Bärbel Barzel · Andreas Büchter · Christian Rütten · Florian Schacht · Stephanie Weskamp-Kleine *Hrsg.*

RESEARCH

Inklusives Lehren und Lernen von Mathematik

Konzepte und Beispiele mit Fokus auf Grund- und Förderschule

Springer Spektrum

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Festschrift für Petra Scherer zum 60. Geburtstag

Vorwort

Bereits Herbart sieht in der „Verschiedenheit der Köpfe“ die große Herausforderung an alle Schulbildung. Dabei ist seit Herbart mehr und mehr ins Bewusstsein gerückt, dass die Verschiedenheit im Klassenzimmer nicht nur die „Köpfe“, sondern ein vielfältiges Heterogenitätsspektrum umfasst. Die daraus erwachsenden Herausforderungen sind einerseits fächerübergreifender Natur, anderseits werden auch dem Fachunterricht Konzepte sowie konkrete unterrichtliche Realisierungen inklusiven Lehrens und Lernens abverlangt. Grundlegend dafür ist u. a. die Verständigung über den Inklusionsbegriff im Fach. Ziel inklusiven Lehrens und Lernens von Mathematik ist im Sinne eines weiten Inklusionsbegriffs die Teilhabe *aller* am mathematischen Diskurs, wobei keine gleichzeitigen Schwerpunktsetzungen hinsichtlich spezifischer Diversitätskategorien in Zusammenhang konkreter Lernanlässe ausgeschlossen werden. Bei entsprechender Individualisierung behält soziales Lernen jedoch seine zentrale Bedeutung. In diesem Zusammenhang bieten substanzielle Lernumgebungen großes Potenzial für den inklusiven Mathematikunterricht. Einerseits ermöglichen sie hinsichtlich aktiv-entdeckenden Lernens allen Lernenden eine gemeinsame Auseinandersetzung mit mathematisch substanziellen Inhalten. Andererseits realisieren sie im Rahmen natürlicher Differenzierung, dass auf individuellen Wegen und auf unterschiedlichen Niveaus gelernt und dieses Lernen kompetenzorientiert durch die Lehrkraft begleitet wird. Für eine solche Begleitung benötigen Lehrkräfte eine entsprechende Vorbereitung auf den inklusiven Mathematikunterricht durch Aus- und Fortbildung. Diese verzahnt sinnvollerweise theoretische Grundlagen mit Praxiserfahrung und zielt auf ein Weiterlernen im Lebenslauf. Besonders um derartige Fortbildungen zur Vorbereitung und Begleitung inklusiven Mathematiklernens gestalten zu können, ist eine Qualifizierung von Multiplikator:innen grundlegend.

Inklusives Lehren und Lernen von Mathematik umfasst also zahlreiche Facetten, zu denen Petra Scherer, Wesentliches beigetragen hat. Sie hat in Austausch und Vernetzung – auch über die Fächer und Disziplinen hinaus – die Sichtweise auf den Umgang mit Heterogenität im inklusiven Unterricht weiter ausdifferenziert und viele Lehrende und Forschende angeregt, Mathematikunterricht – nicht nur in Grund- und Förderschule – inklusionssensibel weiterzuentwickeln. Dieser Band, der als Festschrift anlässlich des 60. Geburtstags von Petra Scherer entstanden ist, versucht mit seinen unterschiedlichen Beiträgen aufzuspannen, was ihre Arbeit prägt und wie sie andere in ihrer Arbeit angeregt und bereichert hat.

Im ersten Teil des Bandes werden unterschiedliche Ideen zum Umgang mit Heterogenität in den Blick genommen. So zeigt zunächst Jahnke, dass ein Blick in die Geschichte der Mathematik vielfältige Zugänge im mathematischen Diskurs sichtbar macht und zugleich zur Wertschätzung kultureller Vielfalt beitragen kann. Kultureller sowie sprachlicher Vielfalt begegnen auch Novotná und Moraová, indem sie Leitideen zur Entwicklung von Unterrichtsmaterialien für den Einsatz in heterogenen Lerngruppen unter Berücksichtigung der Teilhabe nicht muttersprachlicher Schüler:innen entwerfen. Am Beispiel des jahrgangsgemischten Mathematikunterrichts zeigt Matter, dass sich basierend auf dem Verständnis von Mathematik als Wissenschaft der Muster und Strukturen ein gemeinsamer Austausch unter den Lernenden in unterschiedlichen Zonen der Entwicklung entfalten kann. Im Beitrag von Selter und Spiegel wird deutlich, dass sich der Umgang mit Heterogenität nicht nur auf die Verschiedenheit der kognitiven Kompetenzen fokussiert, sondern als wichtige Zieldimension des Mathematikunterrichts auch der sogenannte affektive Bereich berücksichtigt werden muss. Ebenso ist der individuellen Kreativität im Mathematikunterricht Raum zu verschaffen, was nach Bruhn unter anderem durch offene Aufgaben gelingen kann.

Neben den im ersten Teil dargestellten Ideen zum Umgang mit Heterogenität, die sich vorwiegend auf bestimmte Heterogenitätsdimensionen beziehen, fokussiert der zweite Teil eher Lernangebote und ihren konkreten Einsatz in heterogenen bzw. inklusiven Lerngruppen. So nimmt zunächst Roos allgemein das Design inklusiver Lernumgebungen zur Realisierung inklusiven Mathematikunterrichts unter besonderer Berücksichtigung der Perspektive der Schüler:innen in den Blick. Die Bedeutung substanzieller Lernumgebungen für den inklusiven Mathematikunterricht betonen auch Häsel-Weide und Nührenbörger in ihrem Beitrag, wobei aus ihrer Sicht eine differenzierte Begleitung durch die Lehrkraft die Kooperation der Lernenden realisieren hilft, indem die Kinder mit unterschiedlichen Entdeckungen in Kontakt gebracht werden. Bönig und Thöne beleuchten in ihrem Beitrag die Chancen und Grenzen des Einsatzes problemhaltiger Sachaufgaben in inklusiven Settings und diskutieren deren unterrichtliche Realisierungen. Am Beispiel der Behandlung von Symmetrie im dritten Schuljahr machen Graumann und Graumann auf der Basis einer differenzierten, kritischen Sicht auf inklusionsadäquate Lehrmethoden Mut zum inklusiven Unterricht. In einer ebenfalls geometrischen Lernumgebung zu platonischen Körpern für heterogene Lerngruppen im Lehr-Lern-Labor ‚Mathe-Spürnasen‘ entfalten Kaya, Rütten und Weskamp-Kleine vielfältige Lerngelegenheiten nicht nur für Grundschulkinder, sondern auch im Rahmen der Professionalisierung angehender Lehrkräfte. Büchter und Donner schlagen eine substanzielle Lernumgebung vor, bei der Schüler:innen von der Umwandlung von Stammbrüchen in die Dezimalschreibweise ausgehend zu vielfältigen zahlentheoretischen Entdeckungen gelangen können.

Für das gemeinsame Lernen bieten auch digitale Medien zahlreiche Unterstützungsmöglichkeiten. Darauf fokussieren die Beiträge im dritten Teil. Basierend auf einer Darstellung zentraler Konstruktionskriterien digitaler Lernumgebungen stellen Höveler und Mense in ihrem Beitrag Potenziale und Herausforderungen der Apps ‚Kombi‘ und ‚Book Creator‘ zur Konstruktion digital unterstützter kombinatorischer Lernumgebungen dar. Biehler und Frischemeier präsentieren eine digitale Lernumgebung zur Datenanalyse und -exploration für den inklusiven Stochastikunterricht und zeigen dabei, wie digitale Werkzeuge Arbeitsprozesse auslagern und entlasten können sowie durch Interaktivität und Variation der Repräsentationen vielfältige Einsichten in Zusammenhänge ermöglichen. Auf die Herausforderungen einer Bildung in der digitalen Welt müssen zukünftige Lehrkräfte angemessen vorbereitet werden. In diesem Zusammenhang beschreibt der Beitrag von Barzel, Hasebrink, Schacht und Stechemesser Lehr-Lern-Materialien aus dem Projekt ‚DigiMal.nrw‘ zur Weiterentwicklung der universitären Lehramtsausbildung im Fach Mathematik am Beispiel einer Geometrie-Vorlesung.

Dem Diagnostizieren und Fördern im inklusionssensiblen Mathematikunterricht widmet sich der vierte Teil. Von Gaidoschik wird ein konträr zum aktiventdeckenden Lernen stehender Ansatz einer fachdidaktisch und empirisch gestützten Kritik unterzogen, um Argumente dafür zu liefern, warum es ein Fehler und keinesfalls im Interesse von Kindern mit Lernschwierigkeiten ist, Verstehen durch Auswendiglernen ersetzen zu wollen. Breucker und Wember würdigen exemplarisch im Bereich des Sachrechnens die auch durch Petra Scherer vertretene Position, wonach Kinder mit und ohne Lernschwierigkeiten durch aktiventdeckendes Lernen und produktives Üben und in Verzahnung von Diagnose und Förderung gemeinsames Lernen voneinander und miteinander ermöglicht sowie bei Unterstützungsbedarf fundierte Hilfen angeboten werden können. Damit Lehrkräfte ihre Schüler:innen adaptiv unterstützen können, sind u. a. diagnostische Kompetenzen wichtig. Moser Opitz und Wehren-Müller gehen in ihrem Beitrag der Frage nach, welche möglichen Schwierigkeiten eines Kindes Regel- und

Förderlehrkräfte in Fallbeispielen erkennen und mit welchen fachlichen Argumenten sie ihre Annahmen begründen, und fokussieren darüber hinaus die Unterschiede, die sich beim Erkennen und Begründen von Schwierigkeiten zwischen den Regel- und den Förderlehrkräften ergeben. Der Abgleich zwischen den von Lehramtsstudierenden antizipierten und tatsächlichen Schüler:innenbearbeitung aus der Praxis interessiert Jung, Schorein, Spree und Velten im Hinblick auf die (Weiter-)Entwicklung einer universitären Lehrveranstaltung, die versucht, anhand vertiefter fachlicher und fachdidaktischer Auseinandersetzung an beispielhaften Lernumgebungen ohne direkten Praxisbezug Grundlagen für die professionelle Weiterentwicklung im Rahmen von Praxiselementen zu schaffen.

Im abschließenden fünften Teil werden weitere inklusionsorientierte Aspekte zum Mathematiklehren thematisiert. Für die inklusionsbezogene Lehrkräfteprofessionalisierung mit einem vielfalts- und bildungsgerechtigkeitsorientierten Profil ist ein interdisziplinärer Austausch von zentraler Bedeutung. Der Beitrag von van Ackeren-Mindl, Wolfswinkler, Cantone und Gebken arbeitet am Beispiel der Universität Duisburg-Essen heraus, wie vor dem Hintergrund aktueller bildungspolitischer Diskurse und des spezifischen Profils der Universität die inklusionsbezogene Lehrkräfteprofessionalisierung durch die fachdidaktische Entwicklung inklusiver Lernumgebungen in Mathematik substanziell befruchtet werden kann. Pfitzner, Sträter, Gebken und Liersch betrachten in ihrem Beitrag potenzielle Bezüge in der Professionalisierung angehender Sport- und Mathematiklehrkräften. Ein großes Potenzial für die individuelle Professionalisierung angehender Lehrkräfte im Hinblick auf den Umgang mit Heterogenität sehen Rottmann und Wellensiek in einem Exkursionsseminar an eine dänische Folkeskole. Hošpesová und Tichá zeigen, inwiefern Selbst- und besonders Gemeinschaftsreflexionen von Lehrkräften die Möglichkeit bieten, die Qualität des Mathematikunterrichts zu verbessern und eine Vernetzung zwischen Theorie und Praxis zu schaffen. Eine qualitative Untersuchung zur fachspezifischen Kooperation von Grundschullehrpersonen und Sonderpädagog:innen im Rahmen des inklusiven Mathematikunterrichts in Bezug auf die Einstellungen zur Kooperation, die praktizierten Kooperationsformen und -niveaus sowie die Herausforderungen und Erwartungen wird von Geisen vorgestellt. Rösken-Winter, Shure und Penava arbeiten in ihrem Beitrag Potenziale von Scriptwriting als Aktivität und Instrument im Rahmen der Lehrkräfteprofessionalisierung und Qualifizierung von MultiplikatorInnen heraus, wobei das antizipierende Fortsetzen einer (Unterricht- bzw. Fortbildungs-)Szene in schriftlicher Form einer vertieften Auseinandersetzung mit fachlichen und fachdidaktischen Inhalten dient. Die Wahrnehmung und der Umgang mit der Heterogenität der Lehrkräfte in Fortbildungsveranstaltungen durch Multikplikator:innen stehen im Fokus des Beitrags von Bertram, Costa Silva und Rolka.

In diesen fünf Teilen wird das weite Feld deutlich, in dem Petra Scherer in Forschung und Lehre tätig ist. Wir wünschen ihr zu ihrem 60. Geburtstag alles Gute und hoffen, dass sie die Mathematikdidaktik weiterhin noch viele Jahre bereichern wird.

Die Herausgebenden danken allen Autor:innen, die mit ihren Beiträgen diese Festschrift ermöglicht und die unterschiedlichen Facetten in der Arbeit von Petra Scherer im Bereich inklusiven Lehren und Lernens von Mathematik beleuchtet haben.

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Teil I

Umgang mit Heterogenität

Merging Horizons: History of Mathematics and Cultural Diversity

Hans Niels Jahnke

1 Historical Sources in the Mathematics Classroom: An Experience of Cultural Diversity

Reading historical sources in the mathematics classroom is an important field of activity within the HPM community (HPM = History and Pedagogy of Mathematics). It refers to all levels of teaching of mathematics. Some references might highlight the scope and variety of these activities (Arcavi & Isoda, 2007; Laubenbacher & Pengelley, 2000; Kjeldsen & Blomhøj, 2012; Barnett et al., 2014; Jahnke, 2014; Chorlay, 2016). An early overview is given by Jahnke et al. (2000), more recent ones are Jankvist (2014) and Clark et al. (2016).

Working on an historical episode in the mathematics classroom is an experience of cultural diversity. Students are confronted with a world of different motives for doing mathematics, different ways of thinking, and different linguistic and symbolic means. In Germany, most classrooms in which such work is done comprise students of different cultural and religious backgrounds. In this case, one speaks of a *cross-cultural* situation. The most general and well-known approach in this domain is ethnomathematics, a term proposed by Ubiratan d'Ambrosio (2006; also Milton et al., 2017). Ethnomathematics aims at studying mathematical practices and ideas of different cultural groups, in history or at present. Included are mathematical practices of experts as well as of practitioners, such as practical geometries in Islamic countries (Moyon, 2011) or calendars and currency in Icelandic culture

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(Bjarnadóttir, 2015), mental arithmetic as taught in schools in contrast to calculations done by street vendors (Milton et al., 2017, ch. 5). This approach provides also fruitful ideas for the pedagogy of mathematics (Milton et al., 2017, part III).

Two examples may illustrate possible experiences of students. Hosson (2015) reports on teacher students studying early Greek and Chinese cosmologies. An especially striking experience concerns the different interpretations of the same 'shadow observation' in Greek and Chinese texts. Whereas the Greeks used this observation for calculating the circumference of the earth (Eratosthenes' method) the Chinese Chin Shu, a book written around 635 AD under the assumption of a flat earth, took it for calculating the distance of the sun from the earth. Of course, in modern eyes the latter is not correct and, thus, the students have to evaluate this cultural difference, an exciting and demanding task (Hosson, 2015, p. 577ff.).

Glaubitz (2011) studied Al-Khwarizmi's al-jabr (around 800 AD) and his methods of solving quadratic equations with a large number of students (10 classrooms, grade 9). At the end of his thesis (Glaubitz, 2011, p. 357f.), he quotes some remarks of students showing how their image of the Arabic-Islamic culture was changed by this work: "I did not know that the Arabs have done so much for mathematics and science. Today, you would think that they were a bit backward. Therefore, I find it important to learn, what the people there have achieved so early." "Now I think in a different way about the Arab countries and their culture, against which I had prejudices before." "I ask myself: Why have they lost their lead? Why did the Americans land on the moon and not the Arabs?"

Developing Substantial Learning Environments, Rütten et al. (2018) use also historical material as one possible approach for providing different perspectives on a mathematical topic and for interrelating internal as well as external subjects of mathematics (also Rütten & Scherer, 2019).

In studying an historical episode, teachers and students enter a situation in which they are confronted with a multitude of interpretations and points of view. In a certain way, of course, this is the case with any mathematical subject. 'Negotiation of meaning' is the essence of any teaching (Gellert & Krummheuer, 2019). Nevertheless, entering a historical topic poses particular problems. Thus, it will be helpful to discuss shortly what it means to interpret and understand a text.

2 Understanding a Text

What does it mean to interpret a text? Obviously, there is an author (or a group of authors), and it is the task of the reader to understand as best as possible what the author intended to say. It was a great achievement when nineteenth century historicism realized that interpretations of texts necessarily involve considering and carefully studying the historical situation, the context, and the author as an historical person. In the twentieth century, linguistic researchers and philosophers came to understand that interpreting a text depends on the reader and his historical situation as well. Hans Georg Gadamer (1900–2002) in the middle part of his *Wahrheit und Methode* (1990, originally published in 1960; English translation Gadamer, 2004) philosophically elaborated this idea and analysed in detail what a reader and his interpretation add to a text. In consequence, in the triad of author – text – reader today's hermeneutics strongly emphasizes the reader's contribution to an interpretation (Jahnke, 2019 and the literature quoted there).

Gadamer's conception of hermeneutics might be grouped around his three fundamental concepts *application*, *prejudice,* and *hermeneutic circle*. To understand what is meant by application it is best to start with legal and religious texts. Both have the function to tell people how to behave in a concrete situation that is to *apply* them. Obviously, a community can only exist when this is done in every concrete case in the same way. But frequently, it is not clear from the outset what a law or a religious text say to the case in question. A law court or a priest must judge and decide on the 'right' interpretation of the text. Commentaries are written, and a whole system of procedures is established for judging and deciding concrete cases. One can say that any such judgement and decision has added to and changed the original text, or as Gadamer (2004, p. 320) put it:

> This implies that the text, whether law or gospel, if it is to be understood properly – i.e., according to the claim it makes – must be understood at every moment, in every concrete situation, in a new and different way. Understanding here is always application.

Thus, the core of application is the fact that a text is understood "at every moment, in every concrete situation, in a new and different way". Whether we read a love poem or a historical document on an administrative act we inevitably relate it to our situation, our ideas, concepts, emotions, phantasies, former experiences, former studies etc. That is, we apply the text. And by applying the text, we add to it connotations and dimensions of meaning the author necessarily could not have thought of.

Interpreting a text also depends on the expectations, intentions, and questions the reader has in mind and under which he approaches the text. These are determined by the personal intellectual history of the reader which itself is embedded in the culture of his time. Gadamer characterizes this mixture of previous knowledge and intentions by his concept of *prejudice*. This means that we are embedded in tradition, and tradition suggests concepts and questions we pose regarding the texts we study. Contrary to the usual negative connotations of this concept in Gadamer's view prejudices are not an obstacle, but a condition of understanding. However, prejudices become a problem when we remain unconscious of them, and it is our duty to uncover our prejudices as much as possible. This, however, will not lead to an 'extinction of one's self' as historian Leopold von Ranke (1795–1886) had designated the neutrality of a researcher in doing history. In fact, there is no neutrality. Rather, there is a twofold influence of the historical situation in which we read a text. We can best understand this by looking at the human sciences in general. On the one hand, a scientist studying a text would first ask for the state of the art and embed his research problem into this context. On the other hand, there is the 'application' of human sciences which consists, for example, in contributing to societal debates on, say, ethics, aesthetics, politics and culture in general.

In Jahnke (2014, p. 84ff.) and Fried et al. (2016, p. 216ff.), the reader will find a short account of the *hermeneutic circle* applied to reading sources in the mathematics classroom. There is a 'temporal distance' (Gadamer, 2004, p. 303) which separates the reader from the text, and which is to overcome in the act of reading. In the HPM community this distance is frequently called 'dépaysement' (Barbin, 1994) or 'alienation'. In Gadamer's terminology, understanding amounts to a 'merging of horizons' (Gadamer, 2004, p. 310ff.) and he describes the very process by which the merging is achieved by a spiral, the so-called 'hermeneutic circle'. The process needs a point of departure that is an expectation of what the text is about and which questions it might answer. Then, while reading, the reader realizes that some aspects of the expectation do not agree with what is said in the source. Thus, he has to modify the expectation, read again, modify, and so on until he is satisfied with the result.

The hermeneutic circle is a process of adaption. Successful interpretation means that the harmony between the expectations of the reader and the text is step by step enhanced. Gadamer describes this process as a dialectical oscillation between whole and part. This might refer to the interplay between the meaning of a single word and of a phrase in which a word occurs. In further steps, the reader has to take into account the meaning of, for example, a paragraph in its interplay with the whole text. The dialectics of part and whole is a principal problem of understanding, experienced when reading a piece of literature as well as of mathematics.

In a way the hermeneutic circle is quite analogous to the spiral of modelling and can be considered as a process in which a hypothesis is put up, tested against the (empirical) data, modified, tested again and so on until the creator of the model arrives at a satisfactory result. With modelling, too, it is an important point of view that it aims not only at a better and better representation of the problem in question, but that it is also dependent of the situation and the needs of the creator of the model.

In the language of interpretative classroom research, reading a source in a classroom comprises two different interrelated processes, a negotiation of meaning between a student and a text as well as a negotiation of meaning among students and teacher with a possible outcome of merging horizons.

3 The Phases of the Moon: An Interplay of an Historical Source and Naked-Eye Observations

In Sect. 4 we shall sketch a teaching unit in which students are to read two *reading pieces* on earth, moon, and sun in an ancient Greek booklet on astronomy. The idea is to relate their study to naked-eye observations of moon and sun at daytime and night by the students.

The source from which the reading pieces are taken is a booklet of about 80 pages (in the English translation) bearing the title 'The Heavens' (Bowen & Todd, 2004; German transl.: Czwalina, 1927). Its author Cleomedes is not mentioned by any other ancient author. But from astronomers and observations which Cleomedes mentioned in his book, historians of science conclude that he should have lived in the second century AD. *Reading piece one* is from book II, chapter 4, line 21 to 55 on the light of the moon, and *reading piece two* is from book II, chapter 5, line 41 to 80 in which Cleomedes' gives explanations on the phases of the moon.[1]

The idea of the teaching unit is inspired by Martin Wagenschein (1896–1988). Wagenschein himself never has proposed to read historical sources in a classroom at school, but historical texts formed an important part of his own intellectual life. He seems to have read quite a number of writings by famous scientists, f. e. Aristarch, Foucault, Galilei, Leonardo da Vinci, Kepler, Lichtenberg, Newton to mention only a few. He did so for two motives:

The first one can be called '*genetic*'. Wagenschein argued that the 'old scientists' are in fact the young ones, since we can learn from them how they arrived at their results and thus answer the epistemological question of 'How is science possible'. People working in the HPM community are well aware that this optimistic expectation is true only for a small number of sources, but we shall see that at least in the present case this will prove to be true.

The second one can be called a '*language motive*'. All his life, Wagenschein was fighting against 'textbook knowledge' since in his view it prevents students

[1] Line numbers according to the Greek original in the edition (Todd, 1990).

from observing and thinking by themselves. In the case of astronomy, many students and adults know a lot about our solar system but they cannot relate this knowledge to what they observe at the sky themselves. The same is true about most physical phenomena in the world around us. Thus, instead of enlightening, textbook knowledge is 'obscuring knowledge'. To counter this Wagenschein searched for authentic, peculiar, even idiosyncratic linguistic representations of observations and phenomena far off the standardized language of modern science and he read historical texts in the hope to find such authentic phrases and wordings.

Regarding the question of 'How is science possible', Wagenschein attributed a special importance and value to the example of the phases of the moon. He argued that a majority of students and adults being asked to explain this phenomenon would refer to the shadow of the earth as a cause for the crescent figure. But by observing sun and moon when they are simultaneously visible at the sky, it is very easy to realize that there is no shadow of the earth involved and to arrive at the 'right' idea. In addition, one can also directly conclude that the sun must be many times more distant from the earth than the moon. Therefore, time and again he returned to Aristarchos' method of determining the relative distances of sun and moon from the earth (Jahnke, 1998). But he was not so much interested in the numerical side of Aristarchos' method, but in the underlying geometric and qualitative understanding. When we suppose that moon and sun are really physical bodies (which was not obvious to the Greeks) and that the sun is a shining body whereas the moon receives its light from the sun, then we can conclude by thinking and imagination a fundamental fact: **always** is one hemisphere of the moon illuminated by the sun, and one hemisphere is dark. Only on the rare occasions of an eclipse of the moon, this is not the case.

With this fundamental idea in mind, students are asked and guided to observe the pair of moon—sun every day during a certain period of about 2 weeks. When both are visible simultaneously during daytime this is no problem. But what about at night? In this case, we have to add the sun in our mind's eyes to the visible configuration by looking in which direction the illuminated hemisphere of the moon is pointing. In both cases, day or night, we can intuitively conclude that the sun must be distant from the earth many times further than the moon, and, since its apparent size is equal to that of the moon, must be many times larger (Sect. 4). In a combination of imagination, thinking, and observation anybody can get, after some training of his eyes and his imagination, a correct qualitative intuition of the configuration earth – moon – sun without any measurement, any technical instrument, and without any numerical calculation.

All this can be observed and concluded from the two fundamental ideas that the moon receives its light from the sun and that always one hemisphere of the moon is illuminated and the other one is dark. No wonder then, that Wagenschein was interested in authentic descriptions of this phenomenon he found in historical sources. In an Italian – German edition of Leonardo da Vinci's "philosophical diaries", he hit upon a short remark of da Vinci's (Ms. Arundel 94r) on the phases of the moon he liked so much that when quoting it he rearranged it in a way that it looked like a poem (UVeD II [1966], p. 67). Here it is.

"Der Mond hat kein Licht von sich aus, und so viel die Sonne von ihm sieht, soviel beleuchtet sie; und von dieser Beleuchtung sehen wir so viel, wieviel davon uns sieht."	"La luna non ha lume da sè, se non quanto ne vede il sole, tanto l'allumina; della qual luminosità, tanto ne vediamo quanto è quella che vede noi."	"The moon has no light out of herself, and as much as the sun sees of her, as much he illuminates; and of this illumination we see as much as much of it sees us."

What Wagenschein will have fascinated most was the animistic turn in this aphorism of da Vinci's: the sun *sees* a hemisphere of the moon and a part of this hemisphere *sees* us on earth. Sun and moon are considered as intentionally acting living beings. In fact, this is a complete alienation of our usual view in which animate and inanimate nature are sharply contrasted. We shall see in the next section how suggestive and helpful this animistic language can be.

According to his own testimony, the phases of the moon were also an important intellectual companion to Karl Raimund Popper (1902–1994), one of the most influential philosophers of the twentieth century. Popper was interested in and deeply impressed by the Presocratic philosophers since the time when he was 16 years old, and he cultivated this interest all of his life (Popper, 2006, p. 88). But it was only in 1956 that he published a paper entitled 'Back to the Presocratics' in his *Conjectures and Refutations* (1976a; 1st edition in 1956). Since the 1970s, on advice of the later editor A. Petersen, Popper began to think seriously about writing a book on Parmenides and the Presocratics (Popper, 2006, p. 9ff.). He wrote and rewrote several papers including a preface, but the book itself appeared only after his death (Popper, 1998).

Parmenides (520?–450? BC) was a student of Xenophanes, the teacher of Zeno and lived in the newly founded Greek colony of Elea in Southern Italy (Popper, 1998, p. 139). We know of Parmenides' philosophical thinking by a poem in hexameters in the style of Homer and Hesiod of which only 180 lines out of estimated 800 lines have been passed on to us. According to Popper, Parmenides' work "is beset with problems that perhaps will never be solved" (Popper, 1998, p. 139).

Nevertheless, Popper tried to develop an interpretation whose tentative character becomes clear from the fact that his book on Parmenides contains several different attempts.

According to Popper, "Parmenides was the first who consciously placed reality and appearance in opposition and consciously postulated one true unchanging reality behind the changing appearance" (Popper, 1998, p. 140). How did Parmenides arrive at this distinction and to understand its importance? Popper's attempt to answer this question points to the phases of the moon. This is an essential issue in his book. There are three chapters around this idea with slightly modified titles and representing different attempts at describing the idea and its context.

Popper's explanation is rather straightforward. "Parmenides discovered that the observation […] that the Moon – Selene – waxes and wanes during the course of time is false. […] She does not change in any way. Her apparent changes are an illusion." (Popper, 1998, p. 108). "The moon does not change. It is a material sphere of which one half is always illuminated, the other half is always dark." (Popper, 1998, p. 108). In eternity, the moon does not change The changing shape of the moon is mere appearance. It does not really exist, it is 'not being'.

Popper was as excited as Wagenschein about the insight that a hemisphere of the moon is always illuminated. Just like Wagenschein Popper rewrote the respective passage in his source (Parmenides' poem), this time as a love poem. Reporting how, as a boy of 16 years, he hit upon the Presocratics Popper commented: "The verses that I liked best were Parmenides' story of Selene's love for radiant Helios […] before reading Parmenides' story it had not occurred to me to watch how Selene always looks at Helios' rays […]

Bright in the night
with the gift of his light,
Round the Earth she is erring,
Evermore letting her gaze
Turn towards Helios' rays"

(Popper, 1998, p. 88f., Popper's translation of Parmenides). And he added: "Since the day when I first read these lines (in Nestle's translation), 74 or 75 years ago, I have never looked at Selene without working out how her gaze does indeed turn towards Helios' rays (though he is often below the horizon)." Popper, 1998, p. 89). At another place he added: "I personally am indebted to him [Parmenides] for the infinite pleasure of knowing of Selene's longing for Helios […]"(Popper, 1998, p. 130).

To consider Parmenides' phrases as a love poem about goddess Selene (the moon) and god Helios (the sun) is only weakly suggested by the Greek wording. Only the half sentence that Selene is always looking for Helios' rays might be in favor of this interpretation. Therefore, it is a (de)construction by Popper that is similar to Wagenschein's poetic deconstruction of da Vinci's short remark. Both, Wagenschein and Popper, by showing a sensitivity to the possibly artistic quality of their sources, created a personal and individual relation to them.

4 The Phases of the Moon: Sketch of a Teaching Unit

Naked-eye observations of astronomical events presuppose visibility and adequate weather conditions. This makes a scheduling of a teaching unit difficult and a certain flexibility is necessary. It will not always be possible to organically interrelate the reading of the source with observing moon and sun. Sometimes there might be time between both activities. In any case, the teacher has to be attentive for the right conditions over a longer period of time. Therefore, the source and the possible naked-eye observations are described independently of each other and in a second step some relations between them are discussed.

The Source In the following the two reading pieces (for exact details see Sect. 3) are presented and analysed. For reasons of space the first one will only be paraphrased whereas the second will be quoted literally.

In *reading piece one* on the light of the moon, Cleomedes discusses three alternative theories. The first one says that the moon is shining by itself which is refuted by pointing to the eclipses of the moon. The second claims that the moon is reflecting the light of the sun. This is the 'accepted' modern theory which, however, is refuted by Cleomedes. The third one is his own theory which may be considered as a combination of the first and the second. He exposes this theory by comparing the shining moon with a heated piece of iron. When heated by fire the iron will finally glow by itself. Thus, according to Cleomedes, the moon is "altered by the light of the Sun, and through such a blending possesses its own light not intrinsically, but derivatively" (Bowen & Todd, 2004, p. 86). In modern terms, one could say that the light of the sun is seen as sort of a catalysator which causes those spots of the moon hit by sun light to burn by themselves. The proposed reading piece comprises this third theory and two arguments against theory number two. These are: (1) Reflection could take place from solid bodies, even from water, but not from "rarefied bodies". In this case, light would be absorbed like a sponge absorb water. In their commentary, Bowen and Todd (2004, p. 88) say that Cleomedes might have considered the moon as a solid body surrounded by an atmosphere like the earth, whereas the German translation suggests that Cleomedes might have considered the moon as a cloud of gas. (2) If, however, the moon would be a solid body the reflected light could not reach further than two stades (around 360 m) as, according to Cleomedes, experiences show. This, of course, would again refute theory two.

Reading piece one requires of the students to identify the three different theories and to evaluate the arguments in favour and against each theory. A particular point is Cleomedes' refutation of the modern accepted theory. Most students will already 'know' that the moon reflects the light of the sun but very few will ever have asked themselves whether this can be possible at all, a typical case of textbook knowledge. Sure, we can see far distant objects. But can they also illuminate us and our surroundings as moon light does? Are there phenomena in our everyday experience that speak in favour of this theory? Or against it? This is not an easy question, and, thus, in a natural way the reading piece draws our students into ancient discussions and generates in a natural way a feeling for the temporal distance which separates us from the ancient text.

Reading piece two runs as follows:

> The cause of the Moon's having differences in its shapes could be more effectively summarized if we used the following procedure to learn what happens to it. Two circles are conceived of in the Moon: *A*, the one by which its dark part is separated from its illuminated part; *B*, the one by which the part visible to us is separated from the part that is invisible. Each of these circles is smaller than *C*, the circle that can divide the Moon into two equal parts, that is, its great circle. Because the Sun is larger than the Moon, it illuminates more than half of it, and thus *A* (the circle that separates the dark from the illuminated part) is smaller than *C* (the great circle of the Moon). *B* (the circle in our line of sight) is, by the same token, necessarily smaller than *C* (its great circle), since we see less than half of the Moon. The reason is that when a spherical body is seen by two eyes, and the distance between them is less than the diameter of the [sphere] that is being seen, the part [of the sphere] that is seen is less than half. So since *B* divides the Moon not into equal, but into unequal, parts, it too is smaller than *C*, the great circle.
>
> Both *A* and *B*, however, appear as great circles relative to our perception, and while they always have the same size, they still do not maintain the same fixed position, but cause numerous interchanges and configurations relative to one another as at different times they coincide with one another, or slope to intersection at an oblique angle. Most such intersections are minimal interchanges, but, as is the case with a genus, all are of two kinds: a right-angled [intersection], and one in which they intersect obliquely with one another. There are also only two coincidences: when they coincide at conjunction, and at full Moon.
>
> Now when the Moon passes by the Sun after conjunction, circles *A* and *B* distance themselves from one another, and slope to intersection at an oblique angle, so that all

> that is left illuminated, at least in relation to us, is the small [area] between the circumferences of both. This type of transition, from the coincidence of the circles to their intersection, completes the Moon's crescent shape, since as the circles continually move toward intersecting one another at right angles, they also increase the phase of illumination, since the [area] between the intersection of the circles is always illuminated in such a progression.
>
> When the figure of intersection reaches right angles, the Moon is seen at the [first] quarter. But when the circles proceed from this figure to obtuse angles, they cause the deity's gibbous shape, while they cause full Moon by again being fully coincident at opposition. Then by proceeding again from this coincidence to yet another, and by completing the same shapes as they wane, they proceed to the point at which all the luminance disappears when the circles *A* and *B* exactly coincide with the part of the Moon that faces the heavens. That is essentially our discussion concerning the waxings and wanings of the Moon.

The text contains three basic ideas: (1) It introduces the circle *A* which separates the dark and the illuminated part of the moon and the circle *B* in our line of sight and separating the visible (front) part of the moon from the invisible (back) part. In the following we call *A* the dark-light circle and *B* the visibility circle. (2) It is rightly stated that both circles *A* and *B* are smaller than a great circle and that the illuminated part of the moon is always larger than the dark one as well as the visible part is always smaller than the invisible. Since in both cases the difference is very small ("they appear as great circles to our perception"), they are considered as great circles. Note that in defining the dark-light circle *A* it is implicitly said that, except for eclipses of the moon, one half of the moon is **always** illuminated by the sun. We see nearly always only parts of *A* and *B*. While at new moon neither of them is visible, they collapse into one circle at full moon. (3) The last two paragraphs explain how the different phases of the moon from new moon over crescent figures to half and full and back to new moon correspond to the continuous change of the angle α between the planes *A* and *B*.

In Fig. 1, circles *A* and *B* are seen 'from above' (the north pole) as line segments during the first quarter of the waxing moon. We define angle α as that angle between *A* and *B* which measures the counter clockwise rotation of *B* in regard to *A* and is 0 at new moon. Astronomers call α the elongation of the moon. Figure 1 shows three possible positions for α during the first quarter ($\alpha \leq 90°$). The rays of sun light come from right and, because of the enormous distance between sun and moon, are considered as parallel. *P* is an observer on earth. The observer, the moon, and the sun form a cosmic triangle and define a plane. It is easily proved that α is always equal to the angle sun – earth (observer) – moon.

The usual textbook and internet visualizations of the different shapes of the moon during a month are similar to that in Fig. 2 whereas Fig. 1 provides a new

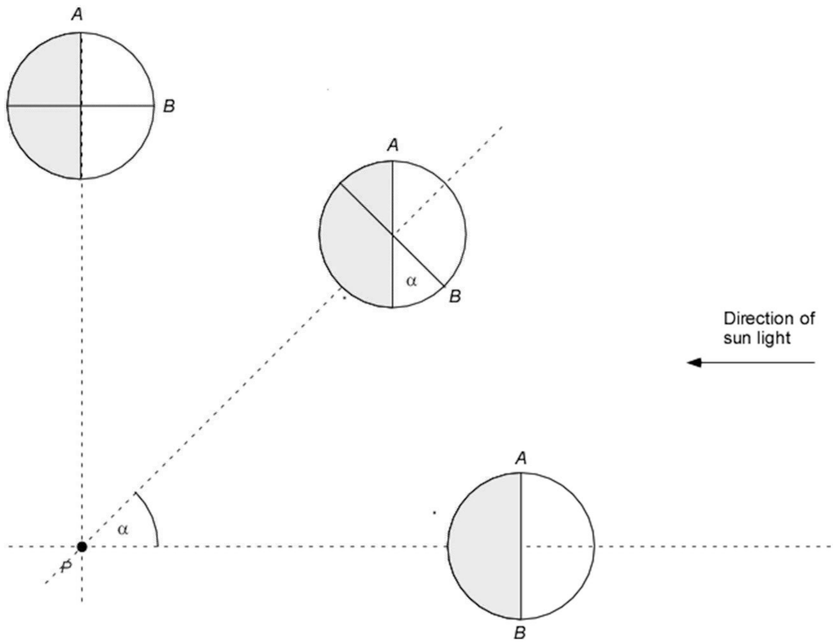
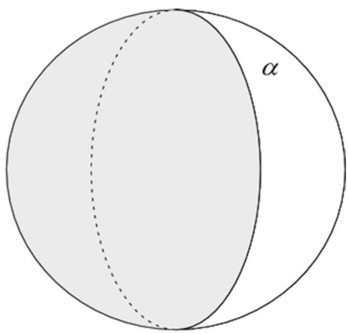


Fig. 1 Relative positions of circles *A* and *B* at three points of time

Fig. 2 Waxing crescent



useful perspective inspired by the source. Figure 2 shows a crescent with α = 45° corresponding to the middle position in Fig. 1. The visibility circle *B* is represented by the outer circumference of the moon whereas the dark-light circle *A* is pictured by the ellipsis with one solid and one dotted arc. Normally, on the real sky the observer sees only the (white) crescent.

In the following we turn to the naked-eye observations and start with **Perspectives on a sphere**. To visualize the moon with its dark and illuminated half sphere, one can draw a great circle on a spherical object (f. e., a ball) and symbolize (by color or any other symbol) which half should represent the illuminated part. In Fig. 3, we see on the left a great circle drawn along the seam of the ball and luckily the valve is exactly in the center of the right half sphere. Thus, we say that this half should represent the illuminated part of our model moon since then the valve points directly to the sun. With such a model moon students can actively work, observe it from different sides and sketch the different shapes of crescents dependent on the elongation α. A possible task for the students could be to complete Fig. 1 to a full turn of the moon and to draw for any value of α the respective waxing or waning crescents.

Observing the Sky Figure 4 is a sketch similar to Wagenschein (without date, 1) and shows sun and a waxing crescent in the daytime. It serves for describing possible activities and ways of argumentation when observing the sky. It does not and should not replace real observations with the students. The plane of projection is the fictitious plane where sun and moon seem to be situated on the sky. The straight line is the horizon of the observer (the small figure). The observer should be imagined as standing far in front of the plane of projection such that he looks obliquely upwards to sun and moon. The observer can conclude: (1) The crescent is not caused by the shadow of the earth since the earth does not stand between sun and

Fig. 3 A ball as a physical model

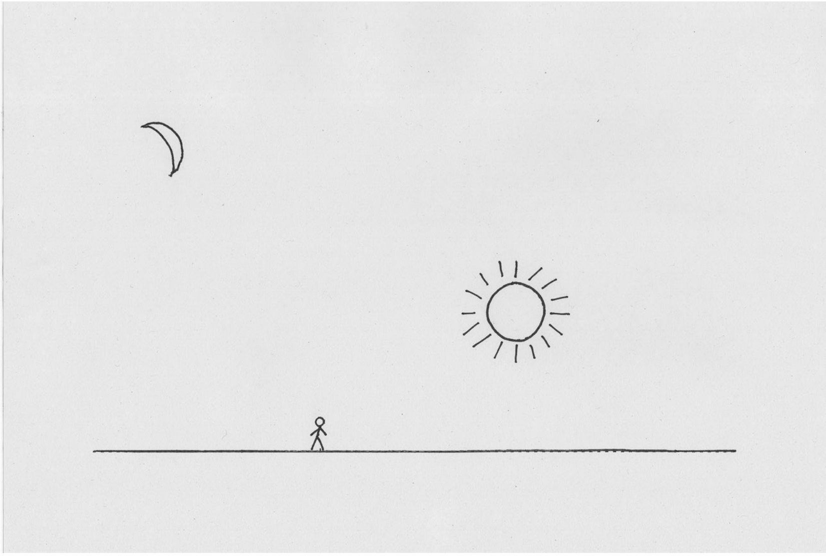


Fig. 4 A typical configuration of a waxing crescent and the sun

moon. (2) Since there is also no other celestial body in the vicinity of moon and sun one half of the moon must necessarily be illuminated by the sun. (3) Thus, it is very plausible to the observer that he sees a part of this illuminated half sphere and that moon light is reflected sun light. When the observer like Cleomedes cannot believe that reflection can produce such bright light on such a large distance he may follow Cleomedes' theory which attempts to explain this. (4) With some thinking the observer will realize that he sees the illuminated half of the moon from obliquely behind. It points neither to us nor to the apparent position of the sun but far away into the cosmos. This is only possible when the sun is also far away and "behind" the moon. To be more precise, the orthogonal to the plane of circle A and the ray from the observer to the apparent sun intersect at the true position of the sun and are nearly parallel. Therefore, the sun is considerably more distant from us than the moon. (5) Since sun and moon as they appear at the sky have the same angular diameter of 0.5°, the sun must be larger than the moon in the same amount as it is more distant. (6) Under the assumption that moon and sun do not change their relative position in the course of a day, we can mentally rotate the configuration until the sun is below horizon and the moon is still visible and, thus, approximately infer where the invisible sun might be posited. The reader is urgently invited to observe the configuration of moon and sun in reality to get a feeling of how compelling the above observations and conclusions are.

Commentary Which simplifying assumptions have been made by Cleomedes and implicitly by the discussion above? In modern speech they might be called modelling assumptions though the term modelling must not necessarily be mentioned in the classroom: (1) Cleomedes considers the dark-light circle *A* and the visibility circle *B* as great circles, though in fact they are smaller. In a commentary, Czwalina (1927, p. 90) says that both circles *A* and *B* have a spherical radius of 89.5° in contrast to the great circle with 90°. This means that when the moon has radius *r* then *A* and *B* have a radius of $\cos(0.5°) \times r = 0.999969 \times r$. (2) In Fig. 1, we have assumed that the light rays of the sun are parallel. This was also a frequent simplifying assumption in ancient astronomy. (3) We assume implicitly that over a day sun and moon do not change their relative position. This means that neither the sun nor the moon do move noticeably. (4) We also assume that the sun does not move over a month though it progresses by around 30° along its yearly orbit. (5) Geometrically, we get the exact position of the sun in Fig. 4 as point of incidence of the half ray from the observer to the apparent sun with the symmetry plane of the two vertices of the crescent.

The Teaching Unit as a Whole: Relations Between Source and Naked-Eye Observations As already said, scheduling the teaching unit is difficult. Ideally, a first outdoor observation of a constellation like in Fig. 4 should be done before reading the source, in any case as early as possible.

Reading piece one on the light of the moon can be set as homework, and 2 h might be sufficient for identifying the three theories and to evaluate them. It is not important to reach a conclusive result. Rather, students should get a feeling about how the questions on the substance of the moon are interrelated with the phenomena we observe and about the temporal distance which separates us from the source we read.

Reading piece two requires probably another 4 h. It should be treated in combination with working on the model moon. Of course, every student should have his own sufficiently large model, ideally a ball or another spherical object of that size. A possible task is to make sketches like Fig. 2 for the different phases new moon, waxing crescent, waxing half-moon, waxing gibbous, full moon, and the respective waning shapes.

Communication during the outdoor observations (eye protection!) requires the development of special means of expression depending on the habits of the students. For example, students could show the position of the dark-light circle *A* by means of the palm of their left hand. The sun stands necessarily on the perpendicular to the palm. The students might then point with their right arms to the sun. The real position of the sun is the point of incidence of the perpendicular to the left hand palm and the direction of the right arm. Since both lines are nearly parallel the point of incidence must be very far distant. Another possibility could be the use of an animistic language: “Where is the moon looking for?” “The moon turns its back on us.” “The moon shows most of its front side.” “The moon looks a little to the right of us past.” Or one can apply the language we have used in explaining Fig. 4, number (4).

There are standard procedures for working on texts well-known from the language lessons as f. e., structuring a text, reproducing the main ideas in one’s own language, or drawing sketches. These methods should also be applied when reading historical sources. The product will be a collection of self-written texts, sketches, and hopefully a good mental image of the different constellations of sun and moon which produce the different shapes of the moon we can observe.

Depending on the classroom, there are numerous possibilities of deepening the inherent geometry, f. e., pointwise construction of a crescent by circle and ruler, or by means of a DGS like Geogebra, or a construction of the visibility circle B. The latter will drastically show that Cleomedes’ simplification of considering it as great circle does make sense only for very large cosmic triangles. Later, this teaching unit could be used as an entrance to 3-dimensional descriptive geometry.

In any case, the students should reflect on what they have done. In fact, without any technical instrument by naked-eye observations they could produce a very suggestive and cogent explanation of the different shapes of the moon in the course a month. Until well into the nineteenth century, there was no possibility for really investigating the questions raised by reading piece one. Whether the moon is a solid body without or with an atmosphere or even a cloud of gas could not be decided. Thus, it was the cogency of the geometrical explanation that led to the acceptance of the idea that the moon reflects the light of the sun. The teaching unit provides a very good opportunity to reflect on how new knowledge is generated or, as Wagenschein had put it in a Kantian style, of “How is science possible?”

Coming back to the theme of cultural diversity one can distinguish between implicit and explicit dimensions. Implicit dimensions refer to the previous knowledge and experiences of the students. Sun and moon are an important part in the daily life of every human being and influence heavily and permanently their perception of time. Sun and moon are also present in children’s and youth literature. There they might appear as means of spatial and temporal orientation as is the case in astronomy and navigation, as appearances arousing fear or romantic feelings, or moon and sun might be personalized as friendly smiling beings or as gods. They are at the same time familiar and unfamiliar. In some religions the course of the moon defines the feast days, especially in the Islamic and Jewish calendar, also the Christian Easter is determined by the moon. A crescent moon is part of the national