CIM Series in Mathematical Sciences 7

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Nonlinear Differential Equations and Applications

Portugal-Italy Conference on NDEA, Évora, Portugal, July 4–6, 2022

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Preface

The Portugal-Italy Conference on Nonlinear Differential Equations and Applications (PICNDEA22) took place on July 4–6, 2022, at the University of Évora, Portugal. Évora is a UNESCO World Heritage city museum since 1986.

The Conference was sponsored by the research centers: CIMA (Centro de Investigação em Matemática e Aplicações), CMAFcIO (Centro de Matemática, Aplicações Fundamentais e Investigação Operacional), and GFM (Grupo de Física Matemática). The Conference was held "face-to-face," with just a couple of online interventions.

The main scientific topics of the conference were Ordinary and Partial Differential Equations, with particular emphasis in non-linear problems originating in applications, and their treatment with the methods of Numerical Analysis.

The fundamental main purpose was to bring together Italian and Portuguese researchers in the above fields, to create new, and amplify previous collaborations, and to follow and discuss new topics in the area.

During these days, 65 participants, including experienced researchers, some leading names in the field, and Ph.D. students, of many nationalities, presented and attended both plenary lectures and parallel sessions, where recent and classical results were illustrated and discussed.

Conference details on participants, scientific program, and contents can be found at https://www.picndea22.uevora.pt/.

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Introduction

We started to plan PICNDEA22 in the summer of 2021, when the recent pandemic crisis was still viewed as a threat. As an unfortunate consequence, scientific meetings, understood as events where people meet in person to exchange ideas, had become a memory of past years.

Scientific events had, instead, found a shelter in the screens and loudspeakers of our laptops and smartphones.

Despite some uncertainty concerning the possibility of effective realization of the project, we went ahead, and we can now say with satisfaction that it was worth it. True, some of the designed participants were hit by a late wave of the epidemic and thus prevented from attending, but only in a relatively small number.

Having the collaboration between Italian and Portuguese mathematicians as a guiding idea seemed only natural, given the volume and importance of the teamwork involving the two mathematical communities, with remarkable increase along the recent decades. The three-day conference at the friendly environment of University of Évora was an opportunity for a number mathematicians (not only Italian or Portuguese), young and senior, to reaffirm that to expose ideas at the blackboard in front of each other has a value worth to be encouraged.

Some of the participants submitted articles to the publication of this volume of proceedings. The articles have been carefully refereed. In some cases, the subject of the papers is essentially the subject of the respective authors' presentation at PICNDEA22. In any case, this volume gives an idea of the quality and variety of the material presented at Évora.

The volume cannot speak for all that PICNDEA22 has meant. In fact, it obviously does not reflect the lively discussions around the presentations. Moreover, we should mention three singular presentations by senior mathematicians that raised big interest from the whole audience:

Alfio Quarteroni, on "Physics-Based and Data-Driven Models for PDEs";

José Francisco Rodrigues, on "José Sebastião e Silva, Mathematician from Mértola to Lisboa, via Evora and Roma" (by the way, José Sebastião e Silva, a leading Portuguese mathematician of the twentieth century, studied in Italy and had a significant contact with Italian mathematicians, as Federigo Enriques and Luigi Fantappiè);

Giuseppe Buttazzo, on a high rank Italian mathematician: “Remembering Ennio De Giorgi.”

We thank all the participants, to whom we owe the success of this conference, and in particular to those who contributed to this volume.

Finally, we would like to thank the support of the International Mathematics Center (CIM) to our editorial initiative, which we dedicate to the memory of the former President of CIM, Professor Isabel Narra de Figueiredo, who recently passed away, for her immediate and continuous enthusiasm dedicated to this initiative.

Contents

Some Optimal Design Problems with Perimeter Penalisation

Ana Cristina Barroso, José Matias, and Elvira Zappale

Abstract We present measure representation and integral representation results for some integral functionals arising in the context of optimal design problems under certain growth conditions and in the presence of a perimeter penalisation term. The functionals in question correspond to the relaxation with respect to a pair (χ, u), where χ is the characteristic function of a set of finite perimeter and u belongs to a suitable function space.

1 Introduction

The search for an optimal shape that minimises a certain cost functional is the basis of an optimal design problem. Given an open, bounded set $\Omega \subset \mathbb{R}^N$, the optimal shape is a subset $E \subset \Omega$ that can be described in terms of its characteristic function

$$E = \{\chi = 1\} \text{ with } \chi : \Omega \to \{0, 1\}.$$

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The purpose of this chapter is to give an overview of the results obtained in [6–8], as well as some others in the literature, where the study of certain relaxed functionals arising in the context of optimal design problems with a perimeter penalisation is undertaken. In particular, the possibility of obtaining a measure representation for these functionals is addressed.

The functionals under consideration are based on energies of the form

$$F(\chi, u) := \int_{\Omega} \chi W_1(\nabla u) + (1 - \chi) W_0(\nabla u)\, dx$$

or

$$F(\chi, u) := \int_{\Omega} \chi W_1(\mathcal{E}u) + (1 - \chi) W_0(\mathcal{E}u)\, dx,$$

where the full gradient ∇u is replaced by its symmetrised counterpart $\mathcal{E}u$. In the above expressions, u belongs to an appropriate function space, χ is a characteristic function of a set of finite perimeter in Ω, and the densities $W_i : \mathbb{R}^{d\times N} \to \mathbb{R}$ (or $W_i : \mathbb{R}^{N\times N}_s \to \mathbb{R}$ in the second case), $i = 0, 1$, are continuous functions satisfying a suitable growth condition.

Hence, we are led to the problem of minimising the previous energies over the pair (χ, u),

$$\min_{(\chi,u)} F(\chi, u),$$

taking into account certain volume constraints, possibly relaxed by means of Lagrange multipliers.

The hypotheses we are assuming (see the following sections) yield the necessary compactness in the u variable; however, minimising sequences $\{\chi_n\} \subset L^\infty(\Omega; \{0, 1\})$ tend to highly oscillate so that in the limit we may no longer obtain a characteristic function. To prevent this from happening, a perimeter term is added to the expression of the energy to be minimised, which now takes the form

$$F(\chi, u) := \int_{\Omega} \chi W_1(\nabla u) + (1 - \chi) W_0(\nabla u)\, dx + |D\chi|(\Omega) \tag{1}$$

or

$$F(\chi, u) := \int_{\Omega} \chi W_1(\mathcal{E}u) + (1 - \chi) W_0(\mathcal{E}u)\, dx + |D\chi|(\Omega). \tag{2}$$

Sections 3 and 4 are devoted to the study of relaxed functionals based on the previous two expressions for the total energy.

Indeed, in Sect. 4, starting from the energy (2), which has a bulk term depending on the symmetrised gradient of u, as well as a perimeter term, we consider the relaxation of $F(\cdot, \cdot)$ with respect to a pair (χ, u), where χ is the characteristic function of a set of finite perimeter, corresponding to the optimal shape, and u is a function of bounded deformation. As pointed out, the perimeter term, which penalises the interface between the two regions $\{\chi = 1\}$ and $\{\chi = 0\}$, is added to ensure compactness of minimising sequences. We obtain a measure representation for this relaxed functional under linear growth conditions (see Barroso, Matias and Zappale [8]).

In Sect. 3, we present the results obtained in Barroso and Zappale [6, 7], where a similar investigation is undertaken in the case of non-standard $p-q$ growth conditions on the original bulk energy densities, which now depend on the full gradient of the u variable, and where the energy (1) also includes a perimeter penalisation term. In this setting, we show in [6] that one of the relaxed functionals under consideration only admits a weak measure representation, whereas for the other a strong measure representation holds. Under some convexity assumptions, we provide a partial characterisation of the corresponding measures, and a full representation is obtained in the one-dimensional setting.

In [7], we further identify some conditions under which the relaxation process gives rise to no concentration effects. In this case, we show that the integral representation in question is composed of a term that is absolutely continuous with respect to the Lebesgue measure, and a perimeter term, but has no additional singular term.

2 Preliminaries

We compile in this section a list of notations that will be used throughout the text, and we recall some results on BV and BD functions for the convenience of the reader:

- $\Omega \subset \mathbb{R}^N$ denotes an open, bounded set with Lipschitz boundary.
- $\mathcal{B}(\Omega)$, $\mathcal{O}(\Omega)$, and $\mathcal{O}_\infty(\Omega)$ represent the families of all Borel, open, and open subsets of Ω with Lipschitz boundary, respectively.
- $\mathcal{M}(\Omega)$ is the set of finite Radon measures on Ω.
- $|\mu|$ stands for the total variation of a measure $\mu \in \mathcal{M}(\Omega)$.
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ stand for the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively.
- The symbol dx will also be used to denote integration with respect to $\mathcal{L}^N$.
- The set of symmetric $N \times N$ matrices is denoted by $\mathbb{R}^{N\times N}_s$.
- Given two vectors $a, b \in \mathbb{R}^N$, $a \odot b$ is the symmetric $N \times N$ matrix defined by $a \odot b := \dfrac{a \otimes b + b \otimes a}{2}$, where $\otimes$ indicates tensor product.
- $B(x, \varepsilon)$ is the open ball in $\mathbb{R}^N$ with centre x and radius ε, $Q(x, \varepsilon)$ is the open cube in $\mathbb{R}^N$ with two of its faces parallel to the unit vector e_N, centre x, and side length ε, whereas $Q_\nu(x, \varepsilon)$ stands for a cube with two of its faces parallel to the unit vector ν; when $x = 0$ and $\varepsilon = 1$, $\nu = e_N$, we simply write B and Q.
- $S^{N-1} := \partial B$ is the unit sphere in $\mathbb{R}^N$.
- $C^\infty_c(\Omega; \mathbb{R}^N)$ and $C^\infty_{\text{per}}(Q; \mathbb{R}^N)$ are the spaces of $\mathbb{R}^N$-valued smooth functions with compact support in Ω and smooth and Q-periodic functions from Q to $\mathbb{R}^N$, respectively.

Given $u \in L^1(\Omega; \mathbb{R}^d)$, we say $x \in \Omega$ is a Lebesgue point of u if there exists $\widetilde{u}(x) \in \mathbb{R}^d$ such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{B(x,\varepsilon)} |u(y) - \widetilde{u}(x)| \, dy = 0,$$

$\widetilde{u}(x)$ is called the approximate limit of u at x. The set of Lebesgue points of u is denoted by Ω_u, and the set $S_u := \Omega \setminus \Omega_u$, which satisfies $\mathcal{L}^N(S_u) = 0$, is called the Lebesgue discontinuity set of u.

The jump set of the function u, denoted by J_u, is the set of points $x \in \Omega \setminus \Omega_u$ for which there exist $a, \ b \in \mathbb{R}^d$ and a unit vector $\nu \in S^{N-1}$, normal to J_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : (y-x)\cdot\nu > 0\}} |u(y) - a| \, dy = 0,$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : (y-x)\cdot\nu < 0\}} |u(y) - b| \, dy = 0.$$

Up to a permutation of (a, b) and a change of sign of ν, the above conditions uniquely determine (a, b, ν), which is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The difference $[u](x) := u^+(x) - u^-(x)$ is called the jump of u at x.

A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of bounded variation, $u \in BV(\Omega; \mathbb{R}^d)$, if all its first-order distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$ for $1 \leq i \leq d$ and $1 \leq j \leq N$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du, and $|Du|$ stands for its total variation.

If $u \in BV(\Omega)$, it is well known that S_u is countably $(N-1)$-rectifiable, and the following decomposition holds:

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u + D^c u,$$

where ∇u denotes the density of the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^N$ and $D^c u$ is the Cantor part of the measure Du.

We say that an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$, E, is a set of finite perimeter in Ω if the perimeter of E in Ω given by

$$P(E; \Omega) := \sup \left\{ \int_E \operatorname{div}\varphi(x) \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^N), \, \|\varphi\|_{L^\infty} \leq 1 \right\}$$

is finite.

A function $u \in L^1(\Omega; \mathbb{R}^N)$ is said to be of bounded deformation, and we write $u \in BD(\Omega)$, if the symmetric part of its distributional derivative Du, $Eu := \dfrac{Du + Du^T}{2}$, is a matrix-valued bounded Radon measure.

If $u \in BD(\Omega)$, then J_u is countably $(N-1)$-rectifiable, and the following decomposition holds:

$$Eu = \mathcal{E}u\mathcal{L}^N \lfloor \Omega + [u] \odot \nu_u \mathcal{H}^{N-1} \lfloor J_u + E^c u,$$

where $\mathcal{E}u$ represents the density of the absolutely continuous part of the measure Eu with respect to the Lebesgue measure and $E^c u$ is the Cantor part of Eu that vanishes on Borel sets B with $\mathcal{H}^{N-1}(B) < +\infty$. The space of special functions of bounded deformation, $SBD(\Omega)$, is comprised of those functions $u \in BD(\Omega)$ for which $E^c u = 0$.

We conclude this section by recalling the notion of symmetric quasiconvexity that will be needed in Sect. 4.

A Borel measurable function $f : \mathbb{R}^{N\times N}_s \to \mathbb{R}$ is said to be symmetric quasiconvex if

$$f(\xi) \leq \int_Q f(\xi + \mathcal{E}\varphi(x))\, dx, \tag{3}$$

for every $\xi \in \mathbb{R}^{N\times N}_s$ and for every $\varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^N)$.

Given $f : \mathbb{R}^{N\times N}_s \to \mathbb{R}$, the symmetric quasiconvex envelope of f, SQf, is defined by

$$SQf(\xi) := \inf\left\{ \int_Q f(\xi + \mathcal{E}\varphi(x))\, dx : \varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^N)\right\}. \tag{4}$$

It turns out that SQf is the greatest symmetric quasiconvex function that is less than or equal to f.

3 Measure Representation Results

In this section, we focus on the case of the energy given in (1). The motivation behind this expression for the energy originates in the optimal design problem proposed by Murat and Tartar [30] and Kohn and Strang [25–27] that consists in identifying the minimal energy configuration of a mixture of two conductive materials present in a container Ω, when only the volume fraction of each one is prescribed. As stated, this problem might not have a solution. However, the introduction, in the energy functional to be minimised, of a term that penalises the perimeter of the sets where the mixture equals one of the conductive materials provides an extra compactness property that is enough to ensure existence of a solution, while also eliminating the case where the two materials are finely mixed. Previous works on this subject include those of Ambrosio and Buttazzo [4], Kohn and Lin [24], and Carita and Zappale [14].

In this setting, the characteristic function χ in (1) corresponds either to one material of a two-component sample or to one of the phases of a single material, W_0 and W_1 are the energy densities associated to each component or phase, and the term $|D\chi|(\Omega)$ penalises the measure of the created interfaces.

Our aim in [6] is to study this optimal design problem within the context of the so-called $p-q$ growth conditions (see (6) and (10) below).

We let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, and we consider two real numbers p and q related through the inequalities

$$1 < p \leq q < \frac{Np}{N-1}. \tag{5}$$

If $N = 1$, we let $1 < p \leq q < +\infty$. With some modifications in the proofs, the case $p = 1$, $1 \leq q < \frac{N}{N-1}$ (or $1 \leq q < +\infty$, if $N = 1$) may also be treated (see [6]).

Let $W_i : \mathbb{R}^{d\times N} \to \mathbb{R}$, $i = 0, 1$, be continuous functions satisfying the following growth condition:

$$\exists \beta > 0 : 0 \leq W_i(\xi) \leq \beta\left(1 + |\xi|^q\right), \quad \forall \xi \in \mathbb{R}^{d\times N}, \tag{6}$$

and define $F : BV(\Omega; \{0, 1\}) \times W^{1,p}(\Omega; \mathbb{R}^d) \to [0, +\infty)$ by

$$\begin{aligned} F(\chi, u) &:= \int_\Omega \chi W_1(\nabla u) + (1-\chi) W_0(\nabla u)\, dx + |D\chi|(\Omega) \\ &:= \int_\Omega f(\chi(x), \nabla u(x))\, dx + |D\chi|(\Omega), \end{aligned} \tag{7}$$

where, for the purpose of simplification of notation, $f : \{0, 1\} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is given by

$$f(b, \xi) := bW_1(\xi) + (1-b)W_0(\xi).$$

We will also consider the following localised version of (7): For every open set $A \subset \Omega$ and every $(\chi, u) \in BV(A; \{0, 1\}) \times W^{1,p}(A; \mathbb{R}^d)$, let

$$F(\chi, u; A) := \int_A f(\chi(x), \nabla u(x))\, dx + |D\chi|(A).$$

The relaxed functionals to be studied in this section are defined by

$$\begin{aligned} \mathcal{F}(\chi, u; A) := \inf\Big\{ \liminf_{n\to+\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}(A; \mathbb{R}^d), \\ \chi_n \in BV(A; \{0, 1\}), u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d), \\ \chi_n \to \chi \text{ in } L^1(A; \{0, 1\}) \Big\} \end{aligned} \tag{8}$$

and

$$\mathcal{F}_{\rm loc}(\chi, u; A) := \inf\Big\{\liminf_{n\to+\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}_{\rm loc}(A; \mathbb{R}^d),$$
$$\chi_n \in BV(A; \{0,1\}), u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d),$$
$$\chi_n \to \chi \text{ in } L^1(A; \{0,1\})\Big\}. \tag{9}$$

Functionals (8) and (9) are defined for fields u belonging to $W^{1,p}(A; \mathbb{R}^d)$; however, the growth condition in (6) ensures boundedness of the energy only in the smaller space $W^{1,q}(A; \mathbb{R}^d)$. Due to this gap between the two spaces, if the following coercivity condition also holds

$$\exists\,\alpha > 0 : W_i(\xi) \geq \alpha|\xi|^p, \quad \forall \xi \in \mathbb{R}^{d\times N}, \tag{10}$$

energy bounded sequences $\{u_n\} \subset W^{1,q}(A; \mathbb{R}^d)$ will be weakly compact in $W^{1,p}(A; \mathbb{R}^d)$, but not necessarily in $W^{1,q}(A; \mathbb{R}^d)$. Hence, it may be possible to energetically approach functions $u \in W^{1,p} \backslash W^{1,q}$, and the above relaxed functionals provide the effective energy associated to such a function u.

Our goal in [6] is to investigate whether $\mathcal{F}(\chi, u; A)$ and $\mathcal{F}_{\rm loc}(\chi, u; A)$ are represented by certain Radon measures defined on the open subsets of $\overline{\Omega}$ and, if so, if these measures can be characterised. As we shall see, a (strong) measure representation holds for (9); however, (8) only admits a weak measure representation.

Let μ be a Radon measure on $\overline{\Omega}$. We recall that μ (strongly) represents a functional $\mathcal{G}(\chi, u; \cdot)$ if $\mu(A) = \mathcal{G}(\chi, u; A)$, for all open sets $A \subset \Omega$. On the other hand, μ weakly represents $\mathcal{G}(\chi, u; \cdot)$ if $\mu(A) \leq \mathcal{G}(\chi, u; A) \leq \mu\left(\overline{A}\right)$, for all open sets $A \subset \Omega$.

When there is no dependence on the χ field and when $p = q$, it is well known that the relaxed functional $\mathcal{F}(u; \cdot)$ admits the integral representation

$$\mathcal{F}(u; A) = \inf\Big\{\liminf_{n\to+\infty} \int_A f(\nabla u_n(x))\,dx : u_n \in W^{1,p}(A; \mathbb{R}^d),$$
$$u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d)\Big\}$$
$$= \int_A Qf(\nabla u(x))\,dx,$$

where Qf denotes the quasiconvex envelope of f.

Assuming that $1 < p \leq q < \frac{Np}{N-1}$ and the following growth condition from above and from below on the density f,

$$\exists\,\alpha, \beta > 0 : \alpha|\xi|^p \leq f(\xi) \leq \beta(1 + |\xi|^q), \forall \xi \in \mathbb{R}^{d\times N}, \tag{11}$$

Fonseca and Malý [21] showed that, when finite, there is a weak measure representation for

$$\mathcal{F}^{q,p}(u;A) := \inf\left\{\liminf_{n\to+\infty}\int_A f(\nabla u_n(x))\,dx : u_n \in W^{1,q}(A;\mathbb{R}^d),\right.$$
$$\left. u_n \rightharpoonup u \text{ in } W^{1,p}(A;\mathbb{R}^d)\right\}.$$

On the other hand, $\mathcal{F}^{q,p}_{\text{loc}}(u;A)$ (defined as above replacing $W^{1,q}$ by $W^{1,q}_{\text{loc}}$) admits a strong measure representation, when finite. If μ_a is the density of the corresponding measure μ with respect to the Lebesgue measure, it was shown in [21] that

$$\mu_a(x_0) \geq Qf(\nabla u(x_0)), \quad \forall u \in W^{1,p}(\Omega;\mathbb{R}^d), \quad \text{a.e. } x_0 \in \Omega.$$

The reverse inequality was proved by Bouchitté, Fonseca and Malý [11], so the bulk part of the measure μ was completely identified.

A key ingredient for the proof in [21] is the following lemma that establishes the existence of a linear operator from $W^{1,p}$ to $W^{1,p}$ that conserves boundary values and improves the integrability of both u and ∇u in a certain subset of Ω, whose Lebesgue measure can be controlled, while maintaining the function values elsewhere.

Lemma 1 *Let p and q satisfy (5). Let $V \subset\subset \Omega$, $W \subset \Omega$ be open sets such that $\Omega = V \cup W$, and let $v \in W^{1,q}(V;\mathbb{R}^d)$, $w \in W^{1,q}(W;\mathbb{R}^d)$. Then, for every $m \in \mathbb{N}$, there exist $z \in W^{1,q}(\Omega;\mathbb{R}^d)$ and open sets $V' \subset V$ and $W' \subset W$, such that $V' \cup W' = \Omega$, $z = v$ in $\Omega \setminus W'$, $z = w$ in $\Omega \setminus V'$,*

$$\mathcal{L}^N(V' \cap W') \leq \frac{C}{m},$$

and

$$\|z\|_{W^{1,q}(V'\cap W')} \leq \frac{C}{m^\tau}\left(\|v\|_{W^{1,p}(V\cap W)} + \|w\|_{W^{1,p}(V\cap W)} + m\,\|w - v\|_{L^p(V\cap W)}\right),$$

where $C = C(p,q,V,W)$ and $\tau = \tau(N,p,q) > 0$.

Other results related to gap problems can be found, for example, in [16, 29, 33], among many other references. In particular, Acerbi, Bouchitté and Fonseca in [1] treated the case of inhomogeneous densities $h(x,\xi)$, where h is convex with respect to ξ and satisfies the growth condition in (11). They showed that

$$\mathcal{F}^{q,p}_{\text{loc}}(u;A) = \int_A h(x,\nabla u(x))\,dx + \mu_s(u;A),$$

where $\mu_s(u,\cdot)$ is a non-negative Radon measure, singular with respect to the Lebesgue measure. It is worth pointing out that, although theirs is a vector-valued problem, the required hypothesis is convexity rather than quasiconvexity. This is related to the fact that, in addition to the gap in the exponents appearing in the growth conditions from above and from below satisfied by the original bulk energy density, h also depends explicitly on x. The measure representation result obtained in [1] (see Theorem 4) was used to prove Theorem 2.

In [28] Mingione and Mucci obtained the relaxation, with respect to the same topology considered in [21], of $\int_\Omega f(x, \nabla u(x))\,dx$ in the non-convex case, provided f satisfies (11) and some further structure assumptions. For integrands f that are sufficiently smooth with respect to the x variable, they showed that the relaxed functional admits the integral representation $\int_\Omega Qf(x, \nabla u(x))\,dx$. An interesting aspect of their analysis is the fact that the required regularity of f is related to the ratio $\frac{q}{p}$ of the gap exponents in the growth and coercivity assumptions: The larger this gap is, the more regular f needs to be. Indeed, if this condition fails to hold, it can be shown that the relaxation process might not even lead to a Radon measure.

In a recent paper, Almi, Reggiani and Solombrino [3] extended previous works to the free discontinuity setting where singularities may appear in the form of jump discontinuities. To take these into account, they studied lower semicontinuity and relaxation of functionals of the form

$$\int_\Omega f(x, \nabla u(x))\,dx + \int_{J_u} g(x, [u](x), \nu_u)\,d\mathcal{H}^{N-1},$$

which, in addition to the bulk term, also include a surface term. In their article, f satisfies a generalised Orlicz growth condition that includes, in particular, the variable exponent case $\xi^{p(x)}$, among others. We point out that their setting does not cover the results addressed in [6, 7] as a certain regularity in the x variable is still required, whereas in our case the variability of the exponent is related to the discontinuous field χ.

The following weak representation result is proved in [6].

Theorem 1 *Let $f(b, \xi) := bW_1(\xi) + (1-b)W_0(\xi)$, where $W_0, W_1 : \mathbb{R}^{d\times N} \to \mathbb{R}$ are continuous functions such that*

$$\exists \beta > 0 : 0 \le W_i(\xi) \le \beta\left(1 + |\xi|^q\right), \quad \forall \xi \in \mathbb{R}^{d\times N}.$$

Let $\chi \in BV(\Omega; \{0,1\})$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ be such that $\mathcal{F}(\chi, u; \Omega) < +\infty$; then there exists a non-negative Radon measure μ on $\overline{\Omega}$ that weakly represents the functional $\mathcal{F}(\chi, u; \cdot)$ given in (8).

The proof rests on the following arguments. The aim is to show that there exists a Radon measure μ such that

$$\mu(A) \le \mathcal{F}(\chi, u; A) \le \mu(\overline{A}), \quad \text{for all open } A \subset \Omega, \tag{12}$$

we begin by proving this result first under the coercivity assumption

$$\exists\alpha > 0 : W_i(\xi) \geq \alpha|\xi|^p, \ \forall \xi \in \mathbb{R}^{d\times N},$$

and this hypothesis is removed in a second step. Indeed, for $\varepsilon > 0$, we consider the auxiliary energy

$$F_\varepsilon(\chi, u; \Omega) := F(\chi, u; \Omega) + \varepsilon \int_\Omega |\nabla u(x)|^p \, dx,$$

and we let $\mathcal{F}_\varepsilon(\chi, u; \Omega)$ denote its relaxed functional. By the first part of the proof, under the coercivity assumption, we conclude that there exists μ_ε that weakly represents $\mathcal{F}_\varepsilon(\chi, u; \cdot)$. Letting $\varepsilon \to 0^+$, it turns out that μ_ε converges, in the sense of measures, to a measure μ that weakly represents $\mathcal{F}(\chi, u; \cdot)$.

By a standard diagonalisation argument, our growth and coercivity assumptions imply that, when finite, the infimum appearing in $\mathcal{F}(\chi, u; \Omega)$ is attained, i.e. there exists $(\chi_n, u_n) \in BV(\Omega; \{0, 1\}) \times W^{1,q}(\Omega; \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $\chi_n \to \chi$ in $L^1(\Omega; \{0, 1\})$, and

$$\lim_{n\to+\infty} F(\chi_n, u_n; \Omega) = \mathcal{F}(\chi, u; \Omega). \tag{13}$$

The next step in the proof is the following nested subadditivity result. If $V, W \subset \Omega$ are open sets such that $V \subset\subset \Omega$ and $\Omega = V \cup W$, $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $\chi \in BV(\Omega; \{0, 1\})$, then

$$\mathcal{F}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; V) + \mathcal{F}(\chi, u; W).$$

To show this, we start by considering sequences v_n, w_n, χ_n, and ζ_n, which are admissible for the relaxed functionals $\mathcal{F}(\chi, u; V)$ and $\mathcal{F}(\chi, u; W)$ and such that

$$\int_V f(\chi_n(x), \nabla v_n(x))\, dx + |D\chi_n|(V) \leq \mathcal{F}(\chi, u; V) + \varepsilon$$

$$\int_W f(\zeta_n(x), \nabla w_n(x))\, dx + |D\zeta_n|(W) \leq \mathcal{F}(\chi, u; W) + \varepsilon.$$

From these sequences, we need to construct z_n and η_n that are admissible for $\mathcal{F}(\chi, u; \Omega)$.

Lemma 1 is essential to complete this stage of our proof. Indeed, we use the trace-preserving operator to connect v_n and w_n across a thin transition layer and to estimate the resulting increase of the energy. Usually, this is done by taking convex combinations using cut-off functions. However, due to the different exponents in (6), this argument does not work in this context since sequences constructed in this way may not remain bounded in $W^{1,q}$.

We can also obtain a transition sequence η_n by piecing together χ_n and ζ_n in such a way that no new interfaces are created. Then z_n and η_n are admissible for $\mathcal{F}(\chi, u; \Omega)$, and our estimates yield

$$\mathcal{F}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; V) + \mathcal{F}(\chi, u; W) + 2\varepsilon$$

from where the nested subadditivity follows.

To conclude the upper bound inequality in (12), we consider the sequence of measures given by

$$\mu_n(E) = \int_{E\cap\Omega} f(\chi_n(x), \nabla u_n(x))\, dx + |D\chi_n|(E \cap \Omega),$$

where χ_n, u_n satisfy (13). The sequence μ_n is bounded so there exists a non-negative Radon measure μ such that (for a subsequence) μ_n converges to μ in the sense of measures. The choice of χ_n, u_n, and Fatou's Lemma allow us to conclude that $\mu(\overline{\Omega}) \leq \mathcal{F}(\chi, u; \Omega)$. On the other hand, the upper semicontinuity of weak $*$ convergence of measures on compact sets gives, for every open set $V \subset \Omega$,

$$\begin{aligned}\mathcal{F}(\chi, u; V) &\leq \liminf_{n\to+\infty} F(\chi_n, u_n; V) = \liminf_{n\to+\infty} \mu_n(V) \\ &\leq \limsup_{n\to+\infty} \mu_n(\overline{V}) \ \leq \mu(\overline{V}).\end{aligned}$$

For the lower bound inequality, we first show that if V is an open set such that $V \subset\subset \Omega$, then

$$\mu(V) \leq \mathcal{F}(\chi, u; V);$$

this is a consequence of the nested subadditivity result. For a general open set, we have

$$\begin{aligned}\mu(V) = \sup\{\mu(O) : O \subset\subset V\} &\leq \sup\{\mathcal{F}(\chi, u; O) : O \subset\subset V\} \\ &\leq \mathcal{F}(\chi, u; V),\end{aligned}$$

and this completes the proof.

Regarding the functional given in (9), the following strong representation result holds (cf. [6]).

Theorem 2 *Let $\chi \in BV(\Omega; \{0, 1\})$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ be such that*

$$\mathcal{F}_{\text{loc}}(\chi, u; \Omega) < +\infty.$$

Then, under the growth condition given in (6), there exists a non-negative finite Radon measure λ on Ω, which strongly represents $\mathcal{F}_{\text{loc}}(\chi, u; \cdot)$.

Moreover, if $f(b,\cdot)$ *is convex for every* $b \in \{0,1\}$*, then, for every open subset* $U \subset \Omega$*, and every* $\chi \in BV(\Omega;\{0,1\})$ *and* $u \in W^{1,p}(\Omega;\mathbb{R}^d)$*,*

$$\mathcal{F}_{\mathrm{loc}}(\chi,u;U) = \int_U f(\chi(x),\nabla u(x))\,dx + |D\chi|(U) + \nu^s(\chi,u;U), \tag{14}$$

where ν^s *is a non-negative Radon measure singular with respect to the Lebesgue measure.*

To prove this theorem, we apply Theorem 1 to conclude the existence of a Radon measure λ in $\overline{\Omega}$ such that

$$\lambda(U) \le \mathcal{F}_{\mathrm{loc}}(\chi,u;U) \le \lambda(\overline{U}),$$

for every open set $U \subset \Omega$. We now want to prove that

$$\lambda(U) \ge \mathcal{F}_{\mathrm{loc}}(\chi,u;U).$$

To this end, we rely on the fact that if λ weakly represents $\mathcal{F}_{\mathrm{loc}}(\chi,u;\cdot)$, then the representation is strong, i.e. $\lambda(U) = \mathcal{F}_{\mathrm{loc}}(\chi,u;U)$, for every U open subset of Ω, provided that

$$\inf_K\{\mathcal{F}_{\mathrm{loc}}(\chi,u;U\setminus K) : K \subset U,\, K \text{ compact}\} = 0.$$

To show that the previous infimum is zero, we consider an increasing sequence of open, bounded, smooth sets $U_h \subset\subset U$, $h \in \mathbb{N}$, such that $\overline{U}_h \subset U_{h+1}$ and $U = \cup_{i=1}^{\infty} U_i$. In each of the sets $U_h \setminus \overline{U}_{h-2}$, we consider admissible sequences for $\mathcal{F}_{\mathrm{loc}}$. The conclusion follows by once again applying Lemma 1 to connect these sequences across transition layers of the form $U_h \setminus \overline{U}_{h-1}$ and by a careful piecing together of the characteristic functions in such a way that no new interfaces are created.

The argument of the proof of (14), for f convex in the second variable, is to show a double inequality and is based on two results: a lower semicontinuity result due to Ioffe [23] (used to obtain the lower bound, see Theorem 3 below) and a representation theorem due to Acerbi, Bouchitté and Fonseca [1] (for the upper bound, c.f. Theorem 4).

Theorem 3 *Let* $g : \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty)$ *be a Borel integrand such that* $g(b,\cdot)$ *is convex for every* $b \in \mathbb{R}^m$*. Then the functional*

$$G(v,u) := \int_\Omega g(v(x),\nabla u(x))\,dx$$

is lower semicontinuous in $L^1(\Omega;\mathbb{R}^m)_{\mathrm{strong}} \times W^{1,1}(\Omega;\mathbb{R}^d)_{\mathrm{weak}}$*.*

Theorem 4 *Let $f : \Omega \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a Carathéodory function such that $f(x, \cdot)$ is convex for a.e. $x \in \Omega$ and*

$$\exists\, C > 0 : \ |\xi|^p \le f(x, \xi) \le C(1 + |\xi|^q), \ \forall \xi \in \mathbb{R}^{d\times N}, \ a.e.\ x \in \Omega,$$

where $1 < p \le q < \frac{Np}{N-1}$. If A is an open subset of Ω and $u \in L^1(A; \mathbb{R}^d)$ is such that $\mathcal{F}^{q,p}_{\mathrm{loc}}(u; A) < +\infty$, where

$$\mathcal{F}^{q,p}_{\mathrm{loc}}(u; A) := \inf \left\{ \liminf_{n\to+\infty} \int_A f(x, \nabla u_n(x))\, dx : u_n \in W^{1,q}_{\mathrm{loc}}, u_n \to u \text{ in } L^1 \right\},$$

then

$$\mathcal{F}^{q,p}_{\mathrm{loc}}(u; U) = \int_U f(x, \nabla u(x))\, dx + \nu^s(u, U), \quad \forall \text{ open } U \subset A,$$

where $\nu^s(u, \cdot)$ is a non-negative Radon measure, singular with respect to the Lebesgue measure.

The outline of the proof of (14) is then as follows.

By the lower semicontinuity theorem of Ioffe, the lower semicontinuity of the total variation, and the superadditivity of the liminf, the functional

$$\int_\Omega f(\chi(x), \nabla u(x))\, dx + |D\chi|(\Omega)$$

is lower semicontinuous with respect to the L^1 strong convergence for χ and the $W^{1,p}$ weak convergence for u. From here, taking the infimum over all admissible sequences for $\mathcal{F}_{\mathrm{loc}}(\chi, u; \Omega)$, we obtain

$$\int_\Omega f(\chi(x), \nabla u(x))\, dx + |D\chi|(\Omega) \le \mathcal{F}_{\mathrm{loc}}(\chi, u; \Omega),$$

and likewise in any open subset U of Ω.

To show the upper bound inequality in (14), we work with a fixed sequence of fields $\chi_n = \chi$. Then, Theorem 4 guarantees the existence of a measure $\nu^s(u, \chi; \cdot)$, singular with respect to the Lebesgue measure, and a sequence $\overline{u}_n \in W^{1,q}_{\mathrm{loc}}(U; \mathbb{R}^d)$ such that $\overline{u}_n \rightharpoonup u$ in $W^{1,p}(U; \mathbb{R}^d)$ and

$$\limsup_{n\to+\infty} \int_U f(\chi(x), \nabla \overline{u}_n(x))\, dx \le \int_U f(\chi(x), \nabla u(x))\, dx + \nu^s(\chi, u; U).$$

This inequality ensures that

$$\mathcal{F}_{\mathrm{loc}}(\chi, u; U) \le \int_U f(\chi(x), \nabla u(x))\, dx + |D\chi|(U) + \nu^s(\chi, u; U).$$

Putting together the lower and upper bounds, and the fact that

$$\mathcal{F}_{\rm loc}(\chi, u; U) = \lambda(U),$$

we conclude that

$$\mathcal{F}_{\rm loc}(\chi, u; U) = \int_U f(\chi(x), \nabla u(x))\, dx + |D\chi|(U) + \nu^s(\chi, u; U).$$

The conclusions of Theorem 2 can be improved in the one-dimensional case where we show that the representing measure is fully identified as no additional singular term arises (see [6]).

Theorem 5 *Let I be an open interval in $\mathbb{R}$, let $f(b, \xi) := bW_1(\xi) + (1-b)W_0(\xi)$, where*

$$\exists\, \beta > 0 : 0 \leq W_i(\xi) \leq \beta \left(1 + |\xi|^q\right), \quad \forall \xi \in \mathbb{R}^d,$$

and let p, q be such that $1 < p \leq q < +\infty$. Let $\chi \in BV(I; \{0, 1\})$ and $u \in W^{1,p}(I; \mathbb{R}^d)$. Then,

$$\mathcal{F}(\chi, u; I) = \int_I f^{**}(\chi(x), u'(x))\, dx + |D\chi|(I).$$

Notice that, in the previous statement, no convexity assumptions are placed on $f(b, \cdot)$, which accounts for the appearance of the convex envelope $f^{**}(b, \cdot)$ as the density of the absolutely continuous part of the limit functional. This theorem also generalises to the optimal design context results due to Ben Belgacem [10] where no x dependence is considered.

The proof of the lower bound follows immediately from Theorem 2 since $f^{**} \leq f$ and f^{**} is convex. For the upper bound, we fix again the sequence $\chi_n = \chi$, and we use a mollification of u, as well as standard relaxation results. It is also important for our arguments that, as we are working in an interval $I \subset \mathbb{R}$, the field χ has finitely many discontinuity points.

The conclusions of Theorem 5 led us to investigate whether we could identify other instances where the absence of the additional singular measure term could be ensured. This cannot be expected to be true in general; indeed, some functionals whose integrands satisfy growth conditions from above and below with a gap in the two exponents do exhibit concentration effects.

The analysis of this problem is undertaken in [7] and relies on the fact that a set E can be approximated from the inside by smooth sets, in such a way that the perimeters also converge, provided the topological boundary of E satisfies a certain mild hypothesis (see Theorem 6 below obtained in [32]).

It is well known that one can always approximate, in measure, a set E of finite perimeter in $\mathbb{R}^N$, with sets E_ε with smooth boundary and such that the convergence of the perimeters also holds, but this approximation, in general, cannot be performed strictly from within. The result of Schmidt shows that the approximation of E is also true with the additional requirement that the smooth sets satisfy $E_\varepsilon \subset\subset E$.

Theorem 6 (Strict interior approximation of the perimeter) *Let E be a bounded open set in $\mathbb{R}^N$ such that*

$$\mathcal{H}^{N-1}(\partial E) = P(E; \mathbb{R}^N). \tag{15}$$

Then, for every $\varepsilon > 0$, there exists an open set E_ε with smooth boundary in $\mathbb{R}^N$ such that

$$E_\varepsilon \subset\subset E, \;\; E \setminus E_\varepsilon \subset N_\varepsilon(\partial E) \cap N_\varepsilon(\partial E_\varepsilon), \;\; P(E_\varepsilon; \mathbb{R}^N) \leq P(E; \mathbb{R}^N) + \varepsilon,$$

where $N_\varepsilon(\cdot)$ stands for ε-neighbourhoods of sets in $\mathbb{R}^N$.

The conditions satisfied by the sets E_ε imply, in particular, that $E = \bigcup_{\varepsilon>0} E_\varepsilon$ and $\lim_{\varepsilon\to 0^+} \mathcal{L}^N(E_\varepsilon) = \mathcal{L}^N(E)$. Also, by the lower semicontinuity of the perimeter and the fact that ∂E_ε are smooth, it follows that

$$\lim_{\varepsilon\to 0^+} \mathcal{H}^{N-1}(\partial E_\varepsilon) = \lim_{\varepsilon\to 0^+} P(E_\varepsilon; \mathbb{R}^N) = P(E; \mathbb{R}^N).$$

This convergence of the perimeters is particularly relevant for our problem since the expression of the energy under consideration includes a perimeter term.

The conclusions of Theorem 6 were already known to hold for bounded Lipschitz domains E (see [32] and the references therein) since these sets satisfy the required boundary assumption. But, if this condition fails to hold, an inner approximation by smooth sets with the above properties may no longer be possible, this is the case if the original set has an internal fracture, for example.

For the result we prove in [7], we still consider $f : \{0, 1\} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ to be defined by

$$f(b, \xi) := bW_1(\xi) + (1 - b)W_0(\xi);$$

however, now the continuous density functions $W_i : \mathbb{R}^{d\times N} \to \mathbb{R}$, $i = 0, 1$, satisfy

$$\exists\, \beta_1 > 0 : \;\; 0 \leq W_1(\xi) \leq \beta_1 \left(1 + |\xi|^p\right), \quad \forall \xi \in \mathbb{R}^{d\times N},$$
$$\exists\, \beta_0 > 0 : \;\; 0 \leq W_0(\xi) \leq \beta_0 \left(1 + |\xi|^q\right), \quad \forall \xi \in \mathbb{R}^{d\times N},$$

so that a stronger hypothesis is imposed on W_1. On the other hand, we now require only that

$$1 < p \leq q < +\infty,$$

and thus the range of admissible exponents is enlarged. Under these assumptions, we obtain the following integral representation result for the relaxed energies $\mathcal{F}_{\mathrm{loc}}(\chi, u; \Omega) = \mathcal{F}(\chi, u; \Omega)$.

Theorem 7 *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open extension domain, let p, q and f satisfy the previous conditions, as well as*

$$f(b, \cdot) \text{ is convex for every } b \in \{0, 1\}.$$

Let χ be the characteristic function of an open, connected set of finite perimeter $E \subset\subset \Omega$ such that

$$\mathcal{H}^{N-1}(\partial E) = P(E; \mathbb{R}^N),\ \mathcal{H}^{N-1}(\partial(\Omega \setminus \overline{E})) = P(\Omega \setminus \overline{E}; \mathbb{R}^N)$$

and let $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ be such that $u \in W^{1,q}(\Omega \setminus \overline{E}; \mathbb{R}^d)$.

Then,

$$\mathcal{F}_{\mathrm{loc}}(\chi, u; \Omega) = \mathcal{F}(\chi, u; \Omega) = \int_\Omega f(\chi(x), \nabla u(x))\, dx + |D\chi|(\Omega). \tag{16}$$

The hypothesis on f and the requirements placed on u and E ensure that

$$\mathcal{F}_{\mathrm{loc}}\,(\chi, u; \Omega) < +\infty.$$

Notice that the assumptions made on u depend on the set E, so, in the above integral representation result, the fields χ and u are not independent of each other. For this reason, Theorem 7 is not a measure representation result, and its proof is obtained directly, by means of a double inequality, and does not depend on our previous representation theorems. This also explains why the exponents p and q can be considered in a wider class, as the need to use Lemma 1 can be bypassed due to the stronger hypothesis on W_1 and the higher regularity of u in the set $\Omega \setminus \overline{E}$. We also mention that the conclusions of Theorem 7 remain valid under somewhat weaker convexity assumptions on the density W_0 (see [7], Remark 3.2).

The main ideas of the proof are the following.

Due to the convexity hypothesis, we conclude the lower bound as before, by an application of Ioffe's Theorem 3.

To obtain the upper bound inequality, we need to construct sequences $u_n \in W^{1,q}(\Omega; \mathbb{R}^d)$ and $\chi_n \in BV(\Omega; \{0, 1\})$, admissible for $\mathcal{F}_{\mathrm{loc}}(\chi, u; \Omega)$, and such that

$$\liminf_{n \to +\infty} F(\chi_n, u_n) \le F(\chi, u).$$

For this construction, the inner approximation result of Theorem 6 is applied to ensure the existence of a layer L_ε, where we connect two different regular sequences both converging to u in L^p, and of a layer F_ε, separating the regions where f has a different growth from above and where u has different integrability properties.

Precisely, the hypothesis on the set E allows us to apply the theorem of Schmidt twice to obtain, for each $\varepsilon > 0$, sets $E_{2\varepsilon} \subset\subset E_\varepsilon \subset\subset E$ such that $\partial E_{2\varepsilon}$ and ∂E_ε are smooth,

$$\lim_{\varepsilon\to 0^+} P(E_{2\varepsilon}; \Omega) = \lim_{\varepsilon\to 0^+} P(E_\varepsilon; \Omega) = P(E; \Omega),$$

and

$$\lim_{\varepsilon\to 0^+} \mathcal{L}^N(E_{2\varepsilon}) = \lim_{\varepsilon\to 0^+} \mathcal{L}^N(E_\varepsilon) = \mathcal{L}^N(E).$$

We may also consider $\Omega_\varepsilon \subset\subset \Omega \setminus \overline{E}$ an inner approximation of $\Omega \setminus \overline{E}$ such that $\partial\Omega_\varepsilon$ is smooth and

$$\lim_{\varepsilon\to 0^+} P(\Omega_\varepsilon; \mathbb{R}^N) = P(\Omega \setminus \overline{E}; \mathbb{R}^N), \quad \lim_{\varepsilon\to 0^+} \mathcal{L}^N(\Omega_\varepsilon) = \mathcal{L}^N(\Omega \setminus \overline{E}).$$

We denote by $L_\varepsilon := E_\varepsilon \setminus \overline{E_{2\varepsilon}}$ and by F_ε the layer between Ω_ε and E. By mollification, we consider two regular sequences $u_{\varepsilon,j}$, $\widetilde{u}_{\varepsilon,j}$, both converging to u in $W^{1,p}(L_\varepsilon; \mathbb{R}^d)$, as $j \to +\infty$, and such that $\widetilde{u}_{\varepsilon,j}$ converges, as $j \to +\infty$, to u in $W^{1,q}(\Omega \setminus \overline{(E \cup F_\varepsilon)}; \mathbb{R}^d)$.

Using a slicing argument, we connect $u_{\varepsilon,j}$ to $\widetilde{u}_{\varepsilon,j}$ across the thin transition set L_ε and are thus able to obtain a sequence $w_{\varepsilon,j} \in W^{1,q}(\Omega; \mathbb{R}^d)$ such that $w_{\varepsilon,j} \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, as $j \to +\infty$. Letting χ_ε denote the characteristic function of the set $E \cup F_\varepsilon$, it turns out that

$$\liminf_{\varepsilon\to 0^+} \liminf_{j\to+\infty} F(\chi_\varepsilon, w_{\varepsilon,j}) \le F(\chi, u),$$

and the conclusion follows by a standard diagonalisation argument.

We end this section by pointing out that a result similar to the one presented in Theorem 7 also holds if one prescribes the volume fraction of each phase, provided $u \in W^{1,q}(\Omega \setminus \widehat{E}; \mathbb{R}^d)$, where $\widehat{E}$ is a compact set such that $\widehat{E} \subset E$.

For $0 < \theta < 1$ and $\chi \in BV(\Omega; \{0, 1\})$ such that $\dfrac{1}{\mathcal{L}^N(\Omega)} \displaystyle\int_\Omega \chi(x)\,dx = \theta$, we let the volume constrained functional be given by

$$\begin{aligned} \mathcal{F}_{\mathrm{vol}}(\chi, u; \Omega) := \inf \Bigg\{ & \liminf_{n\to+\infty} F(\chi_n, u_n; \Omega) : u_n \in W^{1,q}(\Omega; \mathbb{R}^d), \\ & \chi_n \in BV(\Omega; \{0,1\}), u_n \rightharpoonup u \text{ in } W^{1,p}\left(\Omega; \mathbb{R}^d\right), \\ & \chi_n \to \chi \text{ in } L^1(\Omega; \{0,1\}), \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega \chi_n(x)\,dx = \theta \Bigg\}. \end{aligned}$$

Then, under the hypotheses of Theorem 7, it follows that

$$\mathcal{F}_{\mathrm{vol}}(\chi, u; \Omega) = \int_\Omega f(\chi(x), \nabla u(x))\, dx + |D\chi|(\Omega),$$

for every χ characteristic function of an open, connected set of finite perimeter $E \subset\subset \Omega$ such that E and $\Omega \setminus \overline{E}$ satisfy (15) and $\mathcal{L}^N(E) = \theta \mathcal{L}^N(\Omega)$, and for every $u \in W^{1,p}(\Omega; \mathbb{R}^d) \cap W^{1,q}(\Omega \setminus \widehat{E}; \mathbb{R}^d)$.

Indeed, the additional requirement placed on u ensures that the constant sequence $\chi_\varepsilon = \chi$ satisfies the desired volume constraint so it is admissible for $\mathcal{F}_{\mathrm{vol}}\,(\chi, u; \Omega)$. Therefore, the upper bound argument in the proof of Theorem 7 can be simplified, rendering the need to use the layer F_ε unnecessary, whereas the lower bound is clear.

4 Optimal Design Problem in the *BD* Setting

In this section, we focus on an optimal design problem where the cost functional to be minimised is an energy that depends on the symmetrised gradient of the admissible fields.

In fact, in the linear elasticity framework, the cost functional is usually a quadratic energy so we are led to the minimisation problem

$$\min_{(\chi,u)} \int_\Omega \chi(x) W_1(\mathcal{E}u(x)) + (1 - \chi(x)) W_0(\mathcal{E}u(x))),\ dx$$

where $W_0 \geq W_1$ are two elastic energies and $\mathcal{E}u$ is the symmetrised gradient of the displacement u.

When the stress–strain relation ceases to be linear and plasticity occurs, given the lack of reflexivity of the space L^1 and the linear growth of the stored elastic energy, it is necessary to work in a suitable functional space that accounts for fields whose strains are measures. Hence, the problem is set in the space of functions of bounded deformation $BD(\Omega)$ that is composed of integrable vector-valued functions, for which all components $E_{i,j},\ i, j = 1, \dots N$ of the deformation tensor

$$Eu = \frac{(Du + Du^T)}{2},$$

are bounded Radon measures. As mentioned in Section 2, $\mathcal{E}u$ stands for the absolutely continuous part of the symmetrised distributional derivative Eu with respect to the Lebesgue measure $\mathcal{L}^N$.

Even without considering the optimal design context, so when $\chi = \chi_\Omega$, the search for equilibria within the framework of perfect plasticity leads naturally to the study of lower semicontinuity properties and relaxation of energies of the type