration $d(\theta L)/dt^2$ and corresponding restoring inertia force $Md(\theta L)/dt^2 = ML\ddot{\theta}$. Instantaneous moment equilibrium about point p will then require:

$$(ML\ddot{\theta})L + MgL\,sin\,\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + (g/L)\,sin\,\theta = 0$$

which cannot be analytically solved unless we assume θ small, such that $\sin\theta \approx \theta$, in which case $\ddot{\theta} + (g/L)\theta = 0$. Thus, by introducing $\theta = Re\left(a_\theta e^{i\omega t}\right)$, the following is obtained: $g/L - \omega^2 = 0$, from which the eigenfrequency is given by:

$$\omega_n = \sqrt{g/L}$$

The system in Fig. 1.3, as well as all the examples above, contains only one unknown displacement component. We say such systems have one degree of freedom. A more complex system is illustrated at left hand side of Fig. 1.9, showing two masses M_1 and M_2 subject to forces F_1 and F_2. This system has two degrees of freedom, r_1 and r_2, i.e.:

- the number of degrees of freedom in a system is equal to the number of unknown displacement components that are necessary to enable a complete depiction of the position of the system at all times.

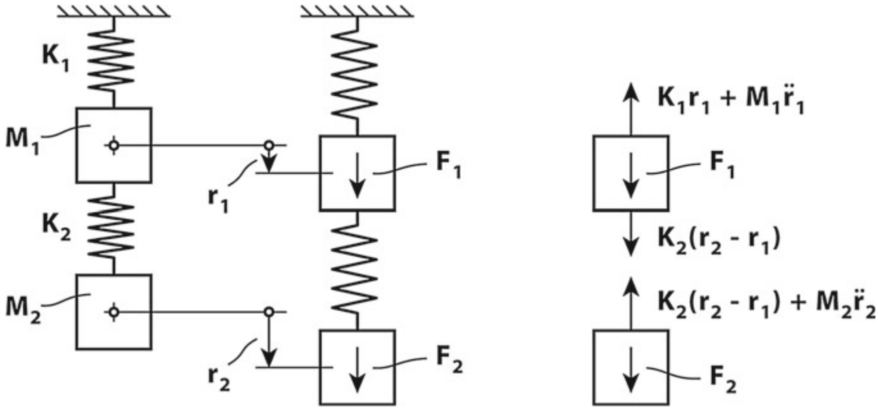

Fig. 1.9 Spring mass system with two degrees of freedom

The equilibrium requirements (see free body diagram of M_1 and M_2 at right hand side of Fig. 1.9) are then given by:

$$\left.\begin{array}{c} K_1 r_1 + M_1 \ddot{r}_1 - F_1 - K_2(r_2 - r_1) = 0 \\ K_2(r_2 - r_1) + M_2 \ddot{r}_2 - F_2 = 0 \end{array}\right\} \tag{1.16}$$

This may more conveniently be written in a matrix–vector format:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{1.17}$$

which, by defining:

$$\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{1.18}$$

may be reduced into the following compact format:

$$\boldsymbol{M}\ddot{\boldsymbol{r}} + \boldsymbol{K}\boldsymbol{r} = \boldsymbol{F} \tag{1.19}$$

If $\boldsymbol{F} = \mathbf{0}$, then the solution is a harmonic motion, which may be described by:

$$\boldsymbol{r} = Re\left(\boldsymbol{\varphi} e^{i\omega t}\right) \quad \text{where: } \boldsymbol{\varphi} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{1.20}$$

By introducing this into Eq. 1.19, then the following requirement is obtained:

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right)\boldsymbol{\varphi} = \mathbf{0} \tag{1.21}$$

A non-trivial solution $\boldsymbol{r} \neq \boldsymbol{0}$ can only be obtained if $\boldsymbol{\varphi} \neq \boldsymbol{0}$, i.e., only if:

$$\begin{aligned} det\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) &= det\begin{bmatrix} (K_1 + K_2) - \omega^2 M_1 & -K_2 \\ -K_2 & K_2 - \omega^2 M_2 \end{bmatrix} \\ &= \left(K_1 + K_2 - \omega^2 M_1\right)\left(K_2 - \omega^2 M_2\right) - K_2^2 = 0 \end{aligned} \tag{1.22}$$

which may be further developed into the polynomial:

$$\omega^4 - [(K_1 + K_2)/M_1 + K_2/M_2]\omega^2 + (K_1/M_1)(K_2/M_2) = 0 \tag{1.23}$$

Its roots are:

$$\omega_{1,2}^2 = \frac{1}{2}\left(\frac{K_1 + K_2}{M_1} + \frac{K_2}{M_2}\right) \mp \sqrt{\frac{1}{4}\left(\frac{K_1 + K_2}{M_1} + \frac{K_2}{M_2}\right)^2 - \frac{K_1}{M_1}\frac{K_2}{M_2}} \tag{1.24}$$

Equation 1.21 is an eigenvalue problem whose solution is given by ω_1 and ω_2. They are the eigenfrequencies of the system. The number of eigenfrequencies will always be the same as the number of degrees of freedom in the system. They are usually presented in ascending order because in almost all practical cases it is a few of the lowest that are of primary interest. For each eigenfrequency ω_n there is a corresponding eigenvector $\boldsymbol{\varphi}_n$. Introducing ω_1 and ω_2, back into Eq. 1.21, one after the other, we obtain:

$$\boldsymbol{\varphi}_1 = a_1\begin{bmatrix} 1 \\ \left(K_1 + K_2 - \omega_1^2 M_1\right)/K_2 \end{bmatrix} \tag{1.25}$$

$$\boldsymbol{\varphi}_2 = a_1\begin{bmatrix} 1 \\ \left(K_1 + K_2 - \omega_2^2 M_1\right)/K_2 \end{bmatrix} \tag{1.26}$$

It is seen that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ may be arbitrarily scaled (e.g., by setting $a_1 = 1$). Thus, they do not represent the actual displacements of the system, only its shape.

We call them the mode shapes of the system. (A displacement response can only be quantified if we have a forcing action or an initial displacement on the system.) Let $K_1 = K_2 = 2 \cdot 10^7$ Nm and $M_1 = M_2 = 10^6$ kg. Then $\omega_1 = 2.76$ rad/s and $\boldsymbol{\varphi}_1 = a_1\left[1 \;\, 1.618\right]^T$, while $\omega_2 = 7.24$ rad/s and $\boldsymbol{\varphi}_2 = a_1\left[1 \;\, -0.618\right]^T$. The motion represented by ω_1 and $\boldsymbol{\varphi}_1$ is shown in the upper diagram in Fig. 1.10. The motion represented by ω_2 and $\boldsymbol{\varphi}_2$ is shown in the lower diagram in Fig. 1.10. In both cases $r_1(t = 0) = 0.5$ and $\dot{r}_1(t = 0) = 0.5$. Note that at ω_1 the two masses are moving in harmony, while at ω_2 they are moving in opposite direction to each other, in compliance with the two eigenmodes.

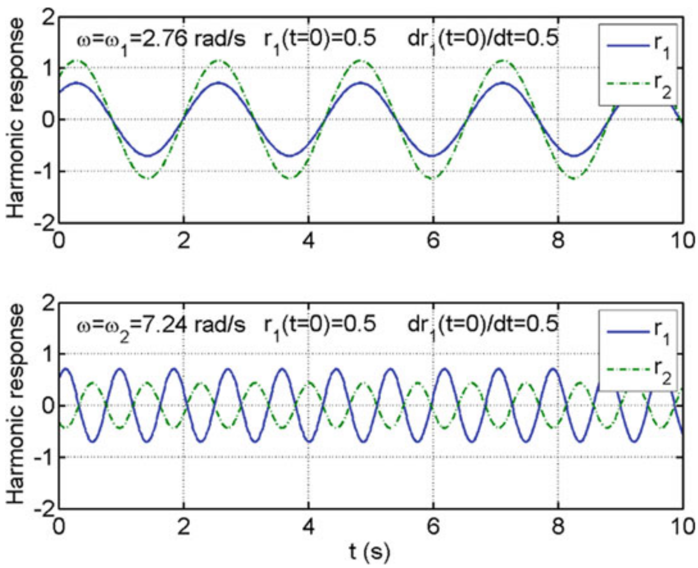

Fig. 1.10 Harmonic motion of two degree of freedom system

Example 1.5: Two Degrees of Freedom Rigid Beam

A beam on flexible supports K_1 and K_2 is shown in Fig. 1.11. For simplicity, it is assumed infinitely rigid, i.e., its bending stiffness is large. The free body diagram at arbitrary displacements $r_1(t) = Re\left(a_1 e^{i\omega t}\right)$ and $r_2(t) = Re\left(a_2 e^{i\omega t}\right)$ is illustrated at right-hand side of Fig. 1.11. In this case it is necessary to demand vertical as well as moment equilibrium. First it is seen that:

- the beam displacement is given by $r(x, t) = r_1 + (r_2 - r_1)x/L$
- while the support forces $R_1 = K_1 r_1$ and $R_2 = K_2 r_2$

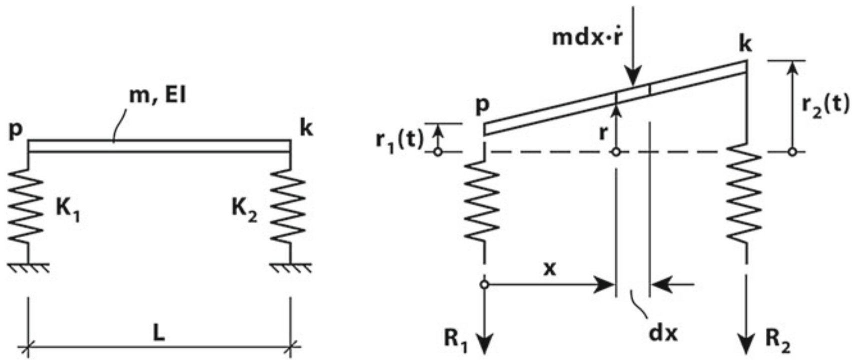

Fig. 1.11 Rigid beam on flexible supports

Thus, vertical equilibrium will require: $R_1 + R_2 + \int_0^L m\ddot{r}dx = 0$, i.e., that:

$$K_1r_1 + K_2r_2 + m\int_0^L \left[\ddot{r}_1 + (\ddot{r}_2 - \ddot{r}_1)\frac{x}{L}\right]dx = K_1r_1 + K_2r_2 + (\ddot{r}_1 + \ddot{r}_2)\frac{mL}{2} = 0$$

while moment equilibrium about the beam end p will require: $R_2L + \int_0^L m\ddot{r}dx = 0$, i.e.:

$$K_2r_2L + m\int_0^L \left[\ddot{r}_1 + (\ddot{r}_2 - \ddot{r}_1)\frac{x}{L}\right]xdx = K_2r_2L + \left(\frac{\ddot{r}_1}{6} + \frac{\ddot{r}_2}{3}\right)mL^2 = 0$$

Introducing $r_1(t) = Re\left(a_1e^{i\omega t}\right)$ and $r_2(t) = Re\left(a_2e^{i\omega t}\right)$, then these equations turn into:

$$K_1a_1 + K_2a_2 - \omega^2(a_1 + a_2)\frac{mL}{2} = 0 \quad and \quad K_2a_2 - \omega^2\left(\frac{a_1}{6} + \frac{a_2}{3}\right)mL = 0$$

which may be written: $\begin{bmatrix} \left(K_1 - \omega^2mL/2\right) & \left(K_2 - \omega^2mL/2\right) \\ -\omega^2mL/6 & \left(K_2 - \omega^2mL/3\right) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Again, this is an eigenvalue problem, whose solution is defined by setting the determinant to the coefficient matrix equal to zero, from which the following is obtained:

$$\begin{gathered}\left(K_1 - \omega^2mL/2\right)\left(K_2 - \omega^2mL/3\right) - \left(-\omega^2mL/6\right)\left(K_2 - \omega^2mL/2\right) \\ = \omega^4 - 4\left(\tfrac{K_1}{mL} + \tfrac{K_2}{mL}\right)\omega^2 + 12\left(\tfrac{K_1}{mL}\right)\left(\tfrac{K_2}{mL}\right) = 0\end{gathered}$$

Thus, the two eigen frequencies (in ascending order) are given by:

$$\omega_{1,2} = \sqrt{2\left[\frac{K_1}{mL} + \frac{K_2}{mL} \mp \sqrt{\left(\frac{K_1}{mL}\right)^2 - \frac{K_1}{mL}\frac{K_2}{mL} + \left(\frac{K_2}{mL}\right)^2}\right]}$$

If, for instance, $K_1 = K_2 = K$, then $\omega_1 = \sqrt{2K/(mL)}$ and $\omega_2 = \sqrt{6K/(mL)}$. Introducing $\omega = \omega_1 = \sqrt{2K/(mL)}$ into the second row of the matrix–vector relationship above, the following is obtained:

$$-\omega_1^2(mL/6)a_1 + \left[K - \omega_1^2(mL/3)\right]a_2 = 0 \quad \Rightarrow \quad a_1 = a_2$$

I.e., the beam displacement is a rigid body motion purely in vertical direction. Introducing $\omega = \omega_2 = \sqrt{6K/(mL)}$ into the second row of the matrix–vector relationship, the following is obtained:

$$-\omega_2^2(mL/6)a_1 + \left[K - \omega_2^2(mL/3)\right]a_2 = 0 \quad \Rightarrow \quad a_1 = -a_2$$

I.e., the beam displacement is a rigid body rotation about its mid-span.

Continuous Systems

A continuous line-like beam subject to a distributed dynamic load $q_z(x,t)$ (with unit N/m) is illustrated in Fig. 1.12a. For such a system the relevant equilibrium requirement is most conveniently established in the form of one or several differential equations. If the system is symmetric about the z-axis, then the response motion is only taking place in the vertical z direction. Since the system is continuous, so is the displacement function $r_z(x,t)$, and therefore, we shall resort to calculus. As shown in Fig. 1.12.b, an incremental element dx will require moment equilibrium (taken about mid-point c, discarding higher order terms):

$$dM_y - V_z dx = 0 \quad \Rightarrow \quad V_z = M_y' \tag{1.27}$$

as well as force equilibrium in z direction:

$$q_z dx - (m_z dx)\ddot{r}_z + dV_z = 0 \quad \Rightarrow \quad V_z' = -q_z + m_z \cdot \ddot{r}_z \tag{1.28}$$

where m_z is mass per unit length (kg/m) of the beam. Thus:

$$M_y'' = -q_z + m_z \cdot \ddot{r}_z \tag{1.29}$$

Since dynamic motion exclusively takes place in the direction of z, the beam cross section is subject to pure bending about the y axis, i.e., $M_y \neq 0$ and $V_z \neq 0$, while all other cross sectional stress resultants are equal to zero. The cross-sectional neutral axis is defined by the axis through zero strain (see Strømmen [46]). Adopting Navier's hypothesis [4, 5] that a cross section perpendicular to the system neutral axis prior to bending remains perpendicular to the neutral axis after bending, then the linear bending strain distribution (see Fig. 1.12c) is given by:

$$\varepsilon_x = \frac{\alpha z_c - (\alpha + d\alpha) z_c}{dx} = -\frac{d\alpha}{dx} z_c \tag{1.30}$$

where z_c is the distance from the neutral axis (CC) to an arbitrary cross-sectional element dA.

Let us assume linear elasticity and take it for granted that displacements are small such that $\alpha \approx r_z'$. Then $\varepsilon_x = -\alpha' z_c = -r_z'' z_c$, and thus:

$$\sigma_x = E\varepsilon_x = E\left(-r_z'' z_c\right) \Rightarrow \begin{cases} N = \int_A \sigma_x \, dA = -r_z'' E \int_A z_c \, dA \\ M_y = \int_A z_c \sigma_x \, dA = -r_z'' E \int_A z_c^2 \, dA \end{cases} \tag{1.31}$$

Requirement $N = 0$ implies that $\int_A z_c \, dA = 0$, which determines the position of the neutral axis (see Eq. 1.1), while $I_y = \int_A z_c^2 dA$ is the cross sectional second area moment. Thus, Eq. 1.29 turns into:

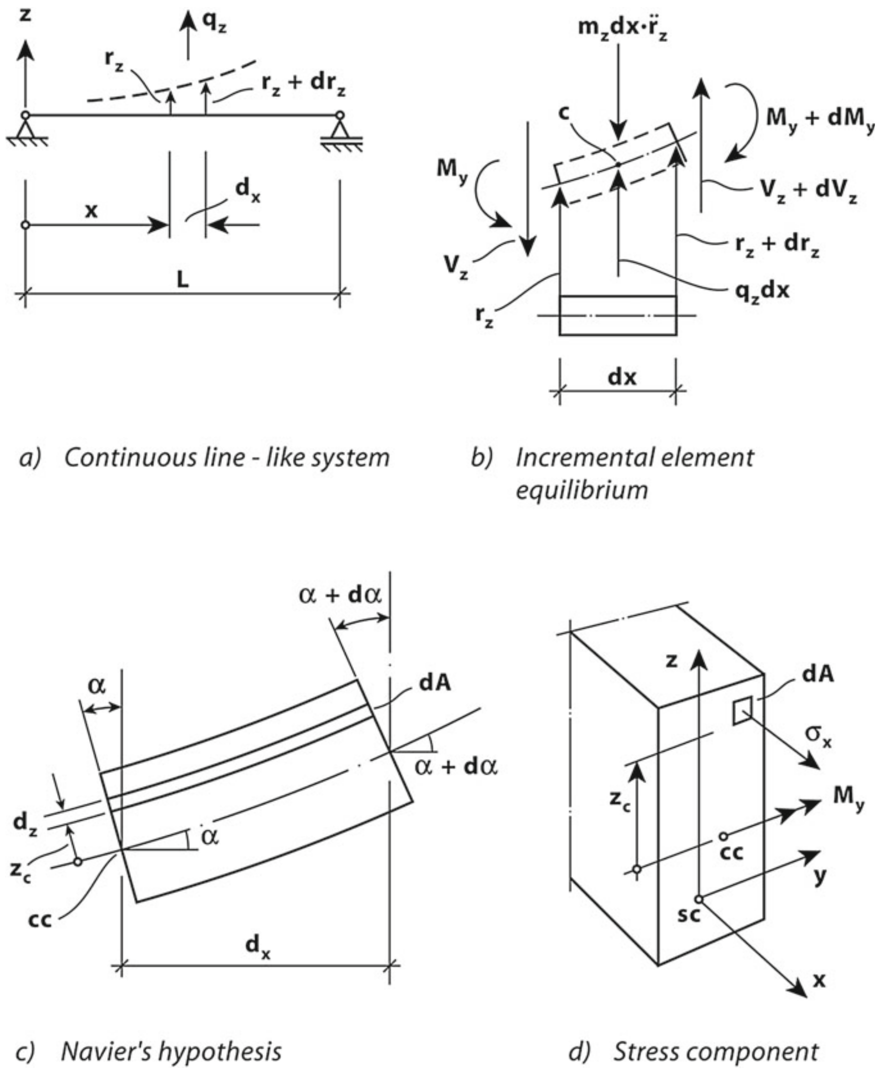

Fig. 1.12 Line-like continuous beam subject to distributed dynamic load

$$\frac{d^2}{dx^2}\left(-r_z'' EI_y\right) = -q_z + m_z \cdot \ddot{r}_z \tag{1.32}$$

which, provided EI_y is constant along the span, may be simplified into:

$$m_z \ddot{r}_z + EI_y r_z'''' = q_z \tag{1.33}$$