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Stéphane Menozzi
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Kolmogorov Operators and Their Applications

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Foreword

Kolmogorov Equations play a pivotal role in bridging the theories of Partial Differential Equations and Stochastic Differential Equations that emerge across various research domains. The objective of the INdAM Meeting on "Kolmogorov Operators and their Applications", held in Cortona from June 13 to 17, 2022, was to convene established researchers whose work revolves around Kolmogorov operators and their applications. This gathering aimed to stimulate the exploration of novel research directions and foster new collaborations.

Degenerate Kolmogorov operators are highly degenerate evolution operators that exhibit invariance with respect to non-Euclidean geometric structures. The simplest instance of this family of differential operators appears when considering the Langevin process $(V_t, X_t)_{t\geq 0}$ in the phase space $\mathbb{R}^d \times \mathbb{R}^d$

$$\begin{cases} V_t = v_0 + W_t, \\ X_t = x_0 + \int_0^t V_s \, ds. \end{cases} \tag{1}$$

Here $(W_t)_{t\geq 0}$ denotes a d-dimensional Wiener process. The density $p = p(t, v, x, v_0, x_0)$ of $(V_t, X_t)_{t\geq 0}$ is the fundamental solution to the strongly degenerate Kolmogorov equation $\mathcal{L}p = 0$, being

$$\mathcal{L}p := \tfrac{1}{2}\Delta_v p + v \cdot \nabla_x p + \partial_t p = 0, \qquad t \geq 0, \quad (v, x) \in \mathbb{R}^{2d}. \tag{2}$$

In 1934 Kolmogorov provided us with the explicit expression of p

$$p(t, v, x, v_0, x_0) = \tfrac{3^{d/2}}{(2\pi t^2)^d} \exp\left(-\tfrac{|v-v_0|^2}{t} - 3\tfrac{(v-v_0)\cdot(x-x_0+tv_0)}{t^2} - 3\tfrac{|x-x_0+tv_0|^2}{t^3}\right),$$

and pointed out that it is a smooth function, despite the strong degeneracy of the opertor $\mathcal{L}$. As it is suggested by the smoothness of the density p, the operator $\mathcal{L}$ is hypoelliptic, that is every distributional solution $f \in L^1_{\text{loc}}(\Omega)$ to the equation

$\mathcal{L}f = g$, in some open set $\Omega \subset \mathbb{R}^{2d+1}$, we have that

$$g \in C^\infty(\Omega) \quad \Rightarrow \quad f \in C^\infty(\Omega).$$

The study of this category of differential operators has seen substantial growth in recent years, driven by both theoretical and practical considerations. They hold significant relevance in the realms of Partial Differential Equations and the theory of Stochastic Processes. In terms of their numerous real-world applications, degenerate Kolmogorov operators find utility in areas such as Kinetic Theory and the Theory of Financial Markets. These research domains are of great interest, featuring numerous open problems and unexplored issues. Currently, researchers are investigating existence, uniqueness, and regularity problems associated with this family of equations, considering both classical and weak theories, as well as the presence of fractional derivatives.

This volume comprises contributions from several speakers at the conference, encompassing a wide array of topics addressed during the event. These topics have transdisciplinary implications, spanning various fields within mathematics. The editors of this volume extend their heartfelt gratitude to the authors for their outstanding contributions.

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Local Regularity for the Landau Equation (with Coulomb Interaction Potential)

François Golse and Cyril Imbert

Abstract This is a survey of our recent work with A. Vasseur [`arXiv:2206.05155`] on the local regularity of some class of weak solutions of the space homogeneous Landau equation with Coulomb singularity. Our main result is that any axisymmetric solution in this class is smooth outside the axis of symmetry.

1 The Landau Equation and the Regularity Problem

The Landau equation discussed here is a variant of the Boltzmann equation of the kinetic theory of gases proposed by Landau [11] in the context of plasma physics. It has been known for quite a long time (see for instance [16]) that the Boltzmann collision integral cannot be defined for particle interacting via a repulsive Coulomb potential, because of a logarithmic divergence, whose coefficient is precisely the Landau collision integral: see for instance §41 in [12], or [3].

As in the case of all kinetic models, the unknown in the Landau equation is the velocity distribution function $f \equiv f(t, x, v)$, that is the number density of particles located at the position x with velocity v at time t. Henceforth, we shall restrict our attention to velocity distribution functions that are independent of the x-variable, a situation referred to as the space-homogeneous case.

The (space-homogeneous) Landau equation with unknown $f \equiv f(t, v) \geq 0$ reads

$$\partial_t f(t, v) = \operatorname{div}_v \int_{\mathbf{R}^3} a(v - w)(\nabla_v - \nabla_w)(f(t, v)f(t, w))dw\,, \quad v \in \mathbf{R}^3\,,$$

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In the right-hand side above, Landau collision kernel a is the matrix-valued function defined by the explicit formula

$$a(z) := \frac{1}{8\pi}\nabla^2|z| = \frac{1}{8\pi|z|}\Pi(z)\,, \qquad \text{where } \Pi(z) := I - \left(\frac{z}{|z|}\right)^{\otimes 2}\,.$$

It is instructive to write a nonconservative form of the Landau equation. It reads

$$\partial_t f(t,v) = \operatorname{trace}\Big((a \star_v f(t,v))\nabla_v^2 f(t,v)\Big) + f(t,v)^2\,,$$

where $\star_v$ designates the convolution product in the v-variable. What is remarkable in this form of the Landau equation is the local term $f(t,v)^2$, which comes from the identity

$$\operatorname{div}(\operatorname{div} a(z)) = -\delta_0(z)\,, \qquad \text{in } \mathcal{D}'(\mathbf{R}^3)\,.$$

The presence of the term $f(t,v)^2$ on the right-hand side obviously raises the following question: does the Cauchy problem with $f\big|_{t=0} = f_{in}$ admit global classical solutions (defined for all $t \geq 0$), or is there a finite-time blow-up for classical solutions of the Landau equation?

If one considers solutions $f(t) > 0$ independent of v, the Landau equation reduces in this case to the Riccati equation $f'(t) = f(t)^2$, whose solution $f(t) = \frac{f_{in}}{1-tf_{in}}$ blows up at time $1/f_{in}$. This is somehow uninteresting because, on physical grounds, the velocity distribution function $f(t,\cdot)$ is expected to be a probability density, or at least an element of $L^1(\mathbf{R}^3)$, which excludes positive constants.

It may seem more relevant to think of the Landau equation as a variant of the semilinear heat equation

$$\partial_t u(t,x) = \Delta_x u(t,x) + \alpha u(t,x)^2\,, \qquad x \in \mathbf{R}^d\,.$$

In this case again, there is a finite-time blow-up phenomenon, which easily follows from Kaplan's clever argument [9]. Pick ϕ to be a ground state for the Dirichlet Laplacian in the unit ball B of $\mathbf{R}^d$:

$$\begin{cases} -\Delta\phi = \lambda_0\phi\,, \qquad \phi > 0 \text{ on } B\,, \\ \phi\big|_{\partial B} = 0\,. \end{cases}$$

Consider the quantity

$$L(t) := \frac{\int_B u(t,x)\phi(x)dx}{\int_B \phi(x)dx}\,.$$

Using the Green formula and the Jensen inequality shows that

$$u(t,\cdot) \geq 0 \text{ on } B \implies \dot{L}(t) \geq -\lambda_0 L(t) + \alpha L^2(t)\,,$$

and a comparison argument based on the Riccati equation again proves that positive classical solutions of the semilinear heat equation for which $L(0) > \lambda_0/\alpha$ blow up in finite time.

However, comparing the Landau equation with the semilinear equation is misleading for the following reason. If $f(t,v)$ grows very fast as $t \to t^* < +\infty$ for v near some point v_0, one can hope that the diffusion matrix $a \star_v f(t,v)$ will grow accordingly, and that the diffusion term trace $\big((a \star_v f(t,v))\nabla_v^2 f(t,v)\big)$ will ultimately offset the effect of the local quadratic nonlinearity $f(t,v)^2$.

For this reason, several authors have considered an "Isotropic" Landau Equation

$$\partial_t u = ((-\Delta)^{-1}u)\Delta u + \alpha u^2\,, \qquad 0 \leq \alpha \leq 1\,.$$

(The term "isotropic" comes from the fact that the matrix field a in Landau's collision integral is replaced with $\frac{1}{\pi|z|}\mathrm{I}_{\mathbf{R}^3}$.) See for instance [10] for a proof of global regularity for all $\alpha \in [0, \frac{74}{75})$. Although this result is an interesting contribution to the understanding of the competition between the smoothing effect of the diffusion term and the promotion of blow-up by the quadratic nonlinearity, the total mass of the solution is dissipated by the dynamics of the isotropic model for $0 \leq \alpha < 1$, which is not very satisfying on physical grounds. More recently, the case $\alpha = 1$ has been considered in [8], where the global regularity of radially symmetric, nonincreasing solutions is proved. This is more satisfying, since the total mass of the solution is conserved by the dynamics of the isotropic Landau equation with $\alpha = 1$, as in the case of the true Landau equation—and yet not fully satisfying because this model fails to conserve the energy of the solution, at variance with the Landau equation. Energy conservation is somehow related to the existence of nontrivial equilibrium solutions of the Landau equation, i.e. solutions of

$$\operatorname{trace}\Big((a \star_v f(v))\nabla_v^2 f(t,v)\Big) + f(v)^2 = 0$$

with finite mass and energy. We shall return to this later, but the existence of such equilibria shows that one cannot argue that either the diffusion term or the quadratic nonlinearity dominates the dynamics.

Other approaches to the regularity issue for the Landau equation itself, instead of a model equation, have been attempted. Silvestre [15] proved the regularity of H-solutions (a class of weak solutions of the Landau equation obtained by Villani in [16]) belonging to $L^\infty((0,+\infty); L^p(\mathbf{R}^3; (1+|v|)^k dv))$ with $p > 3/2$ and $k > 8$. Desvillettes, He and Jiang [5] have obtained a new Lyapunov functional for the Landau equation, and proved the global existence of regular solutions for near equilibrium initial data; they also prove that H-solutions of the Landau equation cannot become singular after some finite time that can be computed explicitly in terms of the initial data. Still another approach to the regularity issue consists in showing that the (potential) singular set of a H-solution must be "small" in some sense. For instance, with Gualdani and Vasseur [6], we proved that the set of singular times of Villani solutions of the Landau equation has Hausdorff dimension $\leq 1/2$. This result is of course reminiscent of the upper bound on the Hausdorff dimension of the set of singular times [13] of Leray solutions of the Navier-Stokes equations in three space dimensions, and of the more recent partial regularity results for the Navier-Stokes equations obtained by Scheffer [14] and Caffarelli-Kohn-Nirenberg [2].

This is survey article with a (hopefully) streamlined presentation of the local regularity results under appropriate symmetry assumptions obtained in collaboration with Vasseur in [7]. There are also partial regularity results in [6, 7] without any additional symmetry assumptions. These results are based in part on the same arguments as the local regularity results discussed here, and on another important ingredient briefly sketched in the last section of the present paper. We have chosen to refrain from giving a thorough presentation of these partial regularity results in the present survey for the sake of brevity. The interested reader is referred to Sect. 7 for a quick description of the partial regularity obtained for the Landau equation with Coulomb interaction, and to [6, 7] for complete proofs of the main results in that section.

2 Fundamental Properties of the Landau Collision Integral

In the first part of section, we recall some very classical properties of the collision integral, such as the conservation laws of mass, momentum and energy, together with the Boltzmann H Theorem, with the expression of the entropy production rate. All these properties have been well known since Landau published his equation in [11].

We shall conclude this section with a much more recent, yet equally fundamental inequality comparing the Landau entropy production rate with the Fisher information.

2.1 Conservation Laws and H-Theorem

Write the Landau collision integral as

$$C(F)(v) := \mathrm{div}_v \int_{\mathbf{R}^3} a(v-w)(\nabla_v - \nabla_w)(F(v)F(w))dw\,.$$

Lemma 1 *For all $F \in C^1(\mathbf{R}^3)$ such that $F(v) \geq 0$ and $\nabla F(v)$ are rapidly decaying as $|v| \to \infty$, the Landau collision integral $C(F)$ is the distribution* *defined by*

$$\langle C(F),\phi\rangle = -\tfrac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3}(\nabla\phi(v)-\nabla\phi(w))\cdot a(v-w)(\nabla_v-\nabla_w)(F(v)F(w))dvdw$$

for all test functions $\phi \in C^\infty(\mathbf{R}^3)$ *such that* $\phi(v)$ *and* $\nabla\phi(v)$ *have at most polynomial growth as* $|v| \to \infty$.

(This formula is based on the definition of the divergence of a vector field in the sense of distributions, viz.

$$\langle C(F),\phi\rangle = -\iint_{\mathbf{R}^3\times\mathbf{R}^3}\nabla\phi(v)\cdot a(v-w)(\nabla_v-\nabla_w)(F(v)F(w))dvdw\,;$$

the conclusion follows from symmetrizing the integrand with the substitution $(v,w)\mapsto(w,v)$.)

Corollary 1 *Under the assumptions of Lemma 1*

$$\langle C(F),1\rangle = \langle C(F),v_1\rangle = \langle C(F),v_2\rangle = \langle C(F),v_3\rangle = \langle C(F),|v|^2\rangle = 0\,.$$

(Indeed, if $\phi \equiv 1$, one has $\nabla\phi = 0$, so that the expression in the right-hand side of the formula in Lemma 1 is obviously 0. By the same token, if $\phi = v_j$ for $j = 1, 2, 3$, then $\nabla\phi$ is a constant vector field, hence $\nabla\phi(v) - \nabla\phi(w) = 0$. Finally, if $\phi(v) = |v|^2$, one has

$$\nabla\phi(v)-\nabla\phi(w) = 2(v-w)$$

and we conclude after observing that $a(v-w)^*\cdot(v-w) = a(v-w)\cdot(v-w) = 0$.)

Corollary 2 (H-Theorem) *Under the assumptions of Lemma 1, if moreover* $F(v) > 0$ *is such that* $\ln F(v)$ *has at most polynomial growth as* $|v| \to \infty$, *then*

$$\begin{aligned}\langle C(F),\ln F\rangle &= -\tfrac{1}{16\pi}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{F(v)F(w)}{|v-w|}\\&\quad\times\left|\Pi(v-w)\left(\frac{\nabla F(v)}{F(v)}-\frac{\nabla F(w)}{F(w)}\right)\right|^2 dvdw\\&= -\iint_{\mathbf{R}^3\times\mathbf{R}^3}\left|\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{F(v)F(w)}\right|^2\frac{dvdw}{4\pi|v-w|}\\&= -\iint_{\mathbf{R}^3\times\mathbf{R}^3}\tfrac{1}{4\pi}\,|\Pi(v-w)(\nabla_v-\nabla_w)\\&\quad\times\sqrt{F(v)F(w)/|v-w|}\Big|^2 dvdw\,.\end{aligned}$$

(The first equality follows from the identity

$$(\nabla_v - \nabla_w)(F(v)F(w)) = F(v)F(w)(\nabla \ln F(v) - \nabla \ln F(w))$$

in the right-hand side of the equality in Lemma 1, and the second from the identity

$$\sqrt{F(v)F(w)}(\nabla \ln F(v) - \nabla \ln F(w)) = 2(\nabla_v - \nabla_w)\sqrt{F(v)F(w)}\,.$$

The last equality is based on observing that $\Pi(v-w)(\nabla_v - \nabla_w)\Phi(|v-w|) = 0$ for all $\Phi \in C^1((0,+\infty))$.)

2.2 *The Entropy Production and the Fisher Information*

In 2015, Desvillettes came up with a remarkable inequality relating the entropy production rate for the Landau equation with the Fisher information.

Theorem 1 ([4]) *For each* $f \in L^1(\mathbf{R}^3; (1+|v|^2)dv)$ *such that* $f \geq 0$ *a.e. on* $\mathbf{R}^3$ *and* $f \ln f \in L^1(\mathbf{R}^3)$*, one has*

$$\int_{\mathbf{R}^3} \frac{\left|\nabla\sqrt{f(v)}\right|^2 dv}{(1+|v|^2)^{3/2}} \leq C_D + C_D \int_{\mathbf{R}^6} \frac{\left|\Pi(v-w)(\nabla_v - \nabla_w)\sqrt{f(v)f(w)}\right|^2}{|v-w|} dv dw$$

for some constant C_D *depending only on the conserved moments of* f*, and on its* H *function:*

$$C_D \equiv C_D\left[\int_{\mathbf{R}^3}(1, v, |v|^2, |\ln f(v)|)f(v)dv\right] > 0\,.$$

Interestingly, the Desvillettes theorem puts the Landau equation in the same class as the Navier-Stokes in three space dimensions, in terms of Lebesgue exponents, as shown by Table 1.

There are several applications of the Desvillettes inequality, including for instance the propagation of moments for weak solutions of the Landau equations,

Table 1 Analogy between the Navier-Stokes equations, with unknown the fluid velocity field $u(t,x) \in \mathbf{R}^3$, and the Landau equation with unknown the distribution function $f(t,v)$

Equations	Unknowns	Dissipation rates
Navier-Stokes	$u \in L^\infty_t L^2_x$	$\nabla_x u \in L^2_t L^2_x$
Landau	$\sqrt{f} \in L^\infty_t L^2((1+|v|)^2 dv)$	$\nabla_v \sqrt{f} \in L^2_t L^2(1+|v|)^{-3} dv)$

i.e. the fact that quantities of the form

$$\int_{\mathbf{R}^3} |v|^m f(t, v) dv$$

remain bounded over finite time intervals if they are bounded for $t = 0$. (We shall not use this fact here, since we are primarily interested in local properties of weak solutions of the Landau equation.)

3 Weak Solutions

In [16], Villani proposed the following notion of weak solution of the Landau equation.

Definition 1 A H-solution of the Landau equation[1] on the time interval $[0, T)$ with initial data $f_{in} \geq 0$ is an element

$$f \in C([0, T), \mathcal{D}'(\mathbf{R}^3)) \cap L^1([0, T); L^1(\mathbf{R}^3; (1 + |v|)^{-1} dv)$$

such that

$$\Pi(v - w)(\nabla_v - \nabla_w)\sqrt{f(t, v) f(t, w)/|v - w|} \in L^2([0, T) \times \mathbf{R}^3_v \times \mathbf{R}^3_w),$$

satisfying the Landau equation in the sense of distributions in the form

$$\int_{\mathbf{R}^3} f_{in}(v)\phi(0, v)dv - \int_{\mathbf{R}^3} f(t, v)\phi(t, v)dv + \int_0^t \int_{\mathbf{R}^3} f(t, v)\partial_t \phi(s, v)dvds$$
$$= \int_0^t \iint_{\mathbf{R}^6} \sqrt{\frac{f(s,v) f(s,w)}{|v-w|}} \Phi(s, v, w) \cdot \Pi(v-w)(\nabla_v - \nabla_w)\sqrt{\frac{f(s,v) f(s,w)}{|v-w|}} dvdwds,$$

for all $\phi \in C_c^\infty([0, +\infty) \times \mathbf{R}^3)$, where $\Phi(t, v, w) := \nabla_v \phi(t, v) - \nabla_w \phi(t, w) \in \mathbf{R}^3$, together with the conservation of mass, momentum and energy, i.e.

$$\int_{\mathbf{R}^3} f(t, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \int_{\mathbf{R}^3} f_{in}(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv \quad \text{for all } t \in [0, T),$$

[1] In the Coulomb interaction case.

and the Boltzmann H-Theorem, in the form

$$f(t,v)\geq 0 \text{ a.e. on } [0,T)\times\mathbf{R}^3\,,\ \text{ and } \int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv$$
$$\leq \int_{\mathbf{R}^3} f_{in}(v)\ln f_{in}(v)dv$$

for all $t\in[0,T)$.

Observe that, by

$$\left(\iint_{\mathbf{R}^6}\sqrt{\tfrac{f(s,v)f(s,w)}{|v-w|}}\Phi(s,v,w)\cdot\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{\tfrac{f(s,v)f(s,w)}{|v-w|}}dvdw\right)^2$$
$$\leq \iint_{\mathbf{R}^6}\left|\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{\tfrac{f(s,v)f(s,w)}{|v-w|}}\right|^2 dvdw$$
$$\times\iint_{\mathbf{R}^6}\tfrac{|\Phi(s,v,w)|^2}{|v-w|}f(s,v)f(s,w)dvdw$$
$$\leq \sup_{s,v,w}\tfrac{|\Phi(s,v,w)|^2}{|v-w|}\|f_{in}\|^2_{L^1(\mathbf{R}^3)}\iint_{\mathbf{R}^6}\left|\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{\tfrac{f(s,v)f(s,w)}{|v-w|}}\right|^2 dvdw$$

by the Cauchy-Schwarz inequality. Since the right-hand side of this last inequality is integrable on $[0,T]$, we conclude from the Cauchy-Schwarz inequality (applied to integrals in the time variable) that, for each H-solution f of the Landau equation

$$t\mapsto\int_{\mathbf{R}^3} f(t,v)\phi(t,v)dv$$

belongs to $C^{0,1/2}([0,+\infty))$ for each $\phi\in C^\infty_c(\mathbf{R}^3)$.

Villani proved the following global existence theorem for the Landau equation (see Theorem 4 (i) of [16]).

Theorem 2 *Let $f_{in}\in L^1(\mathbf{R}^3;(1+|v|^2)dv)$ satisfy*

$$f_{in}\geq 0 \textit{ a.e. on } \mathbf{R}^3\,,\qquad \textit{and } f_{in}\ln f_{in}\in L^1(\mathbf{R}^3)\,.$$

There exists a H-solution f of the Landau equation on $[0,+\infty)$ such that $f\big|_{t=0}=f_{in}$.

We need to explain the basics of Villani's construction. To do this, we introduce a truncating procedure for the Landau collision kernel. Pick $\chi\in C^\infty(\mathbf{R})$ such that

$$\mathbf{1}_{[-1,1]}\leq\chi\leq\mathbf{1}_{[-2,2]}\,,\qquad \text{with } |\chi'|\leq 2\,,$$

and define the truncated collision kernel by the formula

$$a_\delta(z) := (1 - \chi(|z|/\delta))a(z)\,, \quad z \in \mathbf{R}^3$$

for each $\delta > 0$. Set

$$\xi_\delta(v) := \chi(\delta|v|)\,, \quad \text{and} \quad \zeta_\delta(v) := \tfrac{1}{X}\tfrac{1}{\delta^3}\chi\left(\tfrac{|v|}{\delta}\right)\,, \quad \text{where} \quad X := \int_{\mathbf{R}^3}\chi(|v|)dv\,.$$

For each $\delta > 0$, the regularized problem

$$\begin{cases} (\partial_t - \frac{\delta}{2}\Delta_v)f_\delta(t,v) = \operatorname{div}_v\displaystyle\int_{\mathbf{R}^3} a_\delta(v-w)(\nabla_v - \nabla_w)(f_\delta(t,v)f_\delta(t,w))dw\,, \\ f_\delta\big|_{t=0} = \zeta_\delta \star (\xi_\delta f_{in}) + \frac{\delta}{(2\pi)^{3/2}}e^{-|v|^2/2}\,, \end{cases}$$

has a unique smooth solution by standard fixed-point arguments. Besides, the maximum principle implies that

$$f_\delta(t,v) \geq C_\delta(t)e^{-|v|^2/2}$$

so that $\ln f_\delta(t,v) \geq -\frac{1}{2}|v|^2 + \ln C_\delta(t)$, and formal computations with the Boltzmann H-Theorem are legitimate.

Theorem 3 *Under the same assumptions as in Theorem 2, the family f_δ of solutions of the regularized Landau equation is relatively compact in $L^1_{loc}([0,+\infty) \times \mathbf{R}^3)$, and any of its limit points as $\delta \to 0^+$ is a H-solution of the Landau equation with initial data f_{in}.*

Henceforth, we refer to any H-solution of the Landau equation obtained as a limit point of the family of solutions of the regularized Landau equation as in Theorem 3 as a *Villani solution* of the Landau equation. In the sequel, we focus on this specific type of H-solutions to the Landau equation, since they enjoy useful additional properties, which general H-solutions are not known to satisfy (to the best of our knowledge).

4 Local Regularity

Our main results in this paper address the local regularity of Villani solutions of the Landau equation enjoying additional symmetry properties.

Let f be a Villani solution of the Landau equation with initial data f_{in} measurable on $\mathbf{R}^3$ satisfying

$$f_{in} \geq 0 \text{ a.e. and } \quad \int_{\mathbf{R}^3} (1 + |v|^2 + |\ln f_{in}(v)|) f_{in}(v) dv < \infty$$

(see Theorem 3 for the existence of a Villani solution).

Our first result deals with the regularity of axisymmetric Villani solutions of the Landau equation.

Theorem 4 *If f is axisymmetric, i.e. if there exists $\omega \in \mathbf{S}^2$ such that*

$$f(t, v) = f(t, \mathcal{R}(v - (v \cdot \omega)\omega) + (v \cdot \omega)\omega) ,$$

for all $\mathcal{R} \in SO((\mathbf{R}\omega)^{\perp})$, then $f \in C^{\infty}((0, +\infty) \times (\mathbf{R}^3 \setminus \mathbf{R}\omega))$.

In plain words, axisymmetric Villani solutions of the Landau equation are smooth except maybe on the axis of symmetry.

Our next main result is a straightforward consequence of Theorem 4, and treats the case of radial solutions.

Theorem 5 *If f is radially symmetric, i.e. of the form*

$$f(t, v) = F(t, |v|)$$

then $f \in C^{\infty}((0, +\infty) \times (\mathbf{R}^3 \setminus \{0\}))$.

Thus radial Villani solutions of the Landau equation are smooth except maybe at the origin. This is an obvious consequence of Theorem 4, since a radial solution is axisymmetric around any straight line through the origin.

Indeed, according to Theorem 4, the set of singularities of such a solution is included in the intersection of all the straight lines through the origin, which is precisely the origin itself.

After [7] appeared on the `arXiv` preprint server, A. Bobylev wrote a very thorough and interesting discussion [1] of the regularity problem for radial solutions of the Landau equation. The interested reader is strongly advised to study [1] to understand the various scenarios, and competing effects promoting or preventing blow-up for radial solutions of the Landau equation.

5 Mathematical Tools

The proof of Theorem 4 is rather involved. In this section, we shall briefly outline the main steps, and describe the mathematical methods used in this proof.

5.1 Ellipticity Bound

In this brief section, we return to the truncated collision kernel a_δ introduced after the statement of Theorem 2. Obviously, a_δ is bounded and smooth for each $\delta > 0$. Is the ellipticity preserved by this truncation? This is precisely the question addressed in this section.

Lemma 2 *Assume that $f \equiv f(v) \geq 0$ a.e. is a measurable function satisfying the bounds*

$$0 < m_0 \leq \int_{\mathbf{R}^3} f(v)dv \leq M_0\,, \qquad \int_{\mathbf{R}^3} \begin{pmatrix} |v|^2 \\ \ln f(v) \end{pmatrix} f(v)dv \leq \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}.$$

Then, there exists $c_0[m_0, M_0, E_0, H_0] > 0$ and $\delta_0[m_0, M_0, E_0, H_0] \in (0, 1)$ such that

$$0 < \delta < \delta_0 \implies f \star a_\delta(v) \geq \tfrac{c_0}{(1+|v|)^3} I \quad \textit{for all } v \in \mathbf{R}^3\,.$$

5.2 Locally Bounded Solutions Are Smooth

Henceforth, we shall use parabolic cylinders, denoted as follows:

$$Q_r(t_0, v_0) := (t_0 - r^2, t_0] \times B_r(v_0)\,.$$

Because of the parabolic nature of the Landau equation, in order to prove the local smoothness of weak solutions, only some very low regularity is needed. Smoothness follows by some bootstrap procedure.

Lemma 3 *Let f be a Villani solution of the Landau equation. Assume that $f \in L^\infty(Q_R(t_0, v_0))$ for some $R, t_0 > 0$ and some $v_0 \in \mathbf{R}^3$. Then f is a.e. equal to function of class C^∞ on $Q_{R/2}(t_0, v_0)$.*

A consequence of Desvillettes' theorem (Theorem 1) is that Villani solutions of the Landau equation are solutions in the sense of distributions. Thus the Villani solution f satisfies

$$\partial_t f = \mathrm{div}_v(A\nabla_v f + fB)$$

where

$$A(t, v) := f(t, \cdot) \star a(v) \quad \text{and}$$
$$B(t, v) = -f(t, \cdot) \star \mathrm{div}\, a(v) = 2\left(f(t, \cdot) \star \tfrac{z}{|z|^3}\right)(v)\,.$$

Thus, if $0 \leq f \in L^\infty(Q_R(t_0, v_0))$, then

$$f(t,\cdot) \star a = \underbrace{f(t,\cdot) \star a_\delta}_{L^\infty_t L^1_v \star_v L^\infty_v} + \underbrace{f(t,\cdot) \star (a - a_\delta)}_{=(f \mathbf{1}_{Q_R(t_0,v_0)}) \star_v (a - a_\delta)} \in L^\infty(Q_{R-2\delta}(t_0, v_0)),$$

and, by the same token, $fB \in L^\infty(Q_{R-2\delta}(t_0, v_0))$. On the other hand, reducing $\delta > 0$ so that $\delta < \delta_0$ as in Lemma 2, one has

$$f(t,\cdot) \star a(v) \geq f(t,\cdot) \star a_\delta(v) \geq \frac{c_0}{(1+|v|)^3} I, \qquad v \in \mathbf{R}^3,$$

so that f belongs to a parabolic De Giorgi class, and is therefore Hölder continuous, with a bound of the form

$$[f]_{C^\alpha(Q_{\beta_2 R}(t_0,v_0))} \leq C \|f\|_{L^\infty(Q_{\beta_1 R}(t_0,v_0))}(1 + \|B\|_{L^\infty(Q_{\beta_1 R}(t_0,v_0))})$$

for all β_1, β_2 such that $0 < \beta_2 < \beta_1 < 1$. This implies that A and B belong to the Hölder class C^α on $Q_{\beta_3 R}(t_0, v_0)$ for all $\beta_3 \in (0, 1)$, again by decomposing a as $a = (a - a_\delta) + a_\delta$. Then, we can use the Schauder estimates to conclude that $\partial_t f$ and $\nabla_v f$ are of Hölder class C^α on $Q_{\beta_4 R}(t_0, v_0)$ for all $0 < \beta_4 < \beta_3 < 1$. Differentiating both sides of the Landau equation in all variables, the proof of Lemma 3 follows by a standard bootstrap argument.

5.3 Truncated Entropy Inequality

In the sequel, we shall use the De Giorgi method to prove a sufficient condition for local boundedness, and therefore local regularity according to Lemma 3. The De Giorgi method involves considering truncations of the distribution functions. The following notation will be convenient:

$$f_+^\kappa := (f - \kappa)_+ = \max(f - \kappa, 0), \qquad \text{for all } \kappa > 0.$$

We shall also consider the truncated logarithm

$$\ln_+ z := \max(\ln z, 0) = \ln \max(z, 1).$$

Definition 2 For each $g \geq 0$ and each $\kappa > 0$, the truncated entropy generating function is (Fig. 1)

$$h_+^\kappa(g) := \kappa h_+\left(\tfrac{g}{\kappa}\right), \qquad \text{where} \qquad h_+(z) := z \ln_+ z - (z-1)_+.$$

At this point, we shall use a property of Villani solutions of the Landau equation, pertaining to the evolution of the truncated entropy of the distribution

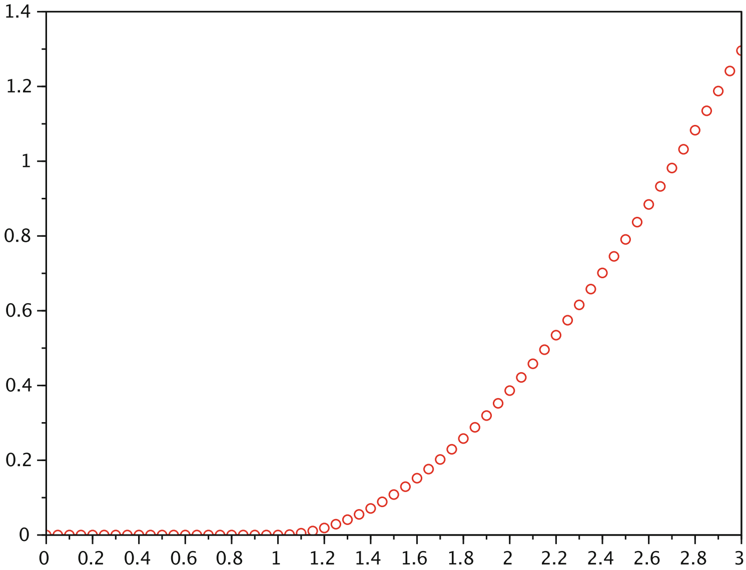

Fig. 1 Graph of the function h_+

function, which we do not know to be satisfied by all H-solutions. This property is fundamental for our analysis, and this is the reason for restricting our attention to Villani solutions.

Let f be a Villani solution of the Landau equation on $\mathbf{R}^3$.

For all $\Psi \in C_c^\infty((0,T)\times\mathbf{R}^3)$, there exists a Lebesgue-negligible set $\mathcal{N} \subset (0,T)$ such that

$$\begin{aligned} &\int_{\mathbf{R}^3} h_+^\kappa(f(t_2,v))\phi(t,v)dv \\ \le &\int_{\mathbf{R}^3} h_+^\kappa(f(t_1,v))\phi(t,v)dv + \int_{t_1}^{t_2}\int_{\mathbf{R}^3} h_+^\kappa(f(t,v))\partial_t\phi(t,v)dvdt \\ &- \int_{t_1}^{t_2}\int_{\mathbf{R}^3}(A\nabla_v f - (\operatorname{div}_v A)f)\Big(\underbrace{\phi\frac{\nabla_v f_+^\kappa}{f}}_{\mathcal{T}_1} + \underbrace{\ln_+(\tfrac{f}{\kappa})\nabla_v\phi}_{\mathcal{T}_2}\Big)(t,v)dvdt \end{aligned}$$

for all $\kappa > 0$ and all $t_1 < t_2 \notin \mathcal{N}$, with

$$\phi := \Psi^2, \qquad A(t,v) := \int_{\mathbf{R}^3} f(t,w)a(v-w)dw, \quad v \in \mathbf{R}^3.$$

This inequality is obtained by writing the evolution of the truncated entropy for solutions of the regularized Landau equation used in Villani's construction of H-solutions, passing to the limit as the regularization parameter $\delta \to 0$.

At this point, we forget about the regularized Landau equation, and think of Villani solutions of the Landau equation as H-solution satisfying the truncated entropy inequality above. In fact, what we are going to use is a consequence of the inequality above, where the test function Ψ is chosen so that

$$\mathbf{1}_{B_r(v_0)} < \Psi < \mathbf{1}_{B_{r+\delta}(v_0)}\,, \quad \text{with} \quad \|\nabla\Psi\|_{L^\infty} \le \frac{c_*}{\delta} \quad \text{and} \quad \|\nabla^2\Psi\|_{L^\infty} \le \frac{c_*}{\delta^2}\,.$$

Choosing Ψ as above, and using the lower bound for the entropy production rate provided by the Desvillettes theorem (Theorem 1), one arrives at the following local estimate, which will be at the core of our analysis of the Landau equation.

Lemma 4 (Key Local Estimate) *Let $v_0 \in \mathbf{R}^3$, and assume that $f_{in} \ge 0$ a.e. on $\mathbf{R}^3$ is a measurable function satisfying*

$$0 < m_0 \le \int_{\mathbf{R}^3} f_{in}(v)dv \le M_0\,, \quad \int_{\mathbf{R}^3} \begin{pmatrix} |v|^2 \\ \ln f(v) \end{pmatrix} f_{in}(v)dv \le \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}.$$

There exists $C_0 \equiv C_0[|v_0|, m_0, M_0, E_0, H_0] \ge 1$ such that any Villani solution of the Landau equation with initial data f_{in} satisfies the inequality

$$\begin{aligned} &\operatorname*{ess\,sup}_{t_0-r^2<t\le t_0} \int_{B_r(v_0)} h_+^\kappa(f(t,v))dv + \int_{Q_r(t_0,v_0)} \frac{|\nabla_v f_+^\kappa(t,v)|^2}{f(t,v)} dtdv \\ &\le C_0 \int_{Q_{r+\delta}(t_0,v_0)} \left(\kappa + \tfrac{1}{\delta^2} + \tfrac{1}{\delta^2} f \star_v \tfrac{1}{|\cdot|}\right) f \left(\ln_+\left(\tfrac{f}{\kappa}\right) + \ln_+\left(\tfrac{f}{\kappa}\right)^2\right) dtdv \\ &\quad + C_0 \int_{Q_{r+\delta}(t_0,v_0)} \left(\tfrac{1}{\delta}(f \wedge \kappa) \star_v \tfrac{1}{|\cdot|^2}\right) f \ln_+\left(\tfrac{f}{\kappa}\right) dtdv \end{aligned}$$

for all $t_0 \in (0,T)$ and all $r, \delta \in (0,1)$, provided that

$$t_0 > (r+\delta)^2\,, \qquad \kappa \in \mathbf{Q} \cap [1, +\infty)\,.$$

5.4 Scalings

We have observed in Table 1 the analogy between H-solutions of the Landau equation and Leray solutions of the Navier-Stokes equations in three space dimensions. However, there is at least one major difference between these equations, namely the scaling invariance.

First, we mention the translation invariance of the Landau equation, which is obvious (and a consequence of the Galilean invariance in classical mechanics).

Thus, for the Landau equation

$$f \; H\text{-solution} \implies f(t_0+\cdot, v_0+\cdot)\; H\text{-solution}$$

for all $t_0 > 0$ and all $v_0 \in \mathbf{R}^3$.

At variance with the Navier-Stokes equation there is a two-parameter group of scaling transformations leaving the Landau equation with Coulomb interaction invariant: for all $\lambda > 0$ and all $\mu > 0$, set

$$f_{\lambda,\mu}(t,v) := \lambda f(\lambda t, \mu v)\,, \qquad (t,v) \in (0,+\infty)\times\mathbf{R}^3\,.$$

Then

$$f \; H\text{-solution of Landau's equation} \implies f_{\lambda,\mu} \; H\text{-solution of Landau's equation}$$

(We recall that if $u \equiv u(t,x) \in \mathbf{R}^3$ is a Leray solution of the Navier-Stokes equations, then, for each $\epsilon > 0$, the rescaled velocity field $u_\epsilon(t,x) := \epsilon u(\epsilon^2 t, \epsilon x)$ is also a Leray solution of the Navier-Stokes equations.)

Here is how the scaling transformation $f \mapsto f_{\lambda,\mu}$ acts on the quantities appearing on the left-hand side of the key local estimate, viz. the truncated entropy and the Fisher information of the truncated distribution function:

$$\int_{B_r(0)} h_+^{\lambda\kappa}(f_{\lambda,\mu}(t,v))dv = \frac{\lambda}{\mu^3}\int_{B_{\mu r}(0)} h_+^{\kappa}(f(\lambda t,\tilde v))d\tilde v\,,$$

$$\int_{-\tau}^0\int_{B_r}\frac{|\nabla_v(f_{\lambda,\mu}(t,v)-\lambda\kappa)_+|^2}{f_{\lambda,\mu}(t,v)}dtdv = \frac{1}{\mu}\int_{-\lambda\tau}^0\int_{B_{\mu r}}\frac{|\nabla_{\tilde v}(f(\tilde t,\tilde v)-\kappa)_+|^2}{f(\tilde t,\tilde v)}d\tilde t d\tilde v\,.$$

It is obviously desirable to keep both these terms of the same order of magnitude under the scaling transforms as one zooms in near a point (t_0, v_0) to study the local regularity of the distribution function. Among all the scaling transformations leaving the Landau equation with Coulomb interaction invariant, we are therefore led to consider the special case $\lambda/\mu^3 = 1/\mu$. Henceforth, we set

$$\mu = \epsilon \quad \text{and} \quad \lambda = \epsilon^2\,.$$

Thus, let $f \equiv f(t,v)$ be a global Villani solution of the Landau equation. Pick $(t_0, v_0) \in (0,+\infty)\times\mathbf{R}^3$. Henceforth, we seek to study this solution near (t_0, v_0) by considering its translated and scaled variant

$$f_\epsilon(t,v) := \epsilon^2 f(t_0+\epsilon^2 t, v_0+\epsilon v)\,, \qquad v\in\mathbf{R}^3\,, \quad t > -t_0/\epsilon^2\,.$$

If one returns to Table 1, one easily checks that this scaling transformation is exactly the same as in the case of the Navier-Stokes equations, since

$$\sqrt{f_\epsilon(t,v)} = \epsilon\sqrt{f(t_0+\epsilon^2 t, v_0+\epsilon v)}$$

is the quantity analogous to the rescaled Navier-Stokes velocity field

$$u_\epsilon(t,x) = \epsilon u(t_0+\epsilon^2 t, x_0+\epsilon x)\,.$$

Set

$$\kappa_\epsilon := \epsilon^2\kappa\,, \qquad \delta_\epsilon := \delta/\epsilon\,, \qquad r_\epsilon := r/\epsilon\,, \qquad f_{\epsilon,+}^{\kappa_\epsilon} := (f_\epsilon - \kappa_\epsilon)_+\,.$$

Assume that

$$\epsilon \in (0, \min(\tfrac{1}{2}, \sqrt{t_0}))\,, \quad \kappa_\epsilon \in [1,2]\cap \mathbf{Q}\,, \quad r_\epsilon \in (0,2]\,, \quad \delta_\epsilon \in (0,1]\,.$$

Corollary 3 (Scaled Local Estimate) *Under the same assumptions and with the same notations as in Lemma 4, there exists* $C_0' \equiv C_0'[|v_0|, m_0, M_0, E_0, H_0] \geq 1$, ***independent*** *of* ϵ*, such that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-r_\epsilon^2<t\leq 0}\int_{B_{r_\epsilon}} h_+^{\kappa_\epsilon}(f_\epsilon(t,v))dv + \int_{Q_{r_\epsilon}} \frac{|\nabla_v f_{\epsilon+}^{\kappa_\epsilon}(t,v)|^2}{f_\epsilon(t,v)}dtdv \\
&\leq C_0'\int_{Q_{r_\epsilon+\delta_\epsilon}}\left(\kappa_\epsilon + \tfrac{1}{\delta_\epsilon^2} + \tfrac{1}{\delta_\epsilon^2} f_\epsilon \star \frac{\mathbf{1}_{B_1^c}}{|\cdot|}\right) f_\epsilon(\ln_+(\tfrac{f_\epsilon}{\kappa_\epsilon}) + \ln_+(\tfrac{f_\epsilon}{\kappa_\epsilon})^2)dtdv \\
&\qquad + \frac{C_0'}{\delta_\epsilon^2}\int_{Q_{r_\epsilon+\delta_\epsilon}} f_\epsilon \star \frac{\mathbf{1}_{B_1}}{|\cdot|} f_\epsilon(\ln_+(\tfrac{f_\epsilon}{\kappa_\epsilon}) + \ln_+(\tfrac{f_\epsilon}{\kappa_\epsilon})^2)dtdv\,.
\end{aligned}$$

This is a rather straightforward consequence of Lemma 4, splitting the term coming from the Landau collision kernel as

$$\frac{1}{|z|} = \frac{\mathbf{1}_{|z|<1}}{|z|} + \frac{\mathbf{1}_{|z|\geq 1}}{|z|}\,,$$

and transforming the integrals by the substitution

$$(t,v) \mapsto (t_0+\epsilon^2 t, v_0+\epsilon v)\,.$$

5.5 *Local Regularity Criterion*

The last mathematical tool used in our study of the local regularity for the Landau equation with Coulomb singularity is the De Giorgi iteration method—specifically, the part of De Giorgi's argument leading to a local bound—and not the reduction of oscillations (in fact, this latter part of De Giorgi's analysis has already been used as a black box procedure in the proof of Lemma 3 before the bootstrap argument based on Schauder's estimates).

Lemma 5 (De Giorgi's First Lemma) *Let $f_\epsilon(t,v) = \epsilon^2 f(t_0+\epsilon^2 t, v_0+\epsilon v)$ be a scaled Villani solution of the Landau equation with initial data $f_{in} := f(0,\cdot)$ satisfying*

$$0 < m_0 \le \int_{\mathbf{R}^3} f_{in}(v)dv \le M_0\,, \qquad \int_{\mathbf{R}^3} \begin{pmatrix} |v|^2 \\ \ln f_{in}(v) \end{pmatrix} f_{in}(v)dv \le \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}.$$

Assume

$$\operatorname*{ess\,sup}_{Q_1(0,0)} f_\epsilon(t,\cdot) \star \frac{\mathbf{1}_{B_1(0)^c}}{|\cdot|} \le Z_\epsilon\,, \qquad \text{where } Z_\epsilon \ge 1\,.$$

Then, there exists $\eta_{DG} \equiv \eta_{DG}[m_0, M_0, E_0, H_0, |v_0|] \in (0,1)$ such that

$$\operatorname*{ess\,sup}_{-4<t\le 0} \int_{B_2} (f_\epsilon(t,v)-1)_+ dv + \int_{Q_2} |\nabla_v \sqrt{f_\epsilon}|^2 \mathbf{1}_{f_\epsilon \ge 1} dt dv \le \frac{\eta_{DG}}{Z_\epsilon^{3/2}}$$
$$\implies f_\epsilon \le 2 \text{ a.e. on } Q_{1/2}\,.$$

Of course, the proof of Lemma 5 is based on simultaneous truncations of the values of the distribution function f, and parabolic zooming transformations near (t_0, v_0). Specifically choose a sequence r_j of radii shrinking from 1 to $\frac{1}{2}$, and a sequence of levels κ_j increasing from 1 to 2, as follows:

$$r_j := \tfrac{1}{2}(1+2^{-j})\,, \qquad \kappa_j := 2-2^{-j}\,, \quad j \ge 0\,.$$

Then we apply Corollary 3 with $r_\epsilon = r_{j+1}$ and $\delta_\epsilon = r_j - r_{j+1}$, while $\kappa_\epsilon = \frac{1}{2}(\kappa_j + \kappa_{j+1})$. Then we proceed with the De Giorgi iteration method, involving a nonlinearization procedure as usual, and arrive at the statement in Lemma 5.

6 A Sketch of the Proof of Theorem 4

The proof of Theorem 4 is based on two ingredients:

(a) a local $L^2_{t,v}$-bound away from the axis of symmetry, and
(b) a local $L^\infty_{t,v}$ bound for the diffusion matrix.

Henceforth we fix $(t_0, v_0) \in (0,+\infty) \times \mathbf{R}^3$ such that v_0 is at a distance $\rho_0 > 0$ from the axis of symmetry of the distribution function f.

6.1 Local $L^2_{t,v}$ Bound

Roughly speaking, away from the axis of symmetry, an axisymmetric function is a function of two variables (the distance to the axis and the height). Therefore, such functions benefit from better Sobolev embeddings than in the generic three dimensional case. This elementary idea is the basis for the following lemma.

Lemma 6 *Let $f \in L^\infty((0,T); L^1(B_R))$ with $\nabla_v f \in L^2((0,T); L^1(B_R))$ be of the form*

$$f(t,v) = g\Big(t, \underbrace{\sqrt{v_1^2+v_2^2}}_{V_1}, \underbrace{v_3}_{V_2}\Big).$$

Let $t_0 \in (0,T)$, let $V^0 = (V_1^0, V_2^0) \in \mathbf{R}^2$, and let $0 < r < V_1^0 - \rho_0$; then

$$\int_{t_0-r^2}^{t_0} \int_{|V-V_0|\le r} g(t,V)^2 dV dt$$
$$\le \tfrac{C_S(B_r)^2}{(2\pi\rho_0)^2} \int_{t_0-r^2}^{t_0} \Big(\|f(t,\cdot)\|^2_{L^1(B_r(0,0,V_2^0))} + \|\nabla f(t,\cdot)\|^2_{L^1(B_r(0,0,V_2^0))}\Big) dt\,,$$

where $C_S(B_r)$ is the Sobolev constant for $W^{1,1}(B_r) \subset L^2(B_r)$ in $\mathbf{R}^2$.

(Notice that the right-hand side of this inequality involves L^1 norms for the 3-dimensional Lebesgue measure over the 3-dimensional ball $B_r(0,0,V_2^0)$, whereas the integral on the left-hand side is over a 2-dimensional domain. This is the reason why the right-hand side is expressed in terms of f, and the left-hand side in terms of the function g.)

6.2 Local $L^\infty_{t,v}$ Bound of the Diffusion Matrix

The quantity $f(t,\cdot) \star \frac{1}{|\cdot|}$ appears twice on the right-hand side of the inequality in Corollary 3. Now this quantity is interesting, because of the following observation. For each $\epsilon > 0$ and each Villani solution of the Landau equation, it holds

$$f_\epsilon(t,\cdot) \star \frac{1}{|\cdot|}(v) = f(t_0 + \epsilon^2 t, \cdot) \star \frac{1}{|\cdot|}(v_0 + \epsilon v)\,,$$

for all $(t_0, v_0) \in (0, +\infty) \times \mathbf{R}^3$, all $v \in \mathbf{R}^3$ and all $t > -t_0/\epsilon^2$, so that

$$\left\| f_\epsilon \star_v \frac{1}{|\cdot|} \right\|_{L^\infty(Q_r(0,0))} = \left\| f \star_v \frac{1}{|\cdot|} \right\|_{L^\infty(Q_{\epsilon r}(t_0,v_0))} .$$

In other words, the quantity $f(t,\cdot) \star \frac{1}{|\cdot|}$ is L^∞-critical.

The next lemma provides an L^∞ control on this quantity for rescaled axisymmetric Villani solutions of the Landau equation.

Lemma 7 *Let $f \geq 0$ be measurable on $(0, T) \times \mathbf{R}^3$ and of the form*

$$f(t, v) := F\left(t, \sqrt{v_1^2 + v_2^2}, v_3\right) .$$

Let $(t_0, v_0) \in (0, T) \times \mathbf{R}^3$ be such that

$$\rho_0 := \sqrt{v_1^2 + v_2^2} > 0 ,$$

and let

$$f_\epsilon(t, v) := \epsilon^2 f(t_0 + \epsilon^2 t, v_0 + \epsilon v) .$$

Then, there exists an absolute constant $C_* > 0$ such that, for $0 < \epsilon < \frac{\rho_0}{6}$, it holds

$$\begin{aligned} \operatorname*{ess\,sup}_{Q_3} f_\epsilon(t,\cdot) \star \frac{1}{|\cdot|} &\leq \frac{C_*}{\rho_0} \int_{\mathbf{R}^3} f(1 + \ln_+ f)(t, v) dv + C_* \rho_0^2 \\ &=: Z_*[M_0, E_0, H_0, \rho_0] . \end{aligned}$$

6.3 Deducing Local Regularity from Lemmas 6 and 7

Write the scaled key local estimate of Corollary 3 with $r_\epsilon = 2$ and $\delta_\epsilon = \kappa_\epsilon = \frac{1}{2}$, using the elementary inequality

$$\ln_+(2y) + (\ln_+(2y))^2 \leq 6y , \qquad y > 0 .$$

Then, by Lemma 7

$$\operatorname*{ess\,sup}_{-4<t\leq 0}\int_{B_2(0)} h_+^{1/2}(f_\epsilon(t,v))dv+\int_{Q_2(0,0)}\frac{|\nabla_v f_{\epsilon+}^{1/2}(t,v)|^2}{f_\epsilon(t,v)}dtdv$$

$$\leq 6C_0'(\tfrac{3}{4}+\tfrac{1}{2}Z_*)\int_{Q_3(0,0)} f_\epsilon(t,v)^2 dtdv\,.$$

Then

$$\int_{Q_3(0,0)} f_\epsilon(t,v)^2 dtdv$$

$$\leq 2\pi(\rho_0+3)\int_{t_0-9\epsilon^2}^{t_0}\int_{\rho_0-3\epsilon}^{\rho_0+3\epsilon}\int_{v_3^0-3\epsilon}^{v_3^0+3\epsilon} g(s,\rho,w_3)^2 dsd\rho dw_3\to 0$$

as $\epsilon\to 0^+$ by Lebesgue's Theorem, since

$$(t_0-9\epsilon^2,t_0)\times(\rho_0-3\epsilon,\rho_0+3\epsilon)\times(v_3^0-3\epsilon,v_3^0+3\epsilon)$$

is a set of set of vanishing measure as $\epsilon\to 0$, and since

$$g\in L^1((t_0-9\epsilon_0^2,t_0)\times(\rho_0-3\epsilon_0,\rho_0+3\epsilon_0)\times(v_3^0-3\epsilon_0,v_3^0+3\epsilon_0))$$

by Lemma 6 for some $\epsilon_0>0$ fixed.

Therefore, there exists $\epsilon>0$ small enough such that

$$\operatorname*{ess\,sup}_{-4<t\leq 0}\int_{B_2(0)} h_+^{1/2}(f_\epsilon(t,v))dv+\int_{Q_2(0,0)}\frac{|\nabla_v f_{\epsilon+}^{1/2}(t,v)|^2}{f_\epsilon(t,v)}dtdv$$

falls below the threshold $\eta_{DG}/Z_*^{3/2}$ in the De Giorgi local regularity criterion (Lemma 5).

Lemma 5 implies that, for this value of ϵ it holds $f_\epsilon\leq 2$ a.e. on $Q_{1/2}(0,0)$, so that

$$f\leq 2/\epsilon \text{ a.e. on } Q_{\epsilon/2}(t_0,v_0)\,.$$

By Lemma 3, this implies that f is of class C^∞ in $Q_{\epsilon/4}(t_0,v_0)$.

7 Conclusion and Final Remarks

We have proved that axisymmetric, or radially symmetric Villani solutions of the Landau-Coulomb equation are regular away from the axis of symmetry, or from the origin in the case of radial solutions.

Our proof based on the De Giorgi method applied to a localized variant of the inequality derived from the H-Theorem for the Landau equation. One key ingredient in this proof is the **upper** bound for the diffusion matrix (Lemma 7 above), which is a critical quantity in $L^\infty_{t,x}$.

The same approach with an additional ingredient, viz. a control of the local mass $\|f_\epsilon\|_{L^\infty_t L^1_v(Q_R(0,0))}$ for $R < \frac{1}{2}$ in terms of $\|\nabla_v \sqrt{f_\epsilon}\|_{L^2(Q_2(0,0))}$ and of $\|\mathfrak{R}_\epsilon\|_{L^2(Q_2(0,0)\times\mathbf{R}^3)}$ with

$$\mathfrak{R}_\epsilon := \Pi(v-w)(\nabla_v - \nabla_w)\sqrt{f_\epsilon(t,v)f_\epsilon(t,w)/|v-w|}$$

shows that, for all Villani solutions f of the Landau equation with Coulomb interaction, the 7/2-dimensional parabolic Hausdorff measure[2] of its singular set satisfies

$$\mathcal{H}^{7/2}_{\text{parabolic}}(\mathbf{S}[f]) = 0\,,$$

where $\mathbf{S}[f]$ is the singular set of f, i.e.

$$\mathbf{S}[f] := \left\{(t,v) \in (0,+\infty)\times\mathbf{R}^3 \text{ s.t. } f \notin L^\infty(Q_r(t,v)) \text{ for all } r < \sqrt{t}\right\}\,.$$

This can be seen as a refinement on our earlier result with M.P. Gualdani and A. Vasseur recalled in the introduction (the fact that the Hausdorff dimension of the set of singular times for any Villani solution of the Landau equation with Coulomb interaction is at most 1/2), also based on the De Giorgi method.

[2] For all $s \in [0,5]$, the s-dimensional parabolic Hausdorff measure of a Borel set $X \subset [0,+\infty)\times\mathbf{R}^3$ is defined as

$$\mathcal{H}^{7/2}_{\text{parabolic}}(X) = \lim_{\delta\to 0^+} \inf\left\{\sum_{k\geq 1} r_k^s \text{ s.t. } X \subset \bigcup_{k\geq 1} Q_{r_k}(t_k,x_k) \text{ with } 0 < r_k < \delta\right\}\,.$$