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Handbook of Geometry and Topology of Singularities VI: Foliations

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Preface

This Volume VI of the *Handbook of Geometry and Topology of Singularities* forms a unit with Volume V, focused on singular holomorphic foliations.

In his foreword to Volume I of this collection, Bernard Teisier wrote:

> In the general scientific culture, Mathematics can appear as quite disconnected. One knows about calculus, complex numbers, Fermat's last theorem, convex optimization, fractals, vector fields and dynamical systems, the law of large numbers, projective geometry, vector bundles, the Fourier transform and wavelets, the stationary phase method, numerical solutions of PDEs, etc., but no connection between them is readily apparent. For the mathematician, however, all these and many others are lineaments of a single landscape. Although he or she may spend most of his or her time studying one area of this landscape, the mathematician is conscious of the possibility of traveling to other places, perhaps at the price of much effort, and bringing back fertile ideas. Some of the results or proofs most appreciated by mathematicians are the result of such fertilizations. I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture.

Foliation Theory also is a multidisciplinary field and a whole area of mathematics in itself, with close connections with dynamical systems, geometry, topology and singularity theory. For instance:

1. The integral lines of a holomorphic vector field on a complex manifold, define a one-dimensional holomorphic foliation with singularities at the zeros of the vector field.
2. A holomorphic 1-form on a manifold determines a codimension-one sub-bundle of the tangent bundle, except at the points where the 1-form vanishes. Under a certain "integrability condition" this gives rise to a codimension-one singular holomorphic foliation.
3. The fibers of a holomorphic function f from a complex $(n+k)$-manifold M into a complex n-manifold N defines a holomorphic foliation with singularities at the critical points of f.
4. Lefschetz pencils in algebraic geometry also are examples of singular holomorphic foliations.

These important examples highlight the deep connections between foliations and singularity theory, and the reasons for having these volumes on foliations as a part of this Handbook, with Felipe Cano, an expert in holomorphic foliations, as a fourth editor for these two volumes.

A foliation means, naively, a partition of a manifold into connected subsets called the leaves, which are immersed manifolds, and one has a local product structure, except that there may be some special points: its singularities.

The theory of holomorphic foliations has its origins in the study of differential equations on the complex plane by C. F. Gauss, A. L. Cauchy, B. Riemann, K. Weierstrass, J. C. Bouquet, C. A. Briot, L. Fuchs, J. Liouville, G. Darboux, P. Painlevé, H. Dulac, I. G. Petrovsky, C. L. Siegel, L. S. Pontryagin and others. At the end of the nineteenth century, Liouville observed that it was not possible to find explicit solutions to most differential equations. A few years later, H. Poincaré stressed the importance of analyzing the topological, geometrical and analytical properties of the solutions of differential equations, even without giving their explicit expressions. This was a landmark for the birth of dynamical systems, for the qualitative theory of differential equations, and eventually for foliation theory.

The concept of "foliation" was formalized in the 1940s in a series of papers by G. Reeb and Ch. Ehresmann. This was inspired by the theory of differential equations, where the phase manifold gets decomposed into one-dimensional real or complex lines, as the case may be. This gives a local partition of the manifold, where at each regular point of the differential equation one has a flow box, or a product decomposition. The new idea was passing from local to global, and having higher dimensional "leaves," as one does, for instance, when considering the fibers of a submersion. Notice that in the complex case, the decomposition obtained by considering the integral lines of a vector field is by one-dimensional complex curves, so these have real dimension two.

After the early first steps, intensive development came into the theory, both in the real and complex cases, and important research schools were developed worldwide.

On the one hand, the deep results for holomorphic foliations by A. Haefliger, B. Malgrange, J. Martinet, J. P. Ramis, R. Moussu, J. F. Mattei, J. Écalle, É. Ghys, D. Cerveau and many others have made of France a *Mecca* for Foliation Theory. Of course this had significant influence in other countries, and particularly in Spain, where J. M. Aroca, F. Cano and others now have a research school with excellent mathematicians all over the country.

Simultaneously, in the former Soviet Union, the geometric theory of complex differential equations was developed in the early 1950s by I. G. Petrovsky and E. M. Landis. This fascinating theory soon attracted the attention of the leading young mathematicians of the 1960s: D. V. Anosov, V. I. Arnold, S. P. Novikov, R. E. Vinogradov, and soon afterwards, Yu. S. Ilyashenko, S. Yu. Yakovenko, S. M. Voronin, A. N. Varchenko, A. G. Khovanskii, A. A. Bolibruch, A. D. Bryuno, D. I. Novikov, G. S. Petrov and many others, that have made of the Soviet Union, and now Russia, a main pole of development for complex foliations.

On the other hand, in the early 1970s, C. Camacho finished his Ph.D. in Berkeley, working with S. Smale on a thesis about smooth group actions, and then moved to the IMPA in Brazil, where, together with P. Sad and A. Lins Neto, and also with M. Soares at Belo Horizonte, he built up a strong research school on holomorphic foliations. Some 10 years later, A. Verjovsky and X. Gómez-Mont started building up a research school on complex foliations in Mexico. The early results of Gómez-Mont were essential to lay down the foundations of deformation theory for complex foliations, and the seminal work by Verjovsky on the uniformization of the leaves of holomorphic one-dimensional foliations has opened an important line of research. The Mexican school has students and collaborators in Russia, France, Spain and Brazil, thus profiting from all those schools. The interaction in Foliation Theory between Mexico, France and Brazil is apparent, for instance, in the area of research known as LVM manifolds, whose genesis is in the classical paper by Camacho, Kuiper and Palis on linear $\mathbb{C}$-actions (see for instance Lopez de Medrano's chapter in Volume II of this Handbook).

We are happy to have in these volumes important contributions from all of these schools, and others.

Let us say a few words about volumes V and VI. These have nine chapters each, and these cover a large scope of the theory of analytic foliations. Volume V starts with a foreword by Professor Yulij Ilyashenko, while Volume VI ends with an epilogue by Professor Jean-Pierre Ramis, two of the main world leaders in the theory of complex foliations.

The foreword by Ilyashenko in Volume V explains some of the most important lines of research in the theory of holomorphic foliations, and it states some major open problems in this area.

The epilogue by Jean-Pierre Ramis, in this volume, is about Stokes phenomena. This is an important legacy for the next generations, as it abounds in deep thoughts and reflections, and it has plenty of historical roots and a vast bibliography. This remarkable work by Ramis describes the Stokes phenomenon, its main significance, and its emergence in various landscapes. It explains how to use Stokes phenomena to enrich the classical dynamics of complex dynamical systems, defining some wild dynamics. It describes too how, adding Stokes multipliers to the classical monodromy, it is possible to classify a lot of complex dynamical systems and to get generalizations of the Riemann-Hilbert correspondence, with plenty of applications. It ends with a short description of many other incarnations of Stokes phenomenon, such as singular perturbations, resurgence, difference and q-difference equations, theoretical physics and others.

Chapter 1 studies the singularities of complete holomorphic vector fields on complex manifolds. It turns out that in the complex case, completeness is not only a property to be understood at infinity, but the singularities of the vector field play an important part. Some 25 years ago, Julio Rebelo noticed and started exploiting the fact that multivaluedness constitutes a local obstruction for a holomorphic vector field to be complete. He formalized this local obstruction to completeness through the notion of semicompleteness, which has been investigated ever since by various authors, including Adolfo Guillot. This chapter reviews this notion, from the foundational definitions to some local and global results, with a special emphasis on manifolds and analytic spaces of dimension two.

Chapter 2 studies the global dynamics of singular holomorphic foliations on complex manifolds of dimension 3. The foliations considered are mostly 1-dimensional, but codimension 1 foliations are also envisaged. The authors notice that the understanding of a germ of 1-dimensional foliation on $\mathbb{C}^n$, $n \geq 3$ passes through the description of a foliation defined on $\mathbb{CP}^{n-1}$ which, as a global object, may exhibit a wild dynamical behavior. In this chapter the authors investigate how the global dynamics of the latter foliation exerts influence on several problems that apparently have a purely local nature. In the course of the discussion, a few recent results and open problems in the area are reviewed.

In Chap. 3, the author studies the irreducible components of algebraic foliations of codimension one on complex projective spaces. A key point explained in this chapter is that every codimension one foliation on $\mathbb{CP}^n$, $n \geq 2$ can be defined on $\mathbb{C}^{n+1}$ by a polynomial 1-form ω with homogeneous components and such that $\omega(R_{n+1}) = 0$, where $R_{n+1} = \sum_{j=0}^{n} z_j \frac{\partial}{\partial z_j}$ is the radial vector field on $\mathbb{C}^{n+1}$. The degree of the foliation is the degree of the corresponding polynomials minus 1. It is thus natural to parametrize the space of such foliations by these forms. This is the space that the author studies.

Chapter 4 is a survey on problems and results on singular holomorphic foliations and Pfaff systems with invariant analytic varieties on complex manifolds. The author focuses on results which are motivated by classical work by Darboux, Poincaré, Painlevé and others, on the problem of algebraic integration of singular polynomial differential equations. The chapter presents results on the Poincaré and Painlevé problem of bounding the degree and genus of an analytic variety invariant by holomorphic foliations and Pfaff systems.

Chapter 5 is concerned with the question of R. Thom about the existence of an invariant hypersurface for germs of holomorphic codimension one foliations. In this chapter, the authors give a panorama of the state of art of this question, where the reduction of singularities plays a central role. The chapter starts with an elementary and detailed study of the final points expected after reduction of singularities, focusing the attention on concepts and properties concerning invariant hypersufaces.

A key concept and tool for studying codimension 1 holomorphic foliations is the transversal pseudo group. The structure of pseudo groups of conformal diffeomorphisms of open subsets of $\mathbb{C}$ with a fixed point at the origin has been intensively studied from various points of view for the last few decades. Chapter 6 recalls the fundamental results on this subject, beginning from the basics, and introduces some recent developments. It provides also novel proofs of several fundamental results and answers partly to a question of Shcherbakov, proving that the linear multipliers of fixed points of a nonsolvable pseudogroup are dense in $\mathbb{C}^*$. In the final section, the author discusses another application to a problem of implicit functions and topological moduli of polynomial maps from $\mathbb{C}^3$ to $\mathbb{C}^2$.

In Chap. 7, the author gives a constructive description of the Zariski-closure of a subgroup of the group $\widehat{\mathrm{Diff}}(\mathbb{C}^n)$ of formal diffeomorphisms. This is the main concept in the algebraic theory of germs of biholomorphisms and formal diffeomorphisms.

For instance, it was used by Etienne Ghys to study nilpotent groups of real analytic diffeomorphisms of the sphere, proving that such groups are always metabelian. The chapter also explains why the Zariski-closure of a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n)$ is meaningful from a geometric viewpoint.

Chapter 8 is an introductory text aimed to graduate students and researchers which already are familiar with the definition of a holomorphic connection. The text both explains the main ideas (while sometimes referring to other texts for more technical details) and remains short enough to allow reader to keep the whole picture in mind. The goal of this chapter is to introduce the Riemann-Hilbert correspondence in the rank two case and irregular setting over the complex numbers $\mathbb{C}$. This establishes a one-to-one correspondence between isomorphism classes of rank two vector bundles with meromorphic linear connections on curves, and some representation of fundamental groups or groupoids on the corresponding Riemann surface with punctures, up to some natural equivalence. The aim of the chapter is to provide a self-contained approach with proofs accessible for a graduate student, in particular including the ramified case.

Finally, Chap. 9 is concerned with the extension of the Riemann-Hilbert correspondence in the irregular case and focuses on Painlevé equations. There is a huge literature on this topic. The leaves of the Painlevé foliations appear as the isomonodromic deformations of a rank 2 linear connection on a moduli space of connections. Therefore, they are the fibers of the Riemann-Hilbert correspondence that sends each connection on its monodromy data, and this correspondence induces a conjugation between the dynamics of the foliation and a dynamic on a space of representations of some fundamental groupoid (a character variety). This one can be identified with a family of cubic surfaces through trace coordinates. The authors describe the dynamics on the character variety related to the Painlevé fifth equation. The chapter presents many consequences of this original approach to study Painlevé equations.

Let us mention briefly the content of Volume V.

Chapter 1 by Bruno Scardua is an introduction to singular holomorphic foliations, with several equivalent definitions of holomorphic foliations and examples. It also presents several of the basic concepts in the theory and gives elementary proofs of several basic results, as for instance the theorem of Mattei-Moussu about the existence of holomorphic first integrals for germs of holomorphic foliations, and the linearization theorem of Camacho-Lins Neto-Sad for foliations in the complex projective plane.

Chapters 2 to 5 are concerned with 1-dimensional holomorphic foliations in dimension 2. Chapter 2 by Ilyashenko is a survey on geometric properties, open problems and conjectures about complex foliations in $\mathbb{C}^2$ and $\mathbb{CP}^2$. Chapter 3 by Laura Ortiz, Ernesto Rosales and Sergei Voronin is a self-contained account that surveys the local understanding of the solutions of differential equations in $(\mathbb{C}^2, 0)$. In Chap. 4, David Marín, Jean-François Mattei and Eliane Salem give a survey on the topology of singularities of holomorphic foliation germs in $(\mathbb{C}^2; 0)$, describing the topology of the leaves, the structure of the leaves space and criteria of conjugacy for foliation germs. Chapter 5 by Nuria Corral uses the classical concept of polar curve in singularity theory, as studied by Teissier, Lê and others, to study foliations and the relations with known results for plane curves.

Chapter 6 by Dmitry Novikov and Sergei Voronin is about generalizations of the classical Rolle theorem, claiming that between any two roots of a real valued differentiable function on a segment must lie a root of its derivative. The authors discuss important generalizations of this theorem for vector-valued and complex analytic functions and for germs of holomorphic maps.

Chapter 7 by Fernando Sanz is a review concerning the study of the local dynamics of a gradient vector field of a real analytic function, starting from the famous Thom's Gradient Conjecture and ending with a discussion in the context of o-minimal geometry.

Chapter 8 by Juan Manuel Aroca is an introduction to the use of techniques originally developed by Newton to study local solutions of algebraic and ordinary differential equations.

In Chap. 9, Jorge Vitório Pereira reviews properties of closed meromorphic 1-forms and of the foliations that these define. It explains classical results from foliation theory, like index theorems, existence of separatrices and resolution of singularities, from the viewpoint of 1-forms and flat connections.

There are many other important contributions to the theory of analytic foliations that could (or even, should) have been included in these volumes. We are grateful to all the very many mathematicians worldwide that have contributed to the theory.

Volumes V and VI of the Handbook complement the previous four volumes of this collection, which is addressed to graduate students and newcomers to singularity theory, as well as to specialists who can use these as guidebooks.

The first four volumes of this collection gathered foundational aspects of the theory, as well as some other important aspects. Some topics are studied in various chapters, and in some cases, also in more than one volume. Each volume contains an index that helps the reader to find what he or she is looking for. The topics studied so far include:

- The combinatorics and topology of plane curves and surface singularities.
- The analytic classification of plane curves.
- Introductions to four of the classical methods for studying the topology and geometry of singular spaces, namely: resolution of singularities, deformation theory, stratifications and slicing the spaces *à la* Lefschetz.
- Milnor's fibration theorem for real and complex singularities, the monodromy, vanishing cycles and Lê cycles.
- Morse theory for stratified spaces and constructible sheaves.
- Limits of tangents to complex varieties, a subject that originates in Whitney's work.
- Zariski's equisingularity and intersection homology.
- Singularities of mappings. Thom-Mather theory.
- The interplay between analytic and topological invariants of complex surface singularities and their relation with modern 3-manifold invariants.
- Indices of vector fields and 1-forms on singular varieties.

– Chern classes and Segre classes for singular varieties.
– Mixed Hodge structures.
– Determinantal singularities.
– Arc spaces.
– Lipschitz geometry in singularity theory;

and many other important subjects.

This collection aims to provide accessible accounts of the state of the art in various aspects of singularity theory, its frontiers and its interactions with other areas of research. This will continue with a Volume VII that will discuss other important aspects of singularity theory and its interactions.

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Chapter 1
On the Singularities of Complete Holomorphic Vector Fields in Dimension Two

Adolfo Guillot

Abstract For a germ of singular holomorphic vector field on a complex manifold to be the local model of a complete one, it is necessary for its solutions to be univalent (and not multivalued). Rebelo formalized this local obstruction to completeness through the notion of *semicompleteness*, which he started studying some twenty-five years ago; it has been investigated ever since by various authors. We here review this notion, from the foundational definitions to some local and global results, with a special emphasis on manifolds and analytic spaces of dimension two.

1.1 Introduction

For holomorphic vector fields on complex manifolds, completeness is not only a property to be understood at infinity, but, as it turns out, the singularities of the vector field play an important part in it. Some twenty-five years ago, Rebelo noticed and started exploiting the fact that multivaluedness constitutes a local obstruction for a holomorphic vector field to be complete. A systematic local study of complete holomorphic vector fields on complex manifolds ensued. For surfaces, a very comprehensive one was made possible by the thorough local understanding we have of holomorphic foliations in dimension two. While the existence of local obstructions for completeness was probably known, or, at least, not completely unexpected, the degree to which multivaluedness is a strong obstruction amid degenerate vector fields was certainly surprising.

The general problem was, of course, not new, as the quest for understanding the algebraic differential equations in the complex domain with single-valued solutions was an important one all throughout the second half of the nineteenth century, with results, to name some, of Briot and Bouquet, Poincaré, Picard and Painlevé (an account up to the turn of the twentieth century is given in Painlevé's *Leçons de*

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Stockholm [60]). While the local nature of Rebelo's point of view is, in principle, not very suitable to directly study transcendence issues (an important point behind Painlevé's motivation), it was a key element permitting its use in nonalgebraic contexts. For instance, an early application of the results obtained by Ghys and Rebelo was the completion of the classification of compact complex surfaces admitting vector fields due to Dloussky et al. [21, 22], which required the local understanding of complete vector fields on some nonalgebraic surfaces.

Rebelo's and Ghys and Rebelo's original efforts were followed by those of Reis and the author. Some of the ideas were successfully developed in other contexts: for other Lie groups, for meromorphic vector fields, in higher dimensions. Over the years we have gained a deeper understanding of some key notions, simplified proofs, realized not only that some ideas were already there (notably, Palais's theory on the integration of infinitesimal actions) but that some had been for a very long time (e.g. work by Briot and Bouquet, Kowalevski, Painlevé, ...).

The aim of this survey is to present, from a personal point of view and with today's hindsight, some of the results concerning the singular points of complete holomorphic vector field in dimension two, both for manifolds and analytic spaces. Many interesting topics have been, regrettably, left outside.

1.2 Semicompleteness

We assume that the reader is acquainted with some basic facts around holomorphic foliations on complex surfaces, like the material covered in the first few chapters of [6] or [11]. All manifolds will be complex; all maps and vector fields, holomorphic, and so on.

We begin by laying the ground for our core definition, that of *semicomplete* or *univalent* vector field (Definition 1.2.3). While the discussion at times is not too far from that in Palais's monograph [61], some objects particular to holomorphic vector fields come into play, and intertwine with the rest of the theory.

Let M be a complex manifold, X a holomorphic vector field on M. We may think of X as an ordinary differential equation and consider *solutions* of X, functions $\phi : U \to M$, $U \subset \mathbf{C}$, such that $\phi'(t) = X|_{\phi(t)}$. The Existence and Uniqueness Theorem guarantees that for every $p \in M$ there exists an open connected subset $U \subset \mathbf{C}$, $0 \in U$, and a solution $\phi : U \to M$ of X such that $\phi(0) = p$; furthermore, two such solutions define the same germ at 0. (We refer the reader to [43, 46] and [45] for basic facts around ordinary differential equations in the complex domain.)

If for every p in M there is a solution $\phi_p : \mathbf{C} \to M$ of X with $\phi_p(0) = p$ we say that X is *complete*. In such a case we may glue together the solutions into a map $\Phi : \mathbf{C} \times M \to M$ defined by $\Phi(t, p) = \phi_p(t)$, that satisfies the conditions

- $\Phi(0, p) = p$,
- $\Phi(t, \Phi(s, p)) = \Phi(t + s, p)$ for all $s, t \in \mathbf{C}$, and for all $p \in M$

(the second one is a consequence of the fact that the ordinary differential equation associated to X is an autonomous one). The map Φ is a holomorphic *flow* or *action of* **C**. Vector fields on compact manifolds are complete, but, in general, completeness is difficult to characterize.

Let $\mathrm{Sing}(X) = \{p \mid X(p) = 0\}$. On $M \setminus \mathrm{Sing}(X)$, when the images of two solutions intersect, their union defines an injectively immersed curve. The maximal curves thus constructed are the *leaves* or *orbits* of X (or of the *foliation* induced by X) on $M \setminus \mathrm{Sing}(X)$. Each one of these curves is tangent to X at each one of its points, and inherits a particular geometry.

A *translation structure* on a complex curve L is an atlas for its complex structure taking values in **C** whose changes of coordinates are given by translations, maps of the form $z \mapsto z + c$. On a curve, the following data are equivalent:

- a holomorphic vector field without zeros,
- a translation structure,
- a holomorphic one-form without zeros.

Indeed, let L be a curve. If X is a nowhere-zero holomorphic vector field on L, if two solutions $\phi_1 : U_1 \to L$ and $\phi_2 : U_2 \to L$ of X have overlapping images, by the uniqueness of solutions, they differ by a translation: there exists $c \in \mathbf{C}$ such that $\phi_1(t) = \phi_2(t+c)$ for every $t \in U_1 \cap (U_2 - c)$, and the local inverses of the solutions of X endow L with a translation structure; we thus go from the vector field to a translation structure. The primitives of a nowhere-vanishing one-form on L are the charts of a translation structure, thus producing a translation structure out of a one-form. To obtain forms and vector fields from a translation structure, notice that since, in **C**, translations preserve both the vector field $\partial/\partial z$ and the one-form $\mathrm{d}z$, these may be pulled back to a curve endowed with a translation structure via its charts, producing a well-defined nowhere-zero vector field and a well-defined nowhere-zero one-form on the curve. Finally, given a nowhere-zero vector field X on L, there is a unique nowhere-zero one-form ω such that $\omega(X) \equiv 1$, the *time form* of X; the inverse procedure produces a vector field starting from a form.

Thus, each orbit of X on $M \setminus \mathrm{Sing}(M)$ is endowed with a translation structure.

Given a translation structure on the curve L with universal covering $\pi : \widetilde{L} \to L$, there is a *developing* map $\mathcal{D} : \widetilde{L} \to \mathbf{C}$, giving a global chart of the projective structure induced on $\widetilde{L}$, and a *monodromy* (or *period*) homomorphism $\mathrm{mon} : \pi_1(L) \to \mathbf{C}$ such that, for every $\alpha \in \pi_1(L)$,

$$\mathcal{D}(\alpha \cdot p) = \mathcal{D}(p) + \mathrm{mon}(\alpha) \tag{1.1}$$

(see [72, Sect. 3.4]). If the translation structure is given by the nowhere-vanishing one-form ω, $\mathcal{D}$ is given by $x \mapsto \int^x \pi^*\omega$ ($\pi^*\omega$ is exact; $\mathcal{D}$ is one of its primitives), and the monodromy is induced by $\gamma \mapsto \int_\gamma \omega$.

A translation structure on a curve L is said to be *uniformizable* if L is isomorphic (as a curve with a translation structure) to the quotient of an open set $\Omega \subset \mathbf{C}$ under a group of translations acting properly discontinuously.

Proposition 1.2.1 *Let M be a complex manifold, X a holomorphic vector field on M, $p \in M \setminus \mathrm{Sing}(X)$, $L \subset M$ the orbit of X through p. The following are equivalent:*

1. *For every solution $\phi : U \to M$ ($U \subset \mathbf{C}$, $0 \in U$) of X with initial condition p and every pair of paths $\gamma_1, \gamma_2 : [0,1] \to \mathbf{C}$ with $\gamma_i(0) = p$, $\gamma_1(1) = \gamma_2(1)$, such that the germ of ϕ at 0 admits analytic continuations along both γ_1 and γ_2, the analytic continuations define the same germ at $\gamma_1(1)$.*
2. *There exists $\Omega \subseteq \mathbf{C}$, $0 \in \Omega$, and $\phi : \Omega \to M$ a solution to X with initial condition p, such that the map $\phi \times i : \Omega \to M \times \mathbf{C}$ given by $t \mapsto (\phi(t), t)$ is proper.*
3. *For every open path $\gamma : [0,1] \to L$, $\gamma(0) = p$, for the time form ω of X on L, $\int_\gamma \omega \neq 0$.*
4. *When considering L with its translation structure, for the covering map $\overline{\pi} : \overline{L} \to L$ associated to the kernel of the monodromy, the map $\overline{\mathcal{D}} : \overline{L} \to \mathbf{C}$ induced by $\mathcal{D}$ is one-to-one.*
5. *The translation structure on L induced by X is uniformizable.*

Proof **(1)** $\Rightarrow$ **(2)**. Let $\phi : U \to M$ a solution of X with initial condition p. Let $\Omega \subset \mathbf{C}$ be the set of all points t for which there exists a path $\gamma : [0,1] \to \mathbf{C}$, joining 0 to t, along which an analytic continuation of ϕ may be defined. By hypothesis, we have a well-defined map $\phi : \Omega \to M$, which is a solution to X (the analytic continuation of a solution is still a solution). It does not have any analytic continuation beyond Ω and is thus is maximal as a holomorphic function from a subset of $\mathbf{C}$ into M. Let $\{t_i\}$ be a sequence of points in Ω converging in $\mathbf{C}$ to a point t_∞ in $\partial\Omega$. Suppose that $q \in M$ is an accumulation point of $\{\phi(t_i)\}$. There exists a neighborhood V of q in M and $\epsilon > 0$ such that, for every $x \in V$, a solution of X with initial condition x is defined in the disk of radius ϵ around 0 in $\mathbf{C}$. But the solution with initial condition $\phi(t_i)$ cannot be defined in a disk of radius greater than $|t_i - t_\infty|$. Thus, the sequence $\{\phi(t_i)\}$ does not have accumulation points in $\mathbf{C}$. This proves that $\phi \times i$ is proper.

(2) $\Rightarrow$ **(3)**. Let $\gamma : [0,1] \to L$ be an open path, $\gamma(0) = p$, and let $\phi : \Omega \to M$ be a solution with initial condition p such that $\phi \times i$ is proper. Observe that ϕ takes values in L and that it is a local biholomorphism. We will prove that it has the path-lifting property, that it is a covering map. Let $s_0 = \sup\{s \in [0,1] \mid \gamma|_{[0,s]}$ can be lifted to $\Omega\}$, and suppose that $s_0 < 1$. Let $\overline{\gamma} : [0, s_0) \to \Omega$ be a lift of the restriction of γ to $[0, s_0)$. We affirm that $\overline{\gamma}$ is proper: otherwise, if $\{\delta_i\}$ is a sequence of positive reals converging to 0 such that $\overline{\gamma}(s_0 - \delta_i)$ converges to $t_0 \in \Omega$ then, since $\gamma(s_0 - \delta_i)$ converges to $\phi(t_0)$, $\overline{\gamma}$ may be extended to s_0 by considering the local biholomorphism between neighborhoods of t_0 and $\phi(t_0)$ given by ϕ, which contradicts our hypothesis. Now, since $\phi \times i$ and $\overline{\gamma}$ are both proper and since $\phi \circ \overline{\gamma}(s) = \gamma(s)$ for all $s < s_0$, the limit of $\phi \circ \overline{\gamma}(s)$ as s approaches s_0 exists, and thus $i \circ \overline{\gamma} : [0, s_0) \to \mathbf{C}$ is also proper. This implies that $\lim_{s \to s_0^-} \int_{\overline{\gamma}|_{[0,s]}} \mathrm{d}t = \infty$, which contradicts the fact that $\int_{\gamma|_{[0,s_0]}} \omega < \infty$ through the identity $\phi^*\omega = \mathrm{d}t$. We have thus proved that $s_0 = 1$, that γ can be lifted to a path $\overline{\gamma} : [0,1] \to \Omega$ (that ϕ is a covering map). Since $\gamma(0) \neq \gamma(1)$, $\overline{\gamma}(0) \neq \overline{\gamma}(1)$, and $\int_\gamma \omega = \int_{\overline{\gamma}} \mathrm{d}t = \overline{\gamma}(1) - \overline{\gamma}(0) \neq 0$.

(**3**) $\Rightarrow$ (**4**). Let $\overline{\pi} : \overline{L} \to L$ be the covering associated to the kernel of the monodromy; it is the smallest covering where the pull-back of the time form ω has no periods. From formula (1.1), there is a well-defined developing map $\overline{\mathcal{D}} : \overline{L} \to \mathbf{C}$, satisfying $\overline{\mathcal{D}} \equiv \mathcal{D} \circ \overline{\pi}$. Let p, q be such that $\overline{\mathcal{D}}(p) = \overline{\mathcal{D}}(q)$. Let $\overline{\gamma} : [0, 1] \to \overline{L}$ be a path joining p and q, and let $\gamma = \overline{\pi} \circ \overline{\gamma}$. Since $\overline{\mathcal{D}}(x) = \int^x \overline{\pi}^*\omega$, $\int_{\overline{\gamma}} \overline{\pi}^*\omega = 0$, and thus $\int_\gamma \omega = 0$. By (3), γ is closed. Since $\int_\gamma \omega = 0$ then, by the defining property of $\overline{L}$, the lift $\overline{\gamma}$ of γ is closed. Thus, $p = q$, and $\overline{\mathcal{D}}$ is one-to-one.

(**4**) $\Rightarrow$ (**5**). Let $\overline{\pi} : \overline{L} \to L$ be the Galois covering associated to the kernel of the monodromy and consider $\overline{\mathcal{D}} : \overline{L} \to \mathbf{C}$ the induced developing map, which is one-to-one by hypothesis. Let $\Omega \subset \mathbf{C}$ denote its image. We have an induced monodromy homomorphism $\overline{\mathrm{mon}} : \mathrm{Gal}(\overline{L}/L) \to \mathbf{C}$, which satisfies $\overline{\mathcal{D}}(\gamma \cdot x) = \overline{\mathcal{D}}(x) + \overline{\mathrm{mon}}(\gamma)$, and which, by the definition of $\overline{L}$, is also one-to-one. In this way, $\overline{\pi} \circ \overline{\mathcal{D}}^{-1} : \Omega \to L$ is a covering map whose group of deck transformations is given by translations, a covering map which uniformizes the translation structure on L.

(**5**) $\Rightarrow$(**1**). Let $U \subseteq \mathbf{C}$, $0 \in U$, and $\phi : U \to M$ a solution of X with initial condition p. By hypothesis, we have an open subset $\Omega \subseteq \mathbf{C}$, $0 \in \Omega$, and a covering map $\overline{\mathcal{D}} : \Omega \to L$ with $\overline{\mathcal{D}}(0) = p$. Up to a translation, we may suppose that the germs of ϕ and $\mathcal{D}$ around 0 coincide. Let us consider the analytic continuation of the germ of ϕ around 0. Let $\gamma : [0, 1] \to \mathbf{C}$, $\gamma(0) = 0$. If the image of γ is contained in Ω, the analytic continuation of the germ of ϕ at 0 along γ at $\gamma(1)$ is given by $\overline{\mathcal{D}}$. If $\gamma(t) \in \Omega$ for $t < t_0$ and $\gamma(t_0) \notin \Omega$, $\overline{\mathcal{D}}$ does not have an analytic continuation at $\gamma(1)$, for $\overline{\mathcal{D}}$, being a covering map, is proper. Thus, if the analytic continuation of ϕ may be defined along the paths $\gamma_1, \gamma_2 : [0, 1] \to \mathbf{C}$, $\gamma_i(0) = 0$, $\gamma_1(1) = \gamma_2(1)$, these analytic continuations coincide at $\gamma_1(1)$, for they are given by $\mathcal{D}$. □

For a holomorphic vector field X on the complex manifold M, the *uniformizability locus of* X is the set of points of M where either X vanishes or where one (hence all) of the above conditions hold. We will see in Theorem 1.3.3 that this locus is closed.

Proposition 1.2.2 *Let X be a holomorphic vector field on the manifold M. The following are equivalent.*

1. *The uniformizability locus of X is all of M.*
2. *There exists an open subset $\Omega \subset \mathbf{C} \times M$, $\{0\} \times M \subset \Omega$, and $\Phi : \Omega \to M$, such that*

   a. *$\Phi(\cdot, p)$ is a solution of X,*
   b. *$\Phi(0, p) = p$,*
   c. *for every p in M, for $\Omega_p = \{t \in \mathbf{C} \mid (t, p) \in \Omega\}$, Ω_p is connected, and the map $j : \Omega_p \to M \times \mathbf{C}$, $t \mapsto (\Phi(t, p), p)$ is proper.*

3. *There exists an open subset $\Omega \subset \mathbf{C} \times M$, $\{0\} \times M \subset \Omega$, and $\Phi : \Omega \to M$, such that*

   a. *$\Phi(\cdot, p)$ is a solution of X.*
   b. *$\Phi(0, p) = p$.*

c. If $(t,p)\in\Omega$ *and* $(s,\Phi(t,p))\in\Omega$ *then* $(t+s,p)\in\Omega$ *and* $\Phi(s,\Phi(t,p))=\Phi(t+s,p)$.

4. *In the nonsingular foliation by curves on* $\mathbf{C}\times M$ *induced by* $-\partial/\partial t\oplus X$*, every leaf that intersects* $\{0\}\times M$ *does so at only one point.*

Proof **(1)** $\Rightarrow$ **(2)**. For $p\in M$, let $\phi_p:\Omega_p\to\mathbf{C}$ be the solution with initial condition p such that $\phi_p\times i$ is proper (Ω_p is connected). Let $\Omega=\cup_{p\in M}(\Omega_p\times\{p\})$. It is an open subset of $\mathbf{C}\times M$ by the continuity of solutions of differential equations with respect to initial conditions. Let $\Phi:\Omega\to M$ be given by $\Phi(t,p)=\phi_p(t)$. By definition, we have (2a) and (2b). Condition (2c) is simply a restatement of the properness condition.

(2) $\Rightarrow$ **(3)**. Let $\Omega\subset\mathbf{C}\times M$ and $\Phi:\Omega\to M$ as in (2). They satisfy conditions (3a) and (3b). Let $t\in\Omega_p$, $s\in\Omega_{\Phi(t,p)}$. Let $\phi_1:\Omega_p\to M$ be given by $\phi_1(u)=\Phi(u,p)$, $\phi_2:(\Omega_{\Phi(t,p)}+t)\to M$ by $\phi_2(u)=\Phi(u-t,\Phi(t,p))$. Let W be the connected component of $\Omega_p\cap(\Omega_{\Phi(t,p)}+t)$ containing t. Since both ϕ_1 and ϕ_2 are solutions of X and $\phi_1(t)=\phi_2(t)$, ϕ_1 and ϕ_2 agree in restriction to W. Let $\{y_i\}$ be a sequence in W converging to a point $y_\infty\in\mathbf{C}$. If $y_\infty\in\partial\Omega_p\cap(\Omega_{\Phi(t,p)}+t)$ then, on the one hand, $\{\phi_1(y_i)\}$ escapes from every compact subset of M, and, on the other, since ϕ_2 is holomorphic in a neighborhood of y_∞, $\{\phi_2(y_i)\}$ converges. Thus, $(\Omega_{\Phi(t,p)}+t)$, which is connected by hypothesis, is contained in Ω_p. Since $s+t\in(\Omega_{\Phi(t,p)}+t)$, $s+t\in\Omega_p$, and $\phi_1(s+t)=\phi_2(s+t)$, this is, $\Phi(s+t,p)=\Phi(s,\Phi(t,p))$.

(3) $\Rightarrow$ **(4)**. On $\mathbf{C}\times M$, consider the vector field such that, for every $t\in\mathbf{C}$, its restriction to $\{t\}\times M$ is X, and let us still denote it by X. Likewise, let $\partial/\partial t$ be the vector field on $\mathbf{C}\times M$ induced by the vector field $\partial/\partial t$ on the factor $\mathbf{C}$. Consider the vector field $-\partial/\partial t\oplus X$ on $\mathbf{C}\times M$ and let $\mathcal{G}$ be the nonsingular foliation by curves that it induces. Consider the action ρ of $\mathbf{C}$ on $\mathbf{C}\times M$ given by translations in the first factor, $\rho(s,(t,p))=(s+t,p)$. It preserves the vector field $-\partial/\partial t\oplus X$ and, a fortiori, the foliation $\mathcal{G}$.

For every $p\in M$, there is a disk $\Delta\subset\mathbf{C}$, centered at 0, where $s\mapsto(-s,\Phi(s,p))$ is defined; it parametrizes a disk in the leaf of $\mathcal{G}$ through $(0,p)$. Using the action ρ, the map $s\mapsto(t-s,\Phi(s,p))$ is defined in the same Δ and parametrizes a disk in the leaf of $\mathcal{G}$ through (t,p).

Let $\Omega'\subset\mathbf{C}\times M$ be the saturated of $\{0\}\times M$ by $\mathcal{G}$. We will begin by proving that $\Omega'\subseteq\Omega$. Let L be a leaf of $\mathcal{G}$ intersecting Ω. Let us prove that $L\subset\Omega$. Since Ω is open, it intersects L in an open subset. Let $(t,p)\in L$ be an accumulation point of points in $L\cap\Omega$ (considering L with its manifold topology). By the previous arguments, there exists $s_0\in\mathbf{C}$ such that $(t-s_0,\Phi(s_0,p))\in L\cap\Omega$. Since (s_0,p) and $(t-s_0,\Phi(s_0,p))$ are both in Ω, hypothesis (3c) implies that $(t,p)\in\Omega$: Ω intersects L in a closed subset, and thus $L\subset\Omega$. Since $\{0\}\times M\subset\Omega$, $\Omega'\subset\Omega$, as stated.

Let $L\subset\Omega'$ be the leaf of $\mathcal{G}$ passing through $(0,q)$. If $(t,p)\in L$ and s is sufficiently small, $s\mapsto(t-s,\Phi(s,p))$ parametrizes a disk in L belonging to Ω and since, again by (3c), $\Phi(t-s,\Phi(s,p))=\Phi(t,p)$, Φ is constant along L. Since $\Phi(0,q)=q$, Φ takes the value q at all points of L. In consequence, L intersects $\{0\}\times M$ at only one point.

(4) $\Rightarrow$ **(1)**. Let us prove that (4) implies condition (3) from Proposition 1.2.1. Let L be a leaf of $\mathcal{F}$ (the foliation induced by X) in $M \setminus \mathrm{Sing}(X)$, ω its time form, $\gamma : [0, 1] \to L$ a curve such that $\int_\gamma \omega = 0$. We will prove that γ is closed. Let $p = \gamma(0)$. We keep the notations from the previous step. Consider the projections $t : \mathbf{C} \times M \to \mathbf{C}$ and $\pi : \mathbf{C}\times \to M$. Let $\widehat{L}$ be the leaf of $\mathcal{G}$ passing through $(0, p)$. The map $\pi|_{\widehat{L}} : \widehat{L} \to L$ is a covering one, and ω lifts via $\pi|_{\widehat{L}}$ to $-\mathrm{d}t|_{\widehat{L}}$. Thus, for the lift $\widehat{\gamma} : [0, 1] \to \widehat{L}$ of γ based at $(0, p)$, $\widehat{\gamma}(1) = \left(-\int_\gamma \omega, \gamma(1)\right) = (0, \gamma(1))$. Since, by hypothesis, $\widehat{L}$ may intersect $\{0\} \times M$ at no point other than $(0, p)$, $\gamma(1) = p$, and γ is closed. □

We arrive to our central definition.

Definition 1.2.3 Let M be a complex manifold, X be a holomorphic vector field on M. We say that X is *univalent* or *semicomplete* if any one (hence all) of the conditions of Proposition 1.2.2 is satisfied.

Notice that *complete vector fields are semicomplete*. Indeed, complete vector fields satisfy condition (3) in Proposition 1.2.2 with $\Omega = \mathbf{C} \times M$. Notice also that *the class of semicomplete vector fields is stable under restrictions*: if X is a semicomplete vector field on M (e.g., a complete vector field) and $U \subset M$ is an open subset, the restriction of X to U is still semicomplete. (This follows directly from condition (4) in Proposition 1.2.2.)

A consequence of the last condition is one of Rebelo's key observations: *it makes sense to speak about germs of semicomplete of vector fields*. If X is a vector field on M and $p \in \mathrm{Sing}(M)$, we say that *the germ of X at p is semicomplete* if there is a neighborhood of p in restriction to which X is semicomplete. In stark contrast with the real case, there exist germs of holomorphic vector fields that are not semicomplete! This opens the door to the local study of complete vector fields.

In the same way as an action can be associated to a complete vector field, a unique global object can be attached to a semicomplete vector field, from which it can be furthermore recovered.

Definition 1.2.4 Let M be a manifold. A *maximum* **C**-*transformation group* or *semiglobal flow* on M is a pair (Ω, Φ), with $\Omega \subset \mathbf{C} \times M$ an open subset containing $\{0\} \times M$, with $\Omega \cap (\mathbf{C} \times \{p\})$ connected for each p, and $\Phi : \Omega \to M$, a map such that

1. $\Phi(0, p) = p$;
2. if $(t, p) \in \Omega$ and $(s, \Phi(t, p)) \in \Omega$ then $(t + s, p) \in \Omega$ and $\Phi(s, \Phi(t, p)) = \Phi(t + s, p)$.

The second condition may be replaced by the tandem:

2a. if (t, p), $(s, \Phi(t, p))$ and $(t + s, p)$ are all three in Ω, $\Phi(s, \Phi(t, p)) = \Phi(t + s, p)$;

2b. for every p in M, for $\Omega_p = \{t \in \mathbf{C} \mid (t, p) \in \Omega\}$, the map $j : \Omega_p \to M \times \mathbf{C}$, $t \mapsto (\Phi(t, p), p)$ is proper.

On its turn, the last condition may be reformulated as:

2b'. for every p in M, for every sequence $\{t_i\}$ such that $\{(p, t_i)\}$ belongs to Ω and converges to a point in $\partial\Omega \subset \mathbf{C} \times M$, the sequence $\{\Phi(t_i, p)\}$ escapes from every compact subset of M.

A standard flow (action of **C**) is naturally such an object. A semiglobal flow can be considered as a Lie groupoid $\Omega \rightrightarrows M$ over M having Φ for target map, and for source one the restriction to Ω of the projection $\mathbf{C} \times M \to M$. This generalizes the standard Lie groupoid associated to an action (see [54, Ex. 1.1.9]).

The definitions of *maximum* **C**-*transformation group* and *univalence* are due to Palais, in the more general setting of infinitesimal Lie group actions on manifolds [61, Ch. III, Defs. VI and VII]); they were rediscovered by Rebelo in the context of holomorphic vector fields under the names of *semiglobal flow* and *semicompleteness* [62, Déf. 2.3]). The connection with condition (3) of Proposition 1.2.1 appears in [62, Prop. 2.7] and [64, Prop. 2.1]. The link with conditions (5) and (4) of the same proposition appears in [34]. Some of these notions also make sense in the topological setting (some even in the set-theoretic one), and for this we refer the to the work of Abadie [1], who studies the equivalent notion of *partial action*.

All the notions in this section and in the following ones can be naturally extended to analytic spaces.

1.3 Semicompleteness for Families of Curves

Let us now present some general results, valid in all dimensions, related to the property of semicompleteness within families of curves. A common underlying principle will be behind all the proofs.

Recall that if L is a leaf of a foliation $\mathcal{F}$ and $p \in L$, if T is a transverse to $\mathcal{F}$ at p, by the process of lifting paths to nearby leaves we obtain a *holonomy* representation $\text{hol} : \pi_1(L, p) \to \text{Diff}(T, p)$ (see, for instance, [45, Ch. I, Sect. 2]). In the present setting we have natural choices for the submersions involved in its definition: given a transversal T through a point p on a leaf L, for a sufficiently small neighborhood $U \subset M$ of p, we may consider the unique submersion $\pi : U \to L$ that has T as one of its fibers and that is a translation in restriction to each leaf (with respect to each leaf's translation structure), and which, in particular, is the identity in restriction to L. Through such a construction, paths in one leaf may be lifted *isometrically* to neighboring ones.

1.3.1 Holonomy

The holonomies of semicomplete vector fields are very special.

Lemma 1.3.1 (Fundamental Lemma) *Let M be a complex manifold, X a univalent vector field on M. Let $p \in M$ be such that $X(p) \neq 0$, let L be the orbit of X through p, considered with the translation structure induced by X. There is a group homomorphism $\rho : \mathrm{mon}(\pi_1(L)) \to \mathrm{hol}(\pi_1(L))$ such that $\mathrm{hol}(\gamma) = \rho \circ \mathrm{mon}(\gamma)$ for all $\gamma \in \pi_1(L)$, this is, the holonomy representation factors through the monodromy one. In particular, the holonomy of L is abelian.*

Proof For γ in $\pi_1(L)$, we must define $\rho(\mathrm{mon}(\gamma))$ as $\mathrm{hol}(\gamma)$, and for this to be well defined it is necessary and sufficient that the holonomy along a path of trivial monodromy is trivial. Let L be a leaf, ω its time form, $p \in L$. Let $\gamma : ([0, 1], 0) \to (L, p)$ be a smooth closed path representing a class in $\pi_1(L)$ with trivial monodromy, a path such that $\int_\gamma \omega = 0$. Let T be a transversal to L at p. For $q \in T$ sufficiently close to p, let L_q the orbit of X through q and ω_q the corresponding time form. Let $\gamma_q : ([0, 1], 0) \to (L_q, q)$ be such that $\omega_q(\gamma_q'(t)) = \omega(\gamma'(t))$ for every $t \in [0, 1]$ (the isometric lift of γ through q). By construction, $\int_{\gamma_q} \omega_q = 0$, and, since X is univalent, γ_q is closed, and intersects T at the point q. The holonomy of γ is trivial. □

The following is one of the early incarnations of the Fundamental Lemma.

Corollary 1.3.2 ([30, Lemma 3.1]) *Let M be a manifold, X a univalent vector field on M inducing the foliation $\mathcal{F}$. Let L be an orbit of X with time form ω and $\gamma : [0, 1] \to L$ a closed path such that $\int_\gamma \omega = 0$. Then the holonomy of $\mathcal{F}$ along γ is trivial.*

1.3.2 Univalence as a Closed Condition

We now present two closely related results that present univalence as a closed condition. This was well known in Painlevé's time, and was fundamental in his research on differential equations with uniform solutions [59, §6] (see also [46, §14.12]).

Theorem 1.3.3 ([41, Cor. 12]) *Let M be a complex manifold, X a vector field on M. The uniformizability locus of X is a closed subset of M (saturated by X). In particular, if the restriction of the vector field to an open and dense subset of M is semicomplete, so is the vector field.*

Proof Let p be a point that is not in the uniformizability locus of X, L_p the orbit of X through p, ω_p the associated time form. Let $\gamma_p : [0, 1] \to L_p$ be an open smooth path such that $\int_{\gamma_p} \omega_p = 0$. For $q \in M$ close to p, let L_q be the orbit of X through q and ω_q the corresponding time form. There exists a smooth path $\gamma_q : [0, 1] \to L_q$, $\gamma_q(0) = q$ such that $\omega_q(\gamma_q'(t)) = \omega_p(\gamma_p'(t))$ for every $t \in [0, 1]$ (an isometric lift). Notice that $\int_{\gamma_q} \omega_q = 0$ but that, since γ_q is close to γ_p, it is an open path, and L_q does not belong to the uniformizability locus. □

As a prototypical application of this principle, we have:

Corollary 1.3.4 ([30, Cor. 2.6]) *Let X be a holomorphic vector field defined on $(\mathbf{C}^n, 0)$. Let $X = X_k + X_{k+1} + \cdots$ be the Taylor series of X at 0, where X_i is a homogeneous vector field of degree i and X_k is not identically zero. If the germ of X at 0 is semicomplete, X_k is semicomplete as a vector field on $\mathbf{C}^n$.*

Proof Let $U \subset \mathbf{C}^n, 0 \in U$ be an open subset where X is semicomplete. For $\alpha \in \mathbf{C}^*$, let $h : \mathbf{C} \to \mathbf{C}$ be the homothety $h_\alpha(z) = \alpha^{-1}z$. The vector field

$$Z_\alpha = \frac{1}{\alpha^{k-1}} h_* X = X_k + \alpha X_{k+1} + \alpha^2 X_{k+2} + \cdots ,$$

defined in $\alpha^{-1}U$, is semicomplete. In the subset $A = (\cup_\alpha(\{\alpha\} \times \alpha^{-1}U)) \cup (\{0\} \times \mathbf{C}^n)$ of $\mathbf{C} \times \mathbf{C}^n$, we have the holomorphic vector field Z given by Z_α in $\{\alpha\} \times \alpha^{-1}U$ and by X_k in $\{0\} \times \mathbf{C}^n$. Since it is semicomplete in $A \cap (\mathbf{C}^* \times \mathbf{C}^n)$, it is semicomplete in its closure A, and is thus semicomplete in restriction to $\{0\} \times \mathbf{C}^n$. □

Theorem 1.3.5 (Ghys-Rebelo [30, Sect. 2.2]) *Let M be a complex manifold. In the space of holomorphic vector fields on M, the univalent ones form a closed subset with respect to the topology of uniform convergence on compact sets.*

Proof The comments at the beginning of this section remain valid for laminated structures other than foliations, and the proof we present will make use of this. Let $\{X_i\}$ a sequence of semicomplete holomorphic vector fields on M converging uniformly on compact subsets to the vector field X_∞ (this gives a continuous vector field in the laminated space $M \times (\mathbf{N} \cup \{\infty\})$—where the topology in $\mathbf{N} \cup \{\infty\}$ is that of $\{0\} \cup \{1/n\}_{n \in \mathbf{N}}$ in $\mathbf{R}$—that equals X_i in restriction to $M \times \{i\}$). Suppose that X_∞ is not semicomplete, that there exists a smooth open path $\gamma_\infty : [0, 1] \to M$ tangent to an orbit L_∞ of M, with time form ω_∞, such that $\int_{\gamma_\infty} \omega_\infty = 0$. Let $p = \gamma_\infty(0)$, L_i the leaf of X_i through p, ω_i the time-form of X_i on L_i. If i is big enough, there exists a smooth path $\gamma_i : [0, 1] \to L_i$, $\gamma_i(0) = p$, such that $\omega_i(\gamma_i'(t)) = \omega_\infty(\gamma_\infty'(t))$ for all $t \in [0, 1]$ (an isometric lift in the laminated space). In particular, $\int_{\gamma_i} \omega_i = \int_{\gamma_\infty} \omega_\infty \neq 0$. Since the curves γ_i depend continuously on i, γ_i is open if i is big enough (for γ_∞ is), and X_i is not semicomplete. □

There exist sequences of complete holomorphic vector fields on manifolds that converge to vector fields which are not complete (see [27, Sect. 3]).

The results of this section can be straightforwardly exported to Palais's more general context [61].

1.4 Semicomplete Vector Fields in Dimension One

For a vector field on a neighborhood of 0 in $\mathbf{C}$ of the form $(\lambda z + \cdots)\partial/\partial z$, $\lambda \neq 0$, its germ at 0 is linearizable (see [51, Prop. 2.3], [45, Thm. 5.5]), and this germ is thus semicomplete. The vector field $z^2\partial/\partial z$ on $\mathbf{C}$ is semicomplete (and, in consequence, so is its germ at 0), for it extends as a holomorphic vector field when