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Shuji Machihara *Editor*

Mathematical Physics and Its Interactions

In Honor of the 60th Birthday of Tohru Ozawa, Tokyo, Japan, August 2021

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Editor

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Preface

This volume is a collection of the contents of lectures that were to be given at the meeting entitled *Mathematical Physics and Its Interactions*, originally scheduled to be held 25–27 August 2021. The meeting would have a special significance, as it was to be a celebration of Professor Tohru Ozawa's 60th birthday.

Preparations had been underway for about 2 years prior to the planned meeting, before the news of COVID-19. In March 2020, many schools, including universities, were closed in Japan. For a long time afterward, many events such as research meetings were canceled. We are grateful to the authors of this volume for their willingness to participate as speakers in 2021 even though news of COVID-19 continued to appear in the media and many people were reluctant to attend. In actuality, however, the reality of vaccines became an issue that could not be ignored, and on 23 April of that year, we announced the cancellation of the meeting. This collection of reports is therefore the only evidence of the existence of this meeting that would have taken place.

Saitama, Japan
July 2023

Shuji Machihara

Contents

Positive Solutions of Superlinear Elliptic Equations with Respect to the Schrödinger Operator

Kentaro Hirata

Dedicated to Professor Tohru Ozawa on the occasion of his sixtieth birthday.

Abstract We study positive solutions of semilinear elliptic equations with respect to the Schrödinger operator in the superlinear case. In a bounded smooth domain, a priori estimates for solutions and their gradients, the Harnack inequality and the boundary Harnack principle are presented. As applications, we also investigate the asymptotic behavior of positive solutions with isolated boundary singularities and the removability of isolated boundary singularities.

Keywords Semilinear elliptic equation · Schrödinger operator · Superharmonic function · Boundary behavior · Boundary Harnack principle · Apriori estimate

2020 Mathematics Subject Classification Primary 35J91 · Secondary 31B10 · 31B35 · 35B09 · 35B45 · 35J10

1 Introduction

This paper is motivated by the following nonlinear Schrödinger equation. Let Ω be a domain in $\mathbb{R}^N$, let Δ be the usual Laplacian on $\mathbb{R}^N$ and let $p > 1$. The standing wave solution $\psi = \psi(x, t) : \Omega \times \mathbb{R} \to \mathbb{C}$ of the nonlinear Schrödinger equation with potential functions $U_1, U_2 : \Omega \to \mathbb{R}$ of the form

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$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + U_1\psi - U_2|\psi|^{p-1}\psi \quad \text{in } \Omega\times\mathbb{R}$$

is given by $\psi(x,t) = e^{i\omega t}u(x)$ with a pair of $\omega\in\mathbb{R}$ and a solution $u:\Omega\to\mathbb{R}$ of the elliptic equation $-\Delta u + (U_1+\omega)u = U_2|u|^{p-1}u$ in Ω. Thus we are interested in revealing properties and the behavior of positive solutions of such elliptic equations.

Throughout this paper, we suppose that Ω is a bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ with a $C^{1,1}$-boundary $\partial\Omega$; roughly speaking, there exists a constant $r_\Omega>0$ such that, in an r_Ω-neighborhood of each point in $\partial\Omega$, we can describe $\partial\Omega$ as the graph of a $C^{1,1}$-function of $(N-1)$-variables by retaking a Cartesian coordinate system if necessary (see Definition 1). We deal with the superlinear elliptic equation with respect to the Schrödinger operator $-\Delta + V_1$ of the form

$$-\Delta u + V_1 u = V_2|u|^{p-1}u \quad \text{in } \Omega, \tag{1}$$

where V_1 and V_2 are real-valued measurable functions on Ω possibly having singularities on $\partial\Omega$. By $\delta_\Omega(x)$, we denote the distance from $x\in\Omega$ to $\partial\Omega$. The letter c stands for a generic positive constant whose value may vary at each occurrence. If necessary, we add the subscript like $c_1, c_2, \ldots$ to specify constants.

Unless otherwise stated explicitly, we always assume that

$$1 < p < \frac{N}{N-2}, \quad \alpha_1 < 2 \quad \text{and} \quad \alpha_2 < N+1-p(N-1) \tag{2}$$

and there exist positive constants c_1, c_2 such that

(V1) $|V_1(x)| \leq c_1\,\delta_\Omega(x)^{-\alpha_1}$ for a.e. $x\in\Omega$,
(V2) $0 \leq V_2(x) \leq c_2\,\delta_\Omega(x)^{-\alpha_2}$ for a.e. $x\in\Omega$.

Note that $V_1, V_2 \in L^\infty_{\mathrm{loc}}(\Omega)$. We say that $u\in L^p_{\mathrm{loc}}(\Omega)$ is a *solution* of (1) if

$$\int_\Omega u(-\Delta\phi)\,dx = \int_\Omega \left\{V_2|u|^{p-1}u - V_1 u\right\}\phi\,dx$$

for all $\phi\in C_c^\infty(\Omega)$; the set of all C^∞-functions with compact support in Ω. Let $c_3>0$ and let x_0 be a fixed point in Ω such that $\delta_\Omega(x_0) > r_\Omega$. By $\mathscr{S}(\Omega)$, we denote the collection of all positive solutions $u\in C(\Omega)$ of (1) such that

$$u(x_0) \leq c_3.$$

Note that $\mathscr{S}(\Omega)$ depends on c_3, p, V_1, V_2, etc., but we do not express them in the symbol $\mathscr{S}(\Omega)$ for simplicity.

In the statements of theorems, the notation $c = c(a,b,\ldots)$ means that the constant c depends at most on the parameters $a, b, \ldots$, while $c = c(\Omega)$ means its dependence on the dimension N and some characters regarding Ω.

Now, let us state main results of this paper. Since we impose no boundary condition, the collection $\mathscr{S}(\Omega)$ may include positive solutions of (1) with singularities on $\partial\Omega$. The following theorem reveals the possible growth rate of such solutions.

Theorem 1 (Boundary growth estimate) *There exists* $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) > 0$ *such that*

$$u(x) \le c\,\delta_\Omega(x)^{1-N} \quad \textit{for all } x \in \Omega, \tag{3}$$

whenever $u \in \mathscr{S}(\Omega)$.

Corollary 1 $\mathscr{S}(\Omega) \subset C^1(\Omega)$. *Moreover, there exists* $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) > 0$ *such that*

$$|\nabla u(x)| \le c\,\delta_\Omega(x)^{-N} \quad \textit{for all } x \in \Omega, \tag{4}$$

whenever $u \in \mathscr{S}(\Omega)$.

Theorem 1 is relevant to the result due to Bidaut-Véron–Vivier [4] for the Lane–Emden equation $-\Delta u = |u|^{p-1}u$ with $1 < p < (N+1)/(N-1)$: for any positive solution u on Ω, there exists a nonnegative harmonic function h on Ω such that

$$h(x) \le u(x) \le c\{h(x) + \delta_\Omega(x)\} \quad \text{for all } x \in \Omega. \tag{5}$$

Taking account of the possible growth rate near $\partial\Omega$ of positive harmonic functions on Ω, we can derive (3) from (5). In our setting of V_1 and V_2, it is hard to show a sharper estimate like (5) for positive solutions of (1). But estimate (3) is sufficient to obtain a variety of nice properties of solutions.

Also, as a special case of Serrin–Zou [19], we know that if $0 \le V_1 \in L^\infty(\Omega)$ and $V_2 \in L^\infty(\Omega)$ has a positive lower bound on Ω, then all positive weak solutions u of (1) with $1 < p < N/(N-2)$ satisfy

$$u(x) \le c\,\delta_\Omega(x)^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega, \tag{6}$$

where the constant c is independent of c_3: the upper bound of $u(x_0)$. See also Dancer [7] and Poláčik–Quittner–Souplet [16] for the case $V_1 \equiv 0$, $V_2 \equiv 1$ and $1 < p < (N+2)/(N-2)$. Note that

$$1 - N > -\frac{2}{p-1} \quad \Longleftrightarrow \quad p < \frac{N+1}{N-1},$$

and also that the growth rate of a positive solution of the Lane–Emden equation with an isolated singularity on $\partial\Omega$ differs among the cases $1 < p < (N+1)/(N-1)$, $p = (N+1)/(N-1)$ and $(N+1)/(N-1) < p < (N+2)/(N-2)$ (see Bidaut-Véron–Ponce–Véron [3] for details). Moreover, since our setting includes the case that V_2 is the indicator function of a set K in Ω (that is, positive solutions are

$(-\Delta + V_1)$-harmonic on $\Omega \setminus \overline{K}$), estimate (3) is sharp even when $(N+1)/(N-1) \le p < N/(N-2)$. Thus the boundary behavior of positive solutions of (1) depends on both of p and V_2.

Remark 1 If V_2 has a positive lower bound on some $B(x_0, r) \subset \Omega$, then we can take the constants c in (3) and (4) independently of c_3. This can be seen by applying (6) with $\Omega = B(x_0, r)$ to u or to $\widetilde{u}$ defined by (36) when V_1 is not nonnegative: In fact, $u(x_0) \le c\widetilde{u}(x_0) \le cr^{-2/(p-1)}$.

Remark 2 Theorem 1 and Corollary 1 are valid for the case $\alpha_2 = N + 1 - p(N-1)$ as well (see their proof). But we see from [8, Theorem 1.5] that Theorem 1 does not hold for $\alpha_2 > N + 1 - p(N-1)$ in general.

The following theorem reveals the vanishing rate of $u \in \mathscr{S}(\Omega)$ vanishing on a portion of $\partial\Omega$. For $0 < r < r_\Omega$ and $\xi \in \partial\Omega$, we let $\xi_r := \xi + r\boldsymbol{n}_\xi$, where $\boldsymbol{n}_\xi$ is the inward unit normal vector to $\partial\Omega$ at ξ.

Theorem 2 *There exist $c_4 = c(\alpha_1, \alpha_2, p, N) \ge 4$, $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) \ge 1$ and $r_1 = r(\alpha_1, \alpha_2, p, \Omega) > 0$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_1$. If $u \in \mathscr{S}(\Omega)$ vanishes continuously on $\partial\Omega \cap B(\xi, c_4 r)$, then*

$$\frac{1}{c}\frac{u(\xi_r)}{r}\,\delta_\Omega(x) \le u(x) \le c\frac{u(\xi_r)}{r}\,\delta_\Omega(x) \quad \text{for all } x \in \Omega \cap B(\xi, r). \tag{7}$$

Corollary 2 *There exists $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) \ge 1$ such that*

$$u(x) \le c\,\delta_\Omega(x) \quad \text{for all } x \in \Omega \tag{8}$$

and

$$|\nabla u(x)| \le c \quad \text{for all } x \in \Omega, \tag{9}$$

whenever $u \in \mathscr{S}(\Omega)$ vanishes continuously on the whole of $\partial\Omega$.

Remark 3 By the same reasoning as in Remark 1, we can take the constants c, appearing in all the estimates in Theorem 2 and Corollary 2, independently of c_3 if V_2 has a positive lower bound on some $B(x_0, r) \subset \Omega$.

As another application of Theorem 2, we present results concerning the asymptotic behavior of solutions in $\mathscr{S}(\Omega)$ having isolated singularities on $\partial\Omega$ and the removability of isolated boundary singularities in Sect. 4.

The above results are new at least for the case $V_1 \not\equiv 0$, whereas those in the case $V_1 \equiv 0$ were proved by the author [8, 10] (Theorem 2 was obtained under $\alpha_2 = 0$), using potential theory and an iteration method.

The plan of this paper is as follows. In Sect. 2, we prove the above theorems in the case $V_1 \le 0$ by using superharmonicity of positive solutions of (1) and modifying the arguments in the case $V_1 \equiv 0$. When V_1 takes positive values, we cannot apply our previous argument. We try to find a positive superharmonic function $\widetilde{u}$ on Ω such that it is comparable to a given $u \in \mathscr{S}(\Omega)$ and the above theorems hold for $\widetilde{u}$, by using the perturbation theory of the Green function and potential theory for the linear Schrödinger equation. Proofs of the above theorems are given in Sect. 3, and their applications and some remarks are given in Sect. 4.

2 The Case $V_1 \le 0$

This section presents proofs of our results in the case $V_1 \le 0$. In light of an application to the case that V_1 takes positive values, we address the problems for a slightly general class which includes $\mathscr{S}(\Omega)$ when $V_1 \le 0$.

2.1 Positive Superharmonic Functions Satisfying a Nonlinear Inequality

A function $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ is called *superharmonic* on Ω if the following conditions are fulfilled:

(i) $u \not\equiv +\infty$,
(ii) u is lower semicontinuous on Ω,
(iii) for any $x \in \Omega$ and $0 < r < \delta_\Omega(x)$, we have

$$u(x) \ge \fint_{B(x,r)} u(y)\, dy,$$

where the symbol $\fint_{B(x,r)}$ stands for the integral average over the open ball $B(x, r)$ of center x and radius r.

For $v \in L^1_{\mathrm{loc}}(\Omega)$, it is known that there exists a superharmonic function u on Ω such that $u = v$ a.e. on Ω if and only if v is a distributional supersolution of the Laplace equation in Ω in the sense that the *distributional Laplacian* of v defined by

$$(-\Delta v)(\phi) := \int_\Omega v(-\Delta\phi)\, dx$$

is nonnegative for all nonnegative $\phi \in C_c^\infty(\Omega)$. Observe that, for continuous functions, the notions of superharmonic functions and distributional supersolutions are equivalent. Also, it is known that a superharmonic function on Ω is in $L^1_{\mathrm{loc}}(\Omega)$ and its distributional Laplacian is regarded as a Radon measure on Ω.

We consider the collection $\mathscr{U}(\Omega)$ of all positive superharmonic functions u on Ω satisfying

(U1) $u(x_0) \le c_3$,

(U2) the distributional Laplacian $-\Delta u \in L^1_{\rm loc}(\Omega)$,
(U3) $-\Delta u(x) \le c_1\, \delta_\Omega(x)^{-\alpha_1} u(x) + c_2\, \delta_\Omega(x)^{-\alpha_2} u(x)^p$ for a.e. $x \in \Omega$.

Though $\mathscr{U}(\Omega)$ depends on the parameters $c_1, c_2, c_3, \alpha_1, \alpha_2, p$ (these are the same as in the introduction), we do not express them in the symbol $\mathscr{U}(\Omega)$ for simplicity. Observe from (V1) and (V2) that $\mathscr{S}(\Omega) \subset \mathscr{U}(\Omega)$ if $V_1 \le 0$.

2.2 Boundary Growth Estimate

In this subsection, we prove the following.

Theorem 3 *There exist* $c = c(c_1, c_2, \alpha_1, \alpha_2, p, \Omega) > 0$ *and* $\beta = \beta(p, N) \ge 1$ *such that*

$$u(x) \le c\, \delta_\Omega(x)^{1-N} \max\{u(x_0), 1\}^\beta \quad \text{for all } x \in \Omega, \tag{10}$$

whenever $u \in \mathscr{U}(\Omega)$.

Corollary 3 $\mathscr{U}(\Omega) \subset C^1(\Omega)$. *Moreover, there exist* $c = c(c_1, c_2, \alpha_1, \alpha_2, p, \Omega) > 0$ *and* $\beta = \beta(p, N) \ge 1$ *such that*

$$|\nabla u(x)| \le c\, \delta_\Omega(x)^{-N} \max\{u(x_0), 1\}^{\beta p} \quad \text{for all } x \in \Omega,$$

whenever $u \in \mathscr{U}(\Omega)$.

Remark 4 All the results regarding $\mathscr{U}(\Omega)$ in Sects. 2.2 and 2.3 are valid under $\alpha_1 \le 2$ and $\alpha_2 \le N + 1 - p(N-1)$.

In the arguments below, we suppose $u \in \mathscr{U}(\Omega)$. A proof of Theorem 3 is based on the author's previous work [8]. The next two lemmas follow from the Riesz decomposition theorem for superharmonic functions, the Martin (Poisson) integral representation of harmonic functions and the Green function estimate.

Lemma 1 ([8, Lemma 2.3]) *There exists* $c = c(\Omega) > 0$ *such that*

$$\int_\Omega \delta_\Omega(y)(-\Delta u(y))\, dy \le c u(x_0).$$

Lemma 2 ([8, Lemma 2.4]) *Let* $j \in \mathbb{N}$. *Then there exists* $c_j = c(j, \Omega) > 0$ *such that*

$$u(x) \le c_j \left\{ \delta_\Omega(z)^{1-N} u(x_0) + \int_{B(z, 2^{-j}\delta_\Omega(z))} \frac{-\Delta u(y)}{|x-y|^{N-2}}\, dy \right\} \tag{11}$$

for all $z \in \Omega$ *and* $x \in B(z, 2^{-j-1}\delta_\Omega(z))$.

For simplicity, we use the symbol $v_1 \lesssim v_2$ if two positive functions v_1 and v_2 satisfy $v_1(x) \le c v_2(x)$ for some constant c independent of variable x. If both of $v_1 \lesssim v_2$ and $v_2 \lesssim v_1$ hold, then we write $v_1 \approx v_2$.

In the arguments below, we fix $z \in \Omega$ and show (10) for $x = z$. For simplicity, we write $B_j := B(0, 2^{-j})$ and

$$\psi(\eta) := \delta_\Omega(z)^{N+1}(-\Delta u(z + \delta_\Omega(z)\eta)) \quad \text{for } \eta \in B_0.$$

Since $-\Delta u \geq 0$ a.e. on Ω and $\delta_\Omega(y) \approx \delta_\Omega(z)$ for all $y \in B(z, \delta_\Omega(z)/2)$, it follows from Lemma 1 and the change of variables that

$$\|\psi\|_{L^1(B_1)} \lesssim u(x_0). \tag{12}$$

Also, we note from (2) and the boundedness of Ω that

$$\delta_\Omega(x)^{-\alpha_1} \lesssim \delta_\Omega(x)^{-2} \quad \text{and} \quad \delta_\Omega(x)^{-\alpha_2} \lesssim \delta_\Omega(x)^{p(N-1)-N-1} \quad \text{for all } x \in \Omega. \tag{13}$$

Lemma 3 *Let*

$$s := \frac{1}{2}\left(p + \frac{N}{N-2}\right) \quad \textit{and} \quad \ell := \left[\frac{\log(s/(s-1))}{\log(s/p)}\right] + 1,$$

where $[\,\cdot\,]$ *is the Gauss symbol. Let* $\kappa \geq 1$. *Then there exists* $c = c(\kappa, c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) > 0$ *such that*

$$\|\psi\|_{L^{\frac{\kappa s}{p}}(B_{j+1})} \leq c\big(\max\{u(x_0), 1\}^p + \|\psi\|^p_{L^\kappa(B_j)}\big) \quad \textit{for all } j \in \{1, \dots, \ell\}.$$

Proof Let $j \in \{1, \dots, \ell\}$ and $c_0 := \max\{c_1, \dots, c_\ell\}$. We write

$$\Psi_j(\eta) := \int_{B_j} \frac{\psi(\zeta)}{|\eta - \zeta|^{N-2}}\, d\zeta.$$

Making the change of variables $x = z + \delta_\Omega(z)\eta$ and $y = z + \delta_\Omega(z)\zeta$ in (11), we have

$$\delta_\Omega(z)^{N-1} u(z + \delta_\Omega(z)\eta) \leq c_0\{u(x_0) + \Psi_j(\eta)\} \quad \text{for all } \eta \in B_{j+1}. \tag{14}$$

It follows from (U3), (13) and (14) that for a.e. $\eta \in B_{j+1}$,

$$\begin{aligned}
\psi(\eta) &\lesssim \delta_\Omega(z)^{N-1} u(z + \delta_\Omega(z)\eta) + \delta_\Omega(z)^{p(N-1)} u(z + \delta_\Omega(z)\eta)^p \\
&\lesssim \{u(x_0) + \Psi_j(\eta)\} + \{u(x_0) + \Psi_j(\eta)\}^p \\
&\lesssim \big(\max\{u(x_0), 1\} + \Psi_j(\eta)\big)^p,
\end{aligned}$$

and therefore

$$\|\psi\|_{L^{\frac{\kappa s}{p}}(B_{j+1})} \lesssim \max\{u(x_0), 1\}^p + \|\Psi_j\|^p_{L^{\kappa s}(B_j)}. \tag{15}$$

To estimate the right hand side, we apply the Jensen inequality. Then

$$\Psi_j(\eta)^\kappa \le \left(\int_{B_0} \frac{d\zeta}{|\eta-\zeta|^{N-2}}\right)^{\kappa-1} \int_{B_j} \frac{\psi(\zeta)^\kappa}{|\eta-\zeta|^{N-2}}\,d\zeta \lesssim \int_{B_j} \frac{\psi(\zeta)^\kappa}{|\eta-\zeta|^{N-2}}\,d\zeta.$$

By the Minkowski inequality for integrals and $s < N/(N-2)$, we have

$$\|\Psi_j\|_{L^{\kappa s}(B_j)} \lesssim \|\psi\|_{L^\kappa(B_j)}.$$

Substituting this into (15), we obtain the desired inequality. □

Proof (*Proof of Theorem* 3) Let s and ℓ be as in Lemma 3, and let $\gamma := s/p$. Then $\gamma > 1$ and $\gamma^\ell/(\gamma^\ell - 1) \le s < N/(N-2)$. By Lemma 2 with $x = z$ and the Hölder inequality,

$$\begin{aligned}\delta_\Omega(z)^{N-1}u(z) &\le c_{\ell+1}\left(u(x_0) + \int_{B_{\ell+1}} \frac{\psi(\zeta)}{|\zeta|^{N-2}}\,d\zeta\right)\\ &\lesssim u(x_0) + \|\psi\|_{L^{\gamma^\ell}(B_{\ell+1})}.\end{aligned}$$

By the way, applying Lemma 3 ℓ times and (12), we have

$$\|\psi\|_{L^{\gamma^\ell}(B_{\ell+1})} \lesssim A + \|\psi\|_{L^1(B_1)}^{p^\ell} \lesssim A,$$

where $A := \max\{u(x_0), 1\}^p + \cdots + \max\{u(x_0), 1\}^{p^\ell}$. Hence we obtain the desired estimate. □

Since $-\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$ by (U3) and Theorem 3, the classical regularity theorem ensures $u \in C^1(\Omega)$. As a consequence of Theorem 3, we obtain the following estimate.

Lemma 4 *There exists* $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) > 0$ *such that*

$$\int_{B(x,r)} |\nabla u|^2\,dy \le cr^{-N} \max\{u(x_0), 1\}^{\beta(p+1)},$$

whenever $u \in \mathscr{U}(\Omega)$ *and* $B(x, 3r) \subset \Omega$.

Proof Since $u \in C^1(\Omega)$ and $-\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, we see from the Riesz representation theorem and (U3) that for all nonnegative $\phi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla\phi\,dy &= \int_\Omega u(-\Delta\phi)\,dy = (-\Delta u)(\phi) = \int_\Omega (-\Delta u)\phi\,dy\\ &\lesssim \int_\Omega \{\delta_\Omega(y)^{-\alpha_1}u(y) + \delta_\Omega(y)^{-\alpha_2}u(y)^p\}\phi(y)\,dy.\end{aligned} \tag{16}$$

Observe from the standard approximation that (16) holds for all nonnegative $\phi \in C_c^1(\Omega)$. Let $B(x, 3r) \subset \Omega$. Take $\psi \in C_c^\infty(B(x, 2r))$ satisfying $0 \le \psi \le 1$, $|\nabla\psi| \le 2/r$ on $B(x, 2r)$ and $\psi = 1$ on $B(x, r)$. Substituting $\phi := u\psi^2 \in C_c^1(\Omega)$ into (16), we have by (13) that

$$\int_{B(x,2r)} |\nabla u|^2 \psi^2 \, dy \le 2 \int_{B(x,2r)} u\psi |\nabla u||\nabla\psi| \, dy \\ + c\left(r^{-2} \int_{B(x,2r)} u^2 \, dy + r^{p(N-1)-N-1} \int_{B(x,2r)} u^{p+1} \, dy \right).$$

Using the Young inequality

$$u\psi|\nabla u||\nabla\psi| \le \frac{(\psi|\nabla u|)^2}{4} + (u|\nabla\psi|)^2 \le \frac{(\psi|\nabla u|)^2}{4} + 4\left(\frac{u}{r}\right)^2,$$

we have by (10) that

$$\int_{B(x,2r)} |\nabla u|^2 \psi^2 \, dy \lesssim r^{-2} \int_{B(x,2r)} u^2 \, dy + r^{p(N-1)-N-1} \int_{B(x,2r)} u^{p+1} \, dy \\ \lesssim r^{-N} \max\{u(x_0), 1\}^{\beta(p+1)}. \tag{17}$$

Thus the lemma follows. □

Then we notice that the following gradient estimate established by Kuusi–Mingione [12] leads to our pointwise estimate for $|\nabla u|$.

Lemma 5 ([12, Theorem 1.1]) *Let $v \in C^1(\Omega)$ be a weak solution of $-\Delta v = \mu$ in Ω, where μ is a finite signed measure on Ω. Then there exists $c = c(N) > 0$ such that*

$$|\nabla v(x)| \le c\left(\int_0^r \frac{|\mu|(B(x,t))}{t^{N-1}} \frac{dt}{t} + \fint_{B(x,r)} |\nabla v| \, dy \right),$$

whenever $B(x, r) \subset \Omega$.

Proof (*Proof of Corollary 3*) As mentioned above, $u \in C^1(\Omega)$. Applying the Hölder inequality and Lemma 4 with $r := \delta_\Omega(x)/3$, we have

$$\fint_{B(x,r)} |\nabla u| \, dy \le \left(\fint_{B(x,r)} |\nabla u|^2 \, dy \right)^{1/2} \lesssim \delta_\Omega(x)^{-N} \max\{u(x_0), 1\}^{\frac{\beta(p+1)}{2}}.$$

Also, (U3), (13) and Theorem 3 yield

$$\int_0^r t^{1-N} \left(\int_{B(x,t)} (-\Delta u) \, dy \right) \frac{dt}{t} \lesssim \delta_\Omega(x)^{-N} \max\{u(x_0), 1\}^{\beta p}.$$

Therefore the conclusion follows from Lemma 5. □

2.3 Harnack Inequality

The Harnack inequality is the interior estimate that

$$\sup_{B(x,r)} u \le c_5 \inf_{B(x,r)} u \quad \text{whenever } B(x,2r) \subset \Omega, \tag{18}$$

where the constant c_5 is independent of x, r and u. Following the previous work [9], let us show this for the class $\mathscr{U}(\Omega)$. Our approach is based on the boundary growth estimate (Theorem 3) and classical potential theory. As a consequence, we see that the lower constraint like $-\Delta u \ge cu^p$ is not need.

Let $u \in \mathscr{U}(\Omega)$. By superharmonicity, we have

$$u(x) \ge \fint_{B(x,r)} u(y)\,dy \ge \fint_{\partial B(x,r)} u\,d\sigma, \tag{19}$$

whenever $\overline{B(x,r)} \subset \Omega$. Here σ is surface area measure on $\partial B(x,r)$. We need the opposite inequalities in some sense to obtain the Harnack inequality.

Let $u \in \mathscr{U}(\Omega)$. By Theorem 3 and (13), we have

$$\delta_\Omega(x)^{-\alpha_2} u(x)^{p-1} \lesssim \delta_\Omega(x)^{-\alpha_2+(p-1)(1-N)} \lesssim \delta_\Omega(x)^{-2} \quad \text{for all } x \in \Omega.$$

Thus, by (U3), there exists $c_6 = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) > 0$ such that

$$-\Delta u(x) \le c_6\, \delta_\Omega(x)^{-2} u(x) \quad \text{for a.e. } x \in \Omega. \tag{20}$$

Theorem 4 (Strong Harnack inequality) *Let* $0 < \kappa < 1$ *and let* c_6 *be as in* (20). *Put*

$$c_\kappa := \kappa \min\left\{\sqrt{\frac{N\kappa}{8c_6}}, \frac{1}{8}\right\}.$$

If u *is a positive superharmonic function on* Ω *with the distributional Laplacian* $-\Delta u \in L^1_{\mathrm{loc}}(\Omega)$ *satisfying* (20)*, then*

$$\frac{1-\kappa}{(1+\kappa)^N} u(x) \le u(y) \le \frac{(1+\kappa)^N}{1-\kappa} u(x) \tag{21}$$

for any pair of $x, y \in \Omega$ *satisfying* $|x-y| \le c_\kappa\, \delta_\Omega(x)$. *In particular, the conclusion is true for all* $u \in \mathscr{U}(\Omega)$.

Proof Let $x \in \Omega$. We first show that

$$(1-\kappa)u(x) \le (1+\kappa)^N u(y) \quad \text{for all } y \in B(x, 2c_\kappa\, \delta_\Omega(x)). \tag{22}$$

To this end, let $r := 2\kappa^{-1} c_\kappa\, \delta_\Omega(x)$. Since $\delta_\Omega(z) \ge \delta_\Omega(x)/2$ for all $z \in B(x,r)$, it follows from the classical theorem by Riesz, (20) and (19) that

$$u(x) - \fint_{\partial B(x,r)} u\, d\sigma = \frac{1}{N}\int_0^r t\left(\fint_{B(x,t)} (-\Delta u(z))\, dz\right) dt$$
$$\leq \frac{2c_6}{N\,\delta_\Omega(x)^2} u(x) \leq \kappa u(x),$$

whence

$$(1-\kappa)u(x) \leq \fint_{\partial B(x,r)} u\, d\sigma. \tag{23}$$

By the way, if $y \in B(x, \kappa r)$, then $B(x,r) \subset B(y, r + |x-y|) \subset B(x, 3\,\delta_\Omega(x)/4) \subset \Omega$, and so

$$\fint_{B(x,r)} u(z)\, dz \leq \left(\frac{r+|x-y|}{r}\right)^N \fint_{B(y,r+|x-y|)} u(z)\, dz$$
$$\leq (1+\kappa)^N u(y). \tag{24}$$

Combining (19), (23) and (24), we obtain (22).

Let $y \in B(x, c_\kappa\, \delta_\Omega(x))$. Then $x \in B(y, 2c_\kappa\, \delta_\Omega(y))$. Changing the position of x and y in (22) yields $(1-\kappa)u(y) \leq (1+\kappa)^N u(x)$. This completes the proof.

Remark 5 By the standard Harnack chain argument, we can obtain (18) from the conclusion of Theorem 4. Moreover, (21) is stronger than (18), since (21) includes the information that $u(y)$ converges uniformly to $u(x)$ as $\kappa \to 0$. This enables us to obtain the result regarding the existence of nontangential boundary limits of $u \in \mathscr{U}(\Omega)$. See [9, Theorem 6.1].

2.4 Boundary Harnack Principle

The aim of this section is to prove the following.

Theorem 5 *There exist $c_4 = c(\alpha_1, \alpha_2, p, N) \geq 4$, $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) \geq 1$ and $r_1 = r(\alpha_1, \alpha_2, p, \Omega) > 0$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_1$. If $u \in \mathscr{U}(\Omega)$ vanishes continuously on $\partial\Omega \cap B(\xi, c_4 r)$, then (7) holds.*

By $\mathscr{H}_+(\Omega)$, we denote the collection of all positive harmonic functions on Ω. The following is an immediate consequence of Theorem 5.

Corollary 4 (Boundary Harnack principle) *The constants c_4, r_1 are the same as in Theorem 5. Let $\xi \in \partial\Omega$ and $0 < r < r_1$. If $u, v \in \mathscr{U}(\Omega) \cup \mathscr{H}_+(\Omega)$ vanish continuously on $\partial\Omega \cap B(\xi, c_4 r)$, then*

$$\frac{1}{c}\frac{u(x)}{v(x)} \leq \frac{u(y)}{v(y)} \leq c\frac{u(x)}{v(x)} \quad \text{for all } x, y \in \Omega \cap B(\xi, r), \tag{25}$$

where $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) > 1$.

In [10], the author could obtain the above results under the assumption that $c_1 = 0$, $\alpha_2 = 0$ and Ω is a bounded Lipschitz domain satisfying the uniform exterior ball condition. Using Lemma 9, we modify our previous arguments.

Let us recall the result for harmonic functions.

Lemma 6 (e.g. [2]) *There exist $c = c(\Omega) \geq 1$ and $r_2 = r(\Omega) > 0$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_2$. If h is a positive harmonic function on $\Omega \cap B(\xi, 4r)$ vanishing continuously on $\partial\Omega \cap B(\xi, 4r)$, then*

$$\frac{1}{c}\frac{h(\xi_r)}{r}\,\delta_\Omega(x) \leq h(x) \leq c\frac{h(\xi_r)}{r}\,\delta_\Omega(x) \quad \text{for all } x \in \Omega \cap B(\xi, r).$$

Remark 6 In [2], the intersections with a bigger ball were considered instead of $B(\xi, 4r)$, but we can modify it by the use of Lemma 18 if $r > 0$ is sufficiently small. For $-\Delta u + Vu = 0$ with an unbounded potential V, estimate (7) is known to hold for positive solutions in the whole of Ω, not in the restricted place $\Omega \cap B(\xi, 4r)$ (see [6, 18]), but this is not applicable to our argument.

The Green function on an open set D for the Dirichlet–Laplacian is denoted by G_D. Recall that $\xi_r := \xi + r\boldsymbol{n}_\xi$ for $\xi \in \partial\Omega$ and $0 < r < r_\Omega$, where $\boldsymbol{n}_\xi$ is the inward unit normal vector to $\partial\Omega$ at ξ.

Lemma 7 *Assume that u is a positive superharmonic function on Ω enjoying the Harnack inequality (18) with constant c_5. Then there exist $c = c(c_5, \Omega) > 0$ and $r_3 = r(\Omega) > 0$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_3$. Then*

$$\frac{u(\xi_r)}{r}\,\delta_\Omega(x) \leq cu(x) \quad \text{for all } x \in \Omega \cap B(\xi, r).$$

Proof Let $G(x) := G_{\Omega\cap B(\xi,6r)}(x, \xi_{4r})$. The Harnack inequality gives

$$\frac{G(x)}{G(\xi_r)} \approx 1 \approx \frac{u(x)}{u(\xi_r)} \quad \text{for all } x \in \partial B(\xi_{4r}, r).$$

It follows from the minimum principle and Lemma 6 that

$$\frac{u(x)}{u(\xi_r)} \gtrsim \frac{G(x)}{G(\xi_r)} \approx \frac{\delta_\Omega(x)}{r} \quad \text{for all } x \in \Omega \cap B(\xi, r).$$

Thus the lemma is proved. □

In order to obtain Theorem 5 for $\alpha_2 > 0$, we need the following estimate, which is also used in the study of removable isolated singularities on $\partial\Omega$ in Sect. 4.2.

Lemma 8 *Assume $0 \in \partial\Omega$. Let $\beta > -2$ and $\gamma > -N - 1 - \beta$. Then there exists $c = c(\beta, \gamma, \Omega) > 0$ such that*

$$\int_\Omega G_\Omega(x, y)\,\delta_\Omega(y)^\beta |y|^\gamma\,dy \leq c\,\delta_\Omega(x)\{J_1(x) + J_2(x)\} \quad \text{for all } x \in \Omega,$$

where

$$J_1(x) := \begin{cases} |x|^{1+\beta+\gamma} & \text{if } \beta > -1, \\ |x|^\gamma \log \dfrac{2|x|}{\delta_\Omega(x)} & \text{if } \beta = -1, \\ \delta_\Omega(x)^{1+\beta}|x|^\gamma & \text{if } \beta < -1, \end{cases}$$

and

$$J_2(x) := \begin{cases} |x|^{1+\beta+\gamma} & \text{if } 1+\beta+\gamma < 0, \\ \log \dfrac{\operatorname{diam}\Omega}{|x|} & \text{if } 1+\beta+\gamma = 0, \\ 1 & \text{if } 1+\beta+\gamma > 0. \end{cases}$$

Proof Fix $x \in \Omega$ and split the range of integration into four parts:

$$E_1 := B(x, \delta_\Omega(x)/2), \qquad E_2 := \Omega \cap B(x, |x|/2) \setminus E_1,$$
$$E_3 := \Omega \cap B(0, 2|x|) \setminus E_2, \quad E_4 := \Omega \setminus (E_1 \cup E_2 \cup E_3).$$

We use the Green function estimate:

$$G_\Omega(x, y) \lesssim \min\left\{ \frac{1}{|x-y|^{N-2}}, \frac{\delta_\Omega(x)\,\delta_\Omega(y)}{|x-y|^N} \right\}.$$

If $y \in E_1$, then $\delta_\Omega(y) \approx \delta_\Omega(x)$ and $|y| \approx |x|$, and so

$$\int_{E_1} G_\Omega(x, y)\,\delta_\Omega(y)^\beta |y|^\gamma\,dy \lesssim \delta_\Omega(x)^{2+\beta}|x|^\gamma.$$

If $y \in E_2$, then $|y| \approx |x|$. Let $A_j := B(x, 2^j\,\delta_\Omega(x)) \setminus B(x, 2^{j-1}\,\delta_\Omega(x))$ and let m be the smallest integer such that $2^m\,\delta_\Omega(x) \geq |x|/2$. Then $A_j \subset B(\eta, 2^{j+1}\,\delta_\Omega(x))$ for some $\eta \in \partial\Omega$. By $\beta > -2$, we have

$$\begin{aligned}
\int_{E_2} G_\Omega(x, y)\,\delta_\Omega(y)^\beta |y|^\gamma\,dy &\lesssim \delta_\Omega(x)|x|^\gamma \sum_{j=0}^{m} \int_{\Omega\cap A_j} \frac{\delta_\Omega(y)^{1+\beta}}{|x-y|^N}\,dy \\
&\lesssim \delta_\Omega(x)|x|^\gamma \sum_{j=0}^{m} (2^j\,\delta_\Omega(x))^{-N} \int_{\Omega\cap B(\eta, 2^{j+1}\,\delta_\Omega(x))} \delta_\Omega(y)^{1+\beta}\,dy \\
&\lesssim \delta_\Omega(x)|x|^\gamma \sum_{j=0}^{m} (2^j\,\delta_\Omega(x))^{1+\beta} \\
&\lesssim \delta_\Omega(x) J_1(x).
\end{aligned}$$

If $y \in E_3$, then $|x-y| \approx |x|$. Let $A'_j := B(0, 2^{-j+1}|x|) \setminus B(0, 2^{-j}|x|)$. Then

$$\begin{aligned}\int_{E_3} G_\Omega(x,y)\,\delta_\Omega(y)^\beta |y|^\gamma\,dy &\lesssim \frac{\delta_\Omega(x)}{|x|^N} \sum_{j=0}^{\infty} \int_{\Omega\cap A'_j} \delta_\Omega(y)^{1+\beta}|y|^\gamma\,dy \\ &\lesssim \frac{\delta_\Omega(x)}{|x|^N} \sum_{j=0}^{\infty} (2^{-j}|x|)^\gamma \int_{\Omega\cap B(0,2^{-j+1}|x|)} \delta_\Omega(y)^{1+\beta}\,dy \\ &\lesssim \delta_\Omega(x)|x|^{1+\beta+\gamma} \sum_{j=0}^{\infty} 2^{-j(N+1+\beta+\gamma)}.\end{aligned}$$

Since $N+1+\beta+\gamma > 0$, the last series converges.

If $y \in E_4$ (assuming $E_4 \neq \emptyset$), then $|x-y| \approx |y|$. Let $A''_j := B(0, 2^{j+1}|x|) \setminus B(0, 2^j|x|)$ and let n be the smallest integer such that $2^{n+1}|x| \geq \operatorname{diam}\Omega$. Then

$$\begin{aligned}\int_{E_4} G_\Omega(x,y)\,\delta_\Omega(y)^\beta |y|^\gamma\,dy &\lesssim \delta_\Omega(x) \sum_{j=1}^{n} \int_{\Omega\cap A''_j} \frac{\delta_\Omega(y)^{1+\beta}}{|y|^{N-\gamma}}\,dy \\ &\lesssim \delta_\Omega(x) \sum_{j=1}^{n} (2^j|x|)^{\gamma-N} \int_{\Omega\cap B(0,2^{j+1}|x|)} \delta_\Omega(y)^{1+\beta}\,dy \\ &\lesssim \delta_\Omega(x)|x|^{1+\beta+\gamma} \sum_{j=1}^{n} 2^{j(1+\beta+\gamma)} \\ &\lesssim \delta_\Omega(x) J_2(x).\end{aligned}$$

Combining all the estimates yields the desired estimate. □

The special case $\gamma = 0$ becomes as follows.

Lemma 9 *Let $\beta > -2$. Then there exists $c = c(\beta, \Omega) > 0$ such that*

$$\int_\Omega G_\Omega(x,y)\,\delta_\Omega(y)^\beta\,dy \leq cJ(x) \quad \text{for all } x \in \Omega,$$

where

$$J(x) := \begin{cases} \delta_\Omega(x) & \text{if } \beta > -1, \\ \delta_\Omega(x) \log \dfrac{2\operatorname{diam}\Omega}{\delta_\Omega(x)} & \text{if } \beta = -1, \\ \delta_\Omega(x)^{\beta+2} & \text{if } \beta < -1. \end{cases}$$

Lemma 10 *There exist $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) > 0$ and $\gamma = \gamma(\alpha_1, \alpha_2, p, \Omega) > 0$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_\Omega/2$. Then*

$$u(x) \le c\left(\frac{|x-\xi|}{r}\right)^{\gamma} \sup_{\Omega\cap B(\xi,r)} u \quad \textit{for all } x \in \Omega \cap B(\xi, r),$$

whenever $u \in \mathscr{U}(\Omega)$ *vanishes continuously on* $\partial\Omega \cap B(\xi, r)$.

Proof We may assume $\xi = 0$. Let $x \in \Omega_r := \Omega \cap B(0, r)$. By the Riesz decomposition theorem, there exists a nonnegative harmonic function h on Ω_r such that

$$u(x) = h(x) + \int_{\Omega_r} G_{\Omega_r}(x, y)(-\Delta u(y))\, dy.$$

As is well known, by comparing h with the Martin kernel $|x|^{\gamma_1} f(x/|x|)$ with pole at ∞ of the complement of the exterior cone of Ω with vertex at 0, we have

$$h(x) \lesssim \left(\frac{|x|}{r}\right)^{\gamma_1} \sup_{\Omega_r} u.$$

Let $\beta := -\max\{\alpha_1, \alpha_2 + (N-1)(p-1)\}$. Then $\beta > -2$ by (2). Also, by (U3) and Theorem 3,

$$-\Delta u(y) \lesssim \delta_\Omega(y)^\beta \sup_{\Omega_r} u \quad \text{for a.e. } y \in \Omega_r.$$

Since $G_{\Omega_r} \le G_\Omega$ on $\Omega_r \times \Omega_r$, it follows from Lemma 9 that

$$\int_{\Omega_r} G_{\Omega_r}(x, y)(-\Delta u(y))\, dy \lesssim \delta_\Omega(x)^{\gamma_2} \sup_{\Omega_r} u \lesssim \left(\frac{|x|}{r}\right)^{\gamma_2} \sup_{\Omega_r} u$$

for some $\gamma_2 > 0$. Thus the lemma follows. □

Lemma 11 (Carleson estimate) *There exist* $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, \Omega) > 0$ *and* $r_4 = r(\Omega) > 0$ *with the following property: Let* $\xi \in \partial\Omega$ *and* $0 < r < r_4$. *If* $u \in \mathscr{U}(\Omega)$ *vanishes continuously on* $\partial\Omega \cap B(\xi, 2r)$*, then*

$$u(x) \le cu(\xi_r) \quad \textit{for all } x \in \Omega \cap B(\xi, r). \tag{26}$$

Proof An idea of the proof is based on [5, 11]. First, we observe from Theorem 4 (Remark 5) and the standard Harnack chain argument (Lemma 18) that there exists a constant $\lambda > 1$ such that

$$u(x) \le \left(\frac{4r}{\delta_\Omega(x)}\right)^{\lambda} u(\xi_r) \quad \text{for all } x \in \Omega \cap B(\xi, 2r),$$

or equivarently

$$\delta_\Omega(x) \le 4\left(\frac{u(\xi_r)}{u(x)}\right)^{\frac{1}{\lambda}} r \quad \text{for all } x \in \Omega \cap B(\xi, 2r). \tag{27}$$

Also, by Lemma 10, there exists $c_7 > 1$ such that

$$u(x) < \frac{1}{2^\lambda} \sup_{\Omega \cap B(\eta_x, c_7\,\delta_\Omega(x))} u, \tag{28}$$

whenever $x \in \Omega \cap B(\xi, 2r)$ satisfies $c_7\,\delta_\Omega(x) < r$ and $\eta_x \in \partial\Omega$ is such that $\delta_\Omega(x) = |\eta_x - x|$ (a similar notation is used in the argument below to denote the nearest boundary point). Put $c_8 := 1/(33c_7)$.

Let us show (26) with $c = 1/c_8^\lambda$. To this end, we suppose to the contrary that there exists $x_1 \in \Omega \cap B(\xi, r)$ such that $c_8^\lambda u(x_1) > u(\xi_r)$. Then $\delta_\Omega(x_1) < 4c_8 r < r/c_7$ by (27). Therefore, by (28), we find $x_2 \in \Omega \cap B(\eta_{x_1}, c_7\,\delta_\Omega(x_1))$ with $2^\lambda u(x_1) \le u(x_2)$. Then

$$\begin{aligned} |x_2 - \xi| &\le |x_2 - \eta_{x_1}| + |\eta_{x_1} - x_1| + |x_1 - \xi| \le 2c_7\,\delta_\Omega(x_1) + r \\ &\le (8c_7c_8 + 1)r \le \frac{3}{2}r \end{aligned}$$

and $c_8^\lambda u(x_2) > 2^\lambda u(\xi_r)$, which implies $\delta_\Omega(x_2) < (4c_8 r)/2 < r/c_7$ by (27). Repeating this process, we obtain a sequence $\{x_k\}$ in Ω such that

$$\begin{aligned} c_8^\lambda u(x_k) &> 2^{\lambda(k-1)} u(\xi_r), \\ \delta_\Omega(x_k) &< \frac{4c_8}{2^{k-1}} r < \frac{r}{c_7} \end{aligned}$$

and

$$\begin{aligned} |x_k - \xi| &\le \sum_{j=2}^{k} |x_j - x_{j-1}| + |x_1 - \xi| \le \sum_{j=2}^{k} 2c_7\,\delta_\Omega(x_{j-1}) + r \\ &\le \sum_{j=2}^{\infty} \frac{8c_7c_8}{2^{j-2}} r + r \le \frac{3}{2}r. \end{aligned}$$

But this contradicts that u vanishes continuously on $\partial\Omega \cap B(\xi, 2r)$. Thus the lemma is proved. □

We are now in a position to prove Theorem 5.

Proof (*Proof of Theorem* 5) Let $c_4 := 4^{m+3}$, where $m \in \mathbb{N}$ is chosen later, and let r_2, r_3, r_4 be as in Lemmas 6, 7, 11, respectively. Put $r_1 := \min\{r_2, r_3, r_4\}/c_4$.

Let $0 < r < r_1$ and $\xi \in \partial\Omega$. For $k \in \{0, 1, \ldots, m+2\}$, we write

$$\Omega_k := \Omega \cap B(\xi, 4^k r) \quad \text{and} \quad G_k := G_{\Omega_k}.$$

Let $u \in \mathscr{U}(\Omega)$ vanish continuously on $\partial\Omega \cap B(\xi, c_4 r)$. In light of the Harnack inequality (Theorem 4) and Lemma 7, it suffices to show the upper estimate:

$$u(x) \lesssim \frac{u(\xi_r)}{r}\, \delta_\Omega(x) \quad \text{for all } x \in \Omega_0. \tag{29}$$

Let $k \in \{1, \dots, m+2\}$. By the Riesz decomposition theorem, there exists a positive harmonic function h_k on Ω_k such that

$$u(x) = h_k(x) + \int_{\Omega_k} G_k(x, y)(-\Delta u(y))\, dy \quad \text{for all } x \in \Omega_k. \tag{30}$$

Since h_k vanishes continuously on $\partial\Omega \cap B(\xi, 4^k r)$, it follows from Lemma 6 and the Harnack inequality that for all $x \in \Omega_{k-1}$,

$$h_k(x) \lesssim \frac{h_k(\xi_{4^{k-1}r})}{r}\, \delta_\Omega(x) \lesssim \frac{h_k(\xi_r)}{r}\, \delta_\Omega(x) \lesssim \frac{u(\xi_r)}{r}\, \delta_\Omega(x). \tag{31}$$

Also, letting $\beta := -\max\{\alpha_1, \alpha_2 + (N-1)(p-1)\} > -2$, we have from (U3) and Theorem 3 that

$$-\Delta u(y) \lesssim \delta_\Omega(y)^\beta u(y) \lesssim \delta_\Omega(y)^\beta u(\xi_r) \quad \text{for a.e. } y \in \Omega_k, \tag{32}$$

since $u(y) \lesssim u(\xi_{4^k r}) \lesssim u(\xi_r)$ by Lemma 11 and the Harnack inequality.

Case 1: $\beta > -1$. Let $x \in \Omega_0$. By (32) and Lemma 9,

$$\int_{\Omega_1} G_1(x, y)(-\Delta u(y))\, dy \lesssim u(\xi_r)\, \delta_\Omega(x) \lesssim \frac{u(\xi_r)}{r}\, \delta_\Omega(x).$$

This, together with (30) and (31), yields (29).

Case 2: $\beta = -1$. Let $z \in \Omega_1$. By (32) and Lemma 9,

$$\int_{\Omega_2} G_2(z, y)(-\Delta u(y))\, dy \lesssim u(\xi_r)\, \delta_\Omega(z) \log \frac{2\operatorname{diam}\Omega}{\delta_\Omega(z)} \lesssim \frac{u(\xi_r)}{r}\, \delta_\Omega(z)^{\frac{1}{2}}.$$

Substituting this and (31) into (30), we have

$$u(z) \lesssim \frac{u(\xi_r)}{r}\, \delta_\Omega(z)^{\frac{1}{2}}.$$

Therefore

$$-\Delta u(z) \lesssim \delta_\Omega(z)^\beta u(z) \lesssim \frac{u(\xi_r)}{r}\, \delta_\Omega(z)^{\beta + \frac{1}{2}} \quad \text{for a.e. } z \in \Omega_1.$$

Again, we apply Lemma 9 to obtain

$$\int_{\Omega_1} G_1(x,z)(-\Delta u(z))\,dz \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(x) \quad \text{for all } x \in \Omega_1.$$

Then (30) and (31) yield (29).

Case 3: $\beta < -1$. Let $\gamma_1 := \beta + 2$. Since $0 < \gamma_1 < 1$, we take the largest $m \in \mathbb{N}$ with $m\gamma_1 < 1$. Define

$$\gamma_k := k\gamma_1 \quad \text{for } k \in \{0, \dots, m+2\}.$$

Note that $\gamma_0 = 0$, $0 < \gamma_k < 1$ for all $k \in \{1, \dots, m\}$ and $\gamma_{m+1} \geq 1$.

We claim that, for each $k \in \{0, \dots, m\}$, we have

$$u(x) \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(x)^{\gamma_k} \quad \text{for all } x \in \Omega_{m+2-k}. \tag{33}$$

Let us show this by induction. The case $k = 0$ follows from Lemma 11 and the Harnack inequality. In fact, for all $x \in \Omega_{m+2}$,

$$u(x) \lesssim u(\xi_{4^{m+2}r}) \lesssim u(\xi_r) \lesssim \frac{u(\xi_r)}{r}.$$

Assume that (33) holds for some $k \in \{0, \dots, m-1\}$. Then, for a.e. $y \in \Omega_{m+2-k}$,

$$-\Delta u(y) \lesssim \delta_\Omega(y)^\beta u(y) \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(y)^{\beta+\gamma_k}. \tag{34}$$

Since $\beta + \gamma_k = \gamma_{k+1} - 2 \in (-2, -1)$, it follows from (34) and Lemma 9 that

$$\int_{\Omega_{m+2-k}} G_{m+2-k}(x,y)(-\Delta u(y))\,dy \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(x)^{\gamma_{k+1}} \quad \text{for all } x \in \Omega_{m+2-k}.$$

This, together with (30) and (31), yields that

$$u(x) \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(x)^{\gamma_{k+1}} \quad \text{for all } x \in \Omega_{m+1-k}.$$

Thus the claim holds.

By (33) with $k = m$, we have

$$-\Delta u(y) \lesssim \delta_\Omega(y)^\beta u(y) \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(y)^{\beta+\gamma_m} \quad \text{for a.e. } y \in \Omega_2$$

and $\beta + \gamma_m = \gamma_{m+1} - 2 \geq -1$. If $\beta + \gamma_m > -1$, then we can apply Lemma 9 to obtain

$$u(x) \lesssim \frac{u(\xi_r)}{r}\,\delta_\Omega(x) \quad \text{for all } x \in \Omega_1 \supset \Omega_0.$$

If $\beta + \gamma_m = -1$, then we apply Lemma 9 twice to obtain (29), as in the Case 2.

This completes the proof. □

3 Proofs of Theorems 1, 2 and Corollaries 1, 2

We develop some results in Sect. 2 to a general case of V_1. But we should note that the results and arguments in Sect. 2 cannot be applied to positive solutions of (1), since they are not necessarily superharmonic on Ω if V_1 takes positive values. Moreover, it seems that there is no counterpart of Lemma 6 for positive solutions of $-\Delta u + V_1^+ u = 0$ in the restricted place $\Omega \cap B(\xi, 4r)$, not in the whole of Ω, whence the argument given in the proof of Theorem 5 does not work well even if we use the decomposition (35) below. We borrow an idea from Cranston–Fabes–Zhao [6] in the study of the linear Schrödinger equation.

For $a \in \mathbb{R}$, we write

$$a^+ := \max\{a, 0\} \quad \text{and} \quad a^- := -\min\{a, 0\}.$$

Using the estimate: if $0 \le \varepsilon \le 1$, then

$$G_\Omega(x, y) \lesssim \frac{\delta_\Omega(x)\,\delta_\Omega(y)^\varepsilon}{|x - y|^{N-1+\varepsilon}} \quad \text{for all } x, y \in \Omega,$$

we see from (V1) and $\alpha_1 < 2$ that for all $x \in \Omega$,

$$\begin{aligned}\int_\Omega \frac{\delta_\Omega(y)}{\delta_\Omega(x)} G_\Omega(x, y) V_1^+(y)\,dy &\lesssim \int_\Omega \frac{\delta_\Omega(y)}{\delta_\Omega(x)} G_\Omega(x, y)\,\delta_\Omega(y)^{-\alpha_1}\,dy \\ &\lesssim \int_\Omega |x - y|^{1-N-(\alpha_1-1)^+}\,dy \lesssim 1.\end{aligned}$$

Thus $V_1^+ \in \mathcal{K}(\Omega)$ in the sense of Riahi [17, 18], by which the Green function $G_{V_1^+,\Omega}$ for the Dirichlet–Schrödinger operator $-\Delta + V_1^+$ on Ω has the following comparison (see also [1, 6, 14, 15, 21]). Note also that $G_{V_1^+,\Omega}$ is positive and continuous on $\Omega \times \Omega$ in the extended sense ([20]). Let M_Ω and $M_{V_1^+,\Omega}$ denote the Martin kernels on Ω (normalized at x_0) for $-\Delta$ and $-\Delta + V_1^+$, respectively.

Lemma 12 *There exists $c_9 = c(V_1^+, \Omega) \ge 1$ such that*

$$\frac{1}{c_9} G_\Omega(x, y) \le G_{V_1^+,\Omega}(x, y) \le G_\Omega(x, y) \quad \textit{for all } x, y \in \Omega$$

and

$$\frac{1}{c_9} M_\Omega(x, y) \le M_{V_1^+,\Omega}(x, y) \le c_9 M_\Omega(x, y) \quad \textit{for all } x \in \Omega \textit{ and } y \in \partial\Omega.$$

Lemma 13 *Assume that $0 \le f \in L^\infty_{\mathrm{loc}}(\Omega)$ for which equation $-\Delta u + V_1^+ u = f$ in Ω has a positive solution $u \in C(\Omega)$. Then there exists a nonnegative solution $h \in C(\Omega)$ of $-\Delta h + V_1^+ h = 0$ in Ω such that*

$$u(x) = h(x) + \int_\Omega G_{V_1^+,\Omega}(x, y) f(y)\, dy \quad \textit{for all } x \in \Omega.$$

Proof This may be known as the Riesz decomposition. For completeness, we give a sketch of the proof. Let $\{\Omega_k\}$ be a sequence of smooth domains such that $\overline{\Omega_k} \subset \Omega_{k+1}$ and $\bigcup_{k=1}^\infty \Omega_k = \Omega$. For $k \in \mathbb{N}$, let $h_k := u - v_k$, where

$$v_k(x) := \int_{\Omega_k} G_{V_1^+,\Omega_k}(x, y) f(y)\, dy \quad \text{for } x \in \Omega_k.$$

By the Fubini theorem, we see that h_k satisfies $-\Delta h_k + V_1^+ h_k = 0$ in Ω_k in the sense of distributions. Moreover, $h_k \in C(\Omega_k)$ by [20]. Since v_k vanishes continuously on $\partial\Omega_k$, the minimum principle ensures that $h_k > 0$ in Ω_k. Also, from the argument in [13, p. 873], we know that $G_{V_1^+,\Omega_k}$ converges increasingly to $G_{V_1^+,\Omega}$ on Ω as $k \to \infty$. By the monotone convergence,

$$\int_\Omega G_{V_1^+,\Omega}(x, y) f(y)\, dy = \lim_{k\to\infty} v_k(x) \le u(x) < \infty \quad \text{for all } x \in \Omega$$

and $\{h_k\}$ converges decreasingly to some nonnegative solution h of $-\Delta h + V_1^+ h = 0$ in Ω. Again, $h \in C(\Omega)$ by [20]. Thus the lemma is proved. □

Let $u \in \mathscr{S}(\Omega)$. By Lemma 13, there is a nonnegative solution $h \in C(\Omega)$ of $-\Delta h + V_1^+ h = 0$ in Ω such that

$$u(x) = h(x) + \int_\Omega G_{V_1^+,\Omega}(x, y)\{V_1^-(y)u(y) + V_2(y)u(y)^p\}\, dy \quad \text{for all } x \in \Omega. \tag{35}$$

By the Martin representation theorem (cf. [18, Theorem 6.1]), there exists a Radon measure μ on $\partial\Omega$ such that

$$h(x) = \int_{\partial\Omega} M_{V_1^+,\Omega}(x, y)\, d\mu(y) \quad \text{for all } x \in \Omega.$$

Using this μ, we define

$$\widetilde{u}(x) := \int_{\partial\Omega} M_\Omega(x, y)\, d\mu(y) + \int_\Omega G_\Omega(x, y)\{V_1^-(y)u(y) + V_2(y)u(y)^p\}\, dy \tag{36}$$

for $x \in \Omega$. Then $\widetilde{u}$ is a positive superharmonic function on Ω with the distributional Laplacian $-\Delta\widetilde{u} = V_1^- u + V_2 u^p \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, by Lemma 12, we have

$$\frac{1}{c_9}\widetilde{u}(x) \le u(x) \le c_9\widetilde{u}(x) \quad \text{for all } x \in \Omega$$

and

$$-\Delta\widetilde{u}(x) \le c_9 V_1^-(x)\widetilde{u}(x) + c_9^p V_2(x)\widetilde{u}(x)^p \quad \text{for a.e. } x \in \Omega.$$

These, together with (V1) and (V2), signify that $\widetilde{u} \in \mathscr{U}(\Omega)$ if replacing the constants c_1, c_2, c_3 in (U1) and (U3) by bigger one, and that $\widetilde{u}$ behaves like u. This is a reason why we considered the class $\mathscr{U}(\Omega)$. Then the results for $u \in \mathscr{S}(\Omega)$ stated in the introduction can be obtained by applying the results in Sect. 2 to $\widetilde{u}$ as follows.

Proof (*Proof of Theorem* 1) Applying Theorem 3 to $\widetilde{u}$, we have

$$u(x) \lesssim \widetilde{u}(x) \lesssim \delta_\Omega(x)^{1-N} \quad \text{for all } x \in \Omega,$$

as desired. □

Proof (*Proof of Corollary* 1) Let $u \in \mathscr{S}(\Omega)$. Since $-\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$ by (V1), (V2) and Theorem 1, the classical regularity theorem ensures $u \in C^1(\Omega)$. Note that we have no information about $|\nabla u| \approx |\nabla\widetilde{u}|$. But, since $-\Delta u$ satisfies (U3) by (V1) and (V2), we observe that (16) holds. Repeating the same argument as in the proof of Lemma 4 with the help of Theorem 1, we can obtain

$$\int_{B(x,r)} |\nabla u|^2\, dy \lesssim r^{-N} \quad \text{whenever } B(x,3r) \subset \Omega.$$

Thus (4) can be proved in the same manner as in the proof of Corollary 3. □

Theorem 6 (Harnack inequality) *There exists* $c_5 = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) \ge 1$ *such that* (18) *holds for all* $u \in \mathscr{S}(\Omega)$.

Proof Let $u \in \mathscr{S}(\Omega)$ and $B(x, 2r) \subset \Omega$. Applying Theorem 4 (Remark 5) to $\widetilde{u}$, we have

$$\sup_{B(x,r)} u \lesssim \sup_{B(x,r)} \widetilde{u} \lesssim \inf_{B(x,r)} \widetilde{u} \lesssim \inf_{B(x,r)} u,$$

as desired. □

Proof (*Proof of Theorem* 2) By the assumption on $u \in \mathscr{S}(\Omega)$, $\widetilde{u}$ vanishes continuously on $\partial\Omega \cap B(\xi, c_4 r)$. Applying Theorem 5 to $\widetilde{u}$, we obtain

$$\frac{u(x)}{u(\xi_r)} \approx \frac{\widetilde{u}(x)}{\widetilde{u}(\xi_r)} \approx \frac{\delta_\Omega(x)}{r} \quad \text{for all } x \in \Omega \cap B(\xi, r),$$

as desired. □

The next corollary follows immediately from Theorem 2.

Corollary 5 *The constants c_4 and r_1 are the same as in Theorem 2. Let $\xi \in \partial\Omega$ and $0 < r < r_1$. If $u, v \in \mathscr{S}(\Omega) \cup \mathscr{H}_+(\Omega)$ vanish continuously on $\partial\Omega \cap B(\xi, c_4 r)$, then (25) holds with $c = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) > 1$.*

Proof (*Proof of Corollary* 2) Using Theorem 2 with $r = r_1/2$ and the Harnack inequality, we have $u(x) \lesssim \delta_\Omega(x) u(x_0) \lesssim c_3\, \delta_\Omega(x)$ for all $x \in \Omega$, whence (8) follows.

To show (9), let $r := \delta_\Omega(x)/3$. Since u satisfies (16), it follows from (17) and (8) that

$$\begin{aligned}\int_{B(x,r)} |\nabla u|^2\, dy &\lesssim r^{-2} \int_{B(x,2r)} u^2\, dy + r^{p(N-1)-N-1} \int_{B(x,2r)} u^{p+1}\, dy \\ &\lesssim r^N + r^{pN} \lesssim r^N.\end{aligned}$$

Also, by (V1), (V2), (2) and (8), we have

$$\int_0^r t^{1-N} \int_{B(x,t)} |-\Delta u(y)|\, dy \frac{dt}{t} \lesssim 1.$$

Appealing to Lemma 5, we can obtain (9). This completes the proof. □

4 Applications

This section gives some applications of our main results stated in the introduction.

4.1 Positive Solutions with Isolated Boundary Singularities

Let us show the asymptotic behavior of a positive solution of (1) near its isolated singularity on $\partial\Omega$. For $p \geq 1$, $\alpha \in \mathbb{R}$ and $\xi \in \partial\Omega$, we let

$$F_{p,\alpha}(x) := \begin{cases} |x-\xi|^{N+1-p(N-1)-\alpha} & \text{if } 1 - p(N-1) - \alpha < 0, \\ |x-\xi|^N \log \dfrac{2\operatorname{diam}\Omega}{|x-\xi|} & \text{if } 1 - p(N-1) - \alpha = 0, \\ |x-\xi|^N & \text{if } 1 - p(N-1) - \alpha > 0. \end{cases}$$

Observe that $F_{p,\alpha}$ vanishes continuously at ξ if $\alpha < N + 1 - p(N-1)$.

Theorem 7 *Let $\xi \in \partial\Omega$. If $u \in \mathscr{S}(\Omega)$ vanishes continuously on $\partial\Omega \setminus \{\xi\}$, then there exist $\lambda = \lambda(u) \geq 0$ and $c_{10} = c(c_1, c_2, c_3, \alpha_1, \alpha_2, p, V_1^+, \Omega) > 0$ such that*

$$0 \leq \frac{u(x)}{M_{V_1^+,\Omega}(x,\xi)} - \lambda \leq c_{10} \max\{F_{1,\alpha_1}(x), F_{p,\alpha_2}(x)\} \quad \text{for all } x \in \Omega. \tag{37}$$