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Handbook of Geometry and Topology of Singularities V: Foliations

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Foreword: Theory of Complex Foliations in Moscow and Outside

The geometric theory of complex differential equations was originated in the early 1950s by Petrovsky and Landis. They introduced complex limit cycles and the analogue of the real Poincaré map, and proved that a generic planar polynomial vector field has no algebraic orbits, a fact known in the West as the Jouanolou theorem. They stated a persistence conjecture for complex limit cycles; this conjecture stays open even now.

There are two approaches to the theory of planar polynomial foliations. One may consider a class of polynomial vector fields that have degree no greater than n in a fixed affine neighborhood of the complex projective plane. These foliations generically have an invariant line at infinity, which contains in general $n+1$ singular points. They give rise to a so called monodromy group at infinity that determines very specific properties of the foliation. The class of these foliations is denoted by $\mathcal{A}_n$. Another class is the class of vector fields that have degree not greater than n in any affine neighborhood of $\mathbb{C}P^2$. This class is denoted by $\mathcal{B}_n$. Generic foliations of this class have no invariant complex lines at all.

Khudai-Verenov, a student of Landis, proved that a generic planar polynomial vector field of class $\mathcal{A}_n$ has the minimality property: all its orbits except for the singular points and the line at infinity are dense. This was the origin of the topological theory of complex foliations. For a while this theory attracted the attention of the leading young mathematicians of the 1960s: Anosov, Arnold, Novikov, Vinogradov and others. A conjecture that a generic planar polynomial vector field is structurally stable was discussed.

Being an undergraduate student, I found a gap in the published proof of the Petrovsky-Landis persistence theorem. Since then I had a dream to find a correct proof: a goal that is not yet achieved. At the same time, complex differential equations became my main subject.

One of the major problems of complex foliations was suggested by Anosov in 1963. *Is it correct that a generic planar polynomial foliation has only simply connected leaves except for no more than a countable number of topological cylinders?* This problem admits the following simplification not yet solved: *Prove that* *a generic planar polynomial foliation has no complex cycles with the monodromy identity.*

A weaker form of this problem was solved in 1995 by Pyartli and I:

Theorem *A generic foliation of class $\mathcal{A}_n$ has no cycle with the holonomy identity on the line at infinity, which is a leaf of the foliation.*

In the late 1970s, it was proved that generic foliation of class $\mathcal{A}_n$ is absolutely rigid (in a sense explained below) and has a countable number of complex limit cycles. It was conjectured that similar results hold in higher dimension.

Partial results in this direction were proved later by Muller, Loray, Rebelo and others. The results for the class $\mathcal{A}_n$ were improved by Shcherbakov, Nakai, Goncharuk, Kudryashov and others. A final version of the rigidity theorem in the quadratic case was proved by Ramírez in 2017.

Theorem *Two quadratic foliations that are topologically equivalent in $\mathbb{C}P^2$ are at the same time affinely equivalent.*

Very recently, Deroin and Alvarez proved an opposite result: there are foliations of class $\mathcal{B}_2$ that are structurally stable (a property opposite to absolute rigidity). In particular, the Jouanolou equation of degree 2 is structurally stable.

Striking results about singularities of linear vector fields in the space of dimension three and higher were discovered in the 1970s by Ladis and independently by Camacho, Kuiper and Palis. It appeared that even for linear vector fields, singularities in the Siegel domain have numeric invariants of topological classification.

In the 1980s, Chaperon proved a complex version of the Grobman-Hartman theorem. It implies the existence of numeric invariants of topological classification of singular points of nonlinear vector fields. For high dimension and degree, the number of the invariants of the polynomial vector fields at their singular points may be greater than the dimension of the space of the vector fields considered.

Thus the absolute rigidity property is expected in at least an open domain within polynomial vector fields of high dimension and degree. It remains to prove that the invariants are in a sense independent, a problem that remains still open.

Roughly speaking, absolute rigidity of vector fields of a certain class means the following: topological equivalence of the vector fields implies their affine equivalence.

In more detail, a vector field of a certain class is absolutely rigid if there exists a neighborhood of this vector field in the class considered, and a neighborhood of the identity in the space of homeomorphisms of the phase space such that if two vector fields from the first neighborhood have topologically equivalent phase portraits conjugated by a homeomorphism from the second neighborhood, then these two vector fields are affinely equivalent.

It seems that in higher dimensions polynomial vector fields form a domain of structurally stable ones as well as one of absolutely rigid vector fields. Are there domains with an intermediate property? To the best of my knowledge, the question is open.

The monodromy group of a polynomial vector field at infinity (a finitely generated group of germs of conformal mappings at a common fixed point) became one of the central objects of the theory. The density of its orbits was discovered by Khudai-Verenov.

It was found in the 1970s that generically these groups have a countable number of isolated periodic orbits, and are topologically rigid: two groups that are topologically equivalent are at the same time holomorphically equivalent. The genericity conditions were improved by many authors: Shcherbakov, Nakai, Sad and others.

The local theory of individual germs played an essential role in these investigations. It starts with easy results about the analytic equivalence between a hyperbolic one-dimensional germ and its linear part, and becomes very involved when the germ is non-hyperbolic.

For the germs whose linear part is an irrational rotation, very delicate results were obtained by Siegel, Bryuno and Yoccoz. Yet the problem about analytic classification of germs with identity linear part remained in the 1970s widely open.

In the late 1930s, Birkhoff, with his universal interest in the whole world of dynamical systems, real and complex, high and low-dimensional, addressed the problem of parabolic germs and found the transition functions between the Fatou coordinates as the moduli of analytic classification of such germs. But he did not succeed to prove that all these moduli may be realized.

In the mid 1970s, I suggested to my graduate student Voronin to find the analytic classification of parabolic germs. We knew nothing about our predecessors, and rediscovered the moduli already found by Birkhoff. Parabolic germs have a very simple formal normal form. For a long time we tried to prove that the corresponding formal series do converge (and Arnold supported these efforts). At last it became clear that, on the contrary, all the moduli may be realized, and the normalizing series diverge in general.

In 1981, the analytic classification of parabolic germs was completed: simultaneously and independently by Écalle and Voronin, though by completely different ways; and also by Malgrange, who delivered Écalle's results at the Bourbaki seminar but who used his own approach, closer to Voronin's than to Écalle's one. As a consequence, the moduli found are now called Écalle-Voronin moduli.

There is a tradition followed by a part of the Russian school that goes back to Petrovsky, who, according to Landis, his student, said: "When I start solving a new problem, I do not try to find what the predecessors have done; if they had found something useful, the problem would be solved." This is certainly debatable. However, later, Voronin came across a book on functional equations with about 400 references to the papers that study parabolic germs. If he tried to learn all this, he would never solve the problem.

Inspired by Malgrange, Martinet and Ramis found the analytic classification of complex saddle-nodes (planar singular points with one zero eigenvalue) and resonant saddles. The latter result was independently obtained, but not completely published, by Elizarov, Voronin and myself. In 1993, *Nonlinear Stokes Phenomena*, the book summarizing all these investigations, was published by the AMS.

In the late 1970s, Arnold discovered that some purely analytic problems of singularity theory contain a sort of hidden dynamics. Thus, the Écalle-Voronin moduli became applicable in singularity theory outside dynamical systems.

The theory of functional moduli is developing even now.

In the last part of their work, Petrovsky and Landis estimated from above the number of complex limit cycles generated by a perturbation of some integrable foliations. The leaves of the foliations they considered were rational curves, that is, Riemann spheres with a finite set of punctures, and the Melnikov integrals were reduced to residues. I remember how frightened I was, still an undergraduate student at that time, when I realized that the problem of perturbation of Hamiltonian foliations whose leaves were Riemann surfaces of arbitrary genus should be studied in full generality.

In two papers of 1969, I investigated the generation of limit cycles under perturbation of Hamiltonian foliations with hyperbolic leaves of arbitrary genus. I got a lower rather than upper estimate of the real limit cycles that might be generated by a perturbation of a Hamiltonian foliation and proved that they may be located in a diversive manner. Between the lines, the Infinitesimal Hilbert 16th problem was stated: *Give an upper estimate of the number of limit cycles that may be generated from the ovals of a Hamiltonian foliation by a perturbation of this foliation.* After 40 years, this problem was solved by Yakovenko and his students Novikov and Binyamini. One should note that the limit cycles born from polycycles are not counted. The problem: *How many real limit cycles may be generated by a perturbation of a real Hamiltonian foliation by a polynomial of the same degree?* still stays open.

In my paper of 1969, there was a theorem whose true statement remained hidden from me. It was Lins Neto who discovered this statement. Some preliminary notes are needed.

It is well known since Poincaré and Hilbert that the set of polynomial vector fields of degree n with a singular point whose linear part is a center and that is an actual center form an algebraic manifold, the so called manifold of centers.

The statement of Lins Neto was: *Hamiltonian vector fields form an irreducible component of the manifold of centers.*

In 2000s, Hossein Movasati, student of Lins Neto, proved that *Polynomial vector fields with a rational first integral and linear center at zero form an irreducible component of the manifold of centers.*

A similar result was proved recently by Christopher and Mardesic for polynomial vector fields with a Darboux first integral.

An open problem is: *Are there any other irreducible components of the manifold of centers different from the three ones named above?*

The *exactness theorem*, proved in 1969, claims:

If the integrals of a polynomial one-form of degree n over the ovals surrounding a singular point of a generic real polynomial of degree $n + 1$ are all zeros then the form is exact.

Khovanskaya in 1997 improved this theorem and replaced ovals surrounding a singular point by arbitrary real ovals.

Developing the exactness theorem, Gavrilov in 1998 invented so called *Petrov moduli*. Given a generic polynomial H, a space of all polynomial 1-forms factorized by exact forms and the forms proportional to dH becomes a finitely generated modulus over a ring of polynomials on H.

The theory of polynomial perturbation of polynomial Hamiltonian vector fields is still a challenging and intensively developing part of the theory of complex foliations.

I see the following major open problems of foliation theory in the complex domain.

Persistence. Do complex limit cycles persist as functions of the parameters of polynomial differential equations?

Uniformization. Where is the boundary between the simultaneously uniformizable and non-uniformizable (algebraic) foliations located?

Anosov problem. Is it true that a generic polynomial foliation has all leaves simply connected except for (a countable number of) topological cylinders?

Rigidity versus structural stability. What is the interplay between structural stability and absolute rigidity? Are there intermediate foliations?

Y. S. Ilyashenko

Preface

This is the fifth volume of the Handbook of Geometry and Topology of Singularities, and this forms a unit together with Volume VI, focused on singular holomorphic foliations.

Singularities are ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. Their scope is vast, their purpose is multifold. Its potential for applications in other areas of mathematics and of knowledge in general is unlimited, and so are its possible sources of inspiration. Singularity theory is a crucible where different types of mathematical problems interact and surprising connections are born.

Foliation Theory is a multidisciplinary field and a whole area of mathematics in itself, with close connections with dynamical systems, geometry, topology and singularity theory. For instance:

(i) The integral lines of a vector field on a manifold, or the orbits of a smooth flow, determine a foliation with singularities at the fixed points of the flow. Holomorphic actions of the complex numbers on complex manifolds, or holomorphic vector fields, define holomorphic one-dimensional foliations with singularities at the zeros of the vector field.

(ii) Differential 1-forms on a manifold determine a codimension-one distribution (a sub-bundle of the tangent bundle), except at the points where the 1-form vanishes. Under a certain "integrability condition," this gives rise to a codimension-one foliation with singularities at the points where the 1-form vanishes. If the 1-form is holomorphic, we get a holomorphic foliation.

(iii) If we consider a holomorphic function f from an $(n + k)$-manifold M into an n-manifold N which is a submersion (like a projection locally) at most points, and we consider the fibers $f^{-1}(y)$ of all points in $y \in N$, we get a codimension-n holomorphic foliation on M, with singularities at the critical points of f.

(iv) Open-books are a special class of foliations, with codimension-one leaves (the pages) and a codimension-two singular set (the binding). The Milnor fibrations associated to isolated critical points of holomorphic maps are examples of open-books.

(v) Lefschetz pencils in algebraic geometry are examples of singular holomorphic foliations.

These important examples highlight the deep connections between foliations and singularity theory, and the reasons for having these volumes on foliations as a part of this Handbook, with Felipe Cano, an expert in holomorphic foliations, as a fourth editor for these two volumes.

A foliation means, naively, a partition of a manifold into connected subsets called the leaves, which are immersed manifolds, and one has a local product structure, except that there may be some special points: its singularities.

The theory of holomorphic foliations has its origins in the study of differential equations on the complex plane by C. F. Gauss, A. L. Cauchy, B. Riemann, K. Weierstrass, J. C. Bouquet, C. A. Briot, L. Fuchs, J. Liouville, G. Darboux, P. Painlevé, H. Dulac, I. G. Petrovsky, C. L. Siegel, L. S. Pontryagin and others. At the end of the nineteenth century, Liouville observed that it was not possible to find explicit solutions to most differential equations. A few years later, H. Poincaré stressed the importance of analyzing the topological, geometrical and analytical properties of the solutions of differential equations, even without giving their explicit expressions. This was a landmark for the birth of dynamical systems, for the qualitative theory of differential equations, and eventually for foliation theory.

The concept of "foliation" was formalized in the 1940s in a series of papers by G. Reeb and Ch. Ehresmann. This was inspired by the theory of differential equations, where the phase manifold gets decomposed into one-dimensional real or complex lines, as the case may be. This gives a local partition of the manifold, where, at each regular point of the differential equation one has a flow box, or a product decomposition. The new idea was passing from local to global, and having higher dimensional "leaves," as one does, for instance, when considering the fibers of a submersion. Notice that in the complex case, the decomposition obtained by considering the integral lines of a vector field is by one-dimensional complex curves, so these have real dimension two.

After the early first steps, intensive development came into the theory, both in the real and complex cases, and important research schools were developed worldwide.

On the one hand, the deep results for holomorphic foliations by A. Haefliger, B. Malgrange, J. Martinet, J. P. Ramis, R. Moussu, J. F. Mattei, J. Écalle, É. Ghys, D. Cerveau and many others have made of France a *Mecca* for Foliation Theory. Of course this had significant influence in other countries, and particularly in Spain, where J. M. Aroca, F. Cano and others now have a research school with excellent mathematicians all over the country.

Simultaneously, in the former Soviet Union, the geometric theory of complex differential equations was developed in the early 1950s by I. G. Petrovsky and E. M. Landis. This fascinating theory soon attracted the attention of the leading young mathematicians of the 1960s: D. V. Anosov, V. I. Arnold, S. P. Novikov, R. E. Vinogradov, and soon afterwards, Yu. S. Ilyashenko, S. Yu. Yakovenko,

S. M. Voronin, A. N. Varchenko, A. G. Khovanskii, A. A. Bolibruch, A. D. Bryuno, D. I. Novikov, G. S. Petrov and many others, that have made of the Soviet Union, and now Russia, a main pole of development for complex foliations.

On the other hand, in the early 1970s, C. Camacho finished his Ph.D. in Berkeley, working with S. Smale on a thesis about smooth group actions, and then moved to the IMPA in Brazil, where, together with P. Sad and A. Lins Neto, and also with M. Soares at Belo Horizonte, built up a strong research school on holomorphic foliations. Some 10 years later, A. Verjovsky and X. Gómez-Mont started building up a research school on complex foliations in Mexico. The early results of Gómez-Mont were essential to lay down the foundations of deformation theory for complex foliations, and the seminal work by Verjovsky on the uniformization of the leaves of holomorphic one-dimensional foliations has opened an important line of research. The Mexican school has students and collaborators in Russia, France, Spain and Brazil, thus profiting from all those schools. The interaction in Foliation Theory between Mexico, France and Brazil is apparent, for instance, in the area of research known as LVM manifolds, whose genesis is in the classical paper by Camacho, Kuiper and Palis on linear $\mathbb{C}$-actions (see for instance Lopez de Medrano's paper in Volume II of this Handbook).

We are happy to have in these volumes important contributions from all of these schools, and others.

Let us say a few words about volumes V and VI. These have nine chapters each, and these cover a large scope of the theory of analytic foliations. Besides these, Volume V starts with a foreword by Professor Yulij Ilyashenko, while Volume VI ends with an epilogue by Professor Jean-Pierre Ramis, two of the main world leaders in the theory of complex foliations.

The foreword by Prof. Ilyashenko explains some of the most important lines of research in the theory of holomorphic foliations and it states some major open problems about:

Persistence. Do complex limit cycles persist as functions of the parameters of polynomial differential equations?

Uniformization. Where is the boundary between the simultaneously uniformizable and non-uniformizable (algebraic) foliations located?

Anosov problem. Is it true that a generic polynomial foliation has all leaves simply connected except for (a countable number of) topological cylinders?

Rigidity versus structural stability. What is the interplay between structural stability and absolute rigidity? Are there intermediate foliations?

Besides the important open problems stated by Prof. Ilyashenko, we would like, as editors of this volumen, to point two other open problems that have been ubiquitous in the Theory along half a century: the problem of reduction of singularities of singular holomorphic foliations and the problem of the existence of a minimal exceptional set for holomorphic foliations of the projective plane.

The epilogue by Jean-Pierre Ramis is about Stokes phenomena, another ubiquitous topic. This is an important legacy for the next generations, as it abounds in deep thoughts and reflections, and it has plenty of historical roots and a vast bibliography.

Chapter 1 of this Handbook is an introduction to singular holomorphic foliations. The chapter introduces the reader to the theory of foliations by proving a couple of important results. The first is the theorem of Mattei-Moussu about the existence of holomorphic first integrals for germs of holomorphic foliations, and the second is the linearization theorem of Camacho-Lins Neto-Sad for foliations in the complex projective plane. The chapter discusses the Frobenius integrability theorem, the concepts of the holonomy group and the Poincaré map, Riccati foliations and Kupka singularities, the blow up method for reducing singularities, the various notions of equivalence for foliations, and it arrives to the linearization theorem of Poincaré-Dulac.

Chapters 2–5 are concerned with one-dimensional holomorphic foliations in dimension two. Chapter 2 is devoted to various geometric properties of complex foliations in $\mathbb{C}^2$ and $\mathbb{CP}^2$. It is a survey of some problems, conjectures and relations between them, and it focuses on two major problems of complex foliations: persistence of complex cycles and simultaneous uniformization of leaves. Chapter 3 is a survey focused on the local understanding of the solutions of differential equations in $(\mathbb{C}^2, 0)$. This is a self-contained account in which the problem of formal-analytic rigidity of vector fields and foliations with dicritical and non-dicritical singularities, their corresponding formal and analytic normal forms and the analytic classification invariants—the Thom invariants—are discussed.

In Chap. 4, the authors give a survey on the topology of singularities of holomorphic foliation germs in $(\mathbb{C}^2; 0)$. Two foliation germs are topologically equivalent (or C^0-conjugate) if there is a homeomorphism between two open neighborhoods that sends leaves into leaves. The final goal of a topological classification would be to obtain a list of foliation germs containing an element of each topological class, with minimal redundancy. The chapter begins with a historical approach to the topological study of singularities of foliation germs. The purpose of this chapter is to describe, giving ideas of the proofs, results obtained by the authors on the topology of the leaves, the structure of the leaves space and criteria of conjugacy for any two foliation germs not necessarily contained in a C^0-trivial deformation.

Chapter 5 uses the classical concept of polar curve in singularity theory, as studied by Teissier, Lê and others, to study foliations. Polar curves can be thought of as being a particular case of the Jacobian curve, which can be defined as the contact curve between two hamiltonian foliations. The aim of this chapter is to describe the results obtained by the author about Polar and Jacobian curves of foliations, explaining the main tools used in the proofs and the relations with known results for plane curves.

Chapter 6 is about generalizations of the classical Rolle theorem, claiming that between any two roots of a real valued differentiable function on a segment must lie a root of its derivative. The authors discuss important generalizations of this theorem for vector-valued and complex analytic functions and for germs of holomorphic maps. The unifying feature for these results lies at the heart of their proofs and springs from the fact that these can be regarded as generalizations of the Rolle theorem. This gives information about various problems in analysis (real and complex) and geometry.

Chapter 7 is a review concerning the study of the local dynamics of a gradient vector field of a real analytic function. Starting from the famous Thom's Gradient Conjecture, the author discusses the state of art of the Finiteness Conjecture on non-oscillation of trajectories and ends with the formulation of the problem in the context of o-minimal geometry.

Chapter 8 is an introduction to the use of techniques originally developed by Newton to study local solutions of algebraic and ordinary differential equations. This beautifully written chapter studies first the application of Newton's polygon to algebraic equations with coefficients in a valued field, and it shows the limitations of its use for valuations of rank greater than one. The Theorem of Kaplansky is the key to have explicit solutions for the rank one case. For differential equations, the lack of a Differential Kaplansky's theorem is an obstacle to have a general theory. Yet, the author studies generalizations to certain ordinary differential equations.

Volume V ends with Chap. 9 that reviews properties of closed meromorphic 1-forms and of the foliations defined by them. It presents and explains classical results from foliation theory, like index theorems, existence of separatrices and resolution of singularities from the viewpoint of the theory of closed meromorphic 1-forms and flat meromorphic connections. The author investigates the algebraicity of separatrices in a semi-global setting (neighborhood of a compact curve contained in the singular set of the foliation), and the geometry of smooth hypersurfaces with numerically trivial normal bundle on compact Kähler manifolds.

Let us mention briefly the content of Volume VI.

The first chapter is by Adolfo Guillot that studies the singularities of complete holomorphic vector fields on complex manifolds. Chapter 2 by Julio Rebelo and Helena Reis studies the global dynamics of singular holomorphic foliations on complex manifolds of dimension three. The foliations considered are mostly one-dimensional, but codimension one foliations are also envisaged.

Chapter 3 by Alcides Lins Neto studies the irreducible components of algebraic foliations of codimension one in complex projective spaces. Chapter 4 is by Maurício Corrêa and it is a survey on problems and results on singular holomorphic foliations and Pfaff systems with invariant analytic varieties on complex manifolds. Chapter 5 by Felipe Cano and Beatriz Molina-Samper is concerned with the question of R. Thom about the existence of an invariant hypersurface for germs of holomorphic codimension one foliations, where the reduction of singularities plays a central role.

A key concept and tool for studying codimension one holomorphic foliations is the transversal pseudo group. Chapter 6 by Isao Nakai recalls the fundamental results on this subject, beginning from the basics. Chapter 7 is by Javier Ribón and it gives a constructive description of the Zariski-closure of subgroups of the group $\widehat{\mathrm{Diff}}(\mathbb{C}^n)$ of formal diffeomorphisms, and it explains why this is meaningful from a geometric viewpoint.

Chapter 8 by Frank Loray is an introductory text to the Riemann-Hilbert correspondence, aimed to graduate students and researchers. The text explains the main ideas and remains short enough to allow readers to keep the whole picture in mind.

Finally, Chap. 9 by Emmanuel Paul and Jean-Pierre Ramis is concerned with the extension of the Riemann-Hilbert correspondence in the irregular case and focuses on Painlevé equations. The authors describe the dynamics on the character variety related to the Painlevé fifth equation, and they present many consequences of this original approach.

There are many other important contributions to the theory of analytic foliations that could (or even, should) have been included in these volumes. We are grateful to all the very many mathematicians worldwide that have contributed to the theory.

Volumes V and VI of the Handbook complement the previous four volumes of this collection, which is addressed to graduate students and newcomers to singularity theory, as well as to specialists who can use these as guidebooks.

The first four volumes of this collection gathered foundational aspects of the theory, as well as some other important aspects. Some topics are studied in various chapters, and in some cases, also in more than one volume. The topics studied so far include:

- The combinatorics and topology of plane curves and surface singularities.
- The classification of plane curves.
- Introductions to four of the classical methods for studying the topology and geometry of singular spaces, namely: resolution of singularities, deformation theory, stratifications and slicing the spaces *à la* Lefschetz.
- Milnor's fibration theorem for real and complex singularities, the monodromy, vanishing and Lê cycles.
- Morse theory for stratified spaces and constructible sheaves.
- Limits of tangents to complex varieties, a subject that originates in Whitney's work.
- Zariski's equisingularity and intersection homology.
- Singularities of mappings. Thom-Mather theory.
- The interplay between analytic and topological invariants of complex surface singularities and their relation with modern three-manifold invariants.
- Indices of vector fields and 1-forms on singular varieties.
- Chern classes and Segre Classes for singular varieties.
- Mixed Hodge structures.
- Determinantal singularities.
- Arc spaces.
- Lipschitz geometry in singularity theory, and many other important subjects.

This collection is aimed to provide accessible accounts of the state-of-the-art in various aspects of singularity theory, its frontiers and its interactions with other areas of research. This will continue with a Volume VII discussing other important areas of singularity theory and its interactions.

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Chapter 1
Holomorphic Foliations: Singularities and Local Geometric Aspects

Bruno Scárdua

Abstract This text has been written with the aim of providing a fast introduction to the framework of holomorphic foliations with singularities. Our methodology is based in proving a couple of important results. The first is the theorem of Mattei-Moussu about existence of holomorphic first integrals for germs of holomorphic foliations. The second is the linearization theorem of Camacho-Lins Neto-Sad, for foliations in the complex projective plane. With these we address both aspects, local and global, of this interesting subject. This text is highly influenced by the author's personal interests and it is not intended to exhaust the subject, nor to be a complete full introduction to this beautiful field. Indeed, I also recommend various other texts as, for instance, by D. Cerveau and J.-F. Mattei, Y. Ilyashenko and F. Loray (see references below). I hope these notes will be helpful to those interested in this interesting field in mathematics.

1.1 Introduction

The theory of foliations is one of those subjects in mathematics that gathers several distinct domains such as topology, dynamical systems and geometry, among others. Its origins go back to the works of C. Ehresmann and Shih ([25, 26]) and G. Reeb ([65, 66]). It provides an interesting and valuable approach to the qualitative study of dynamics and ordinary differential equations on manifolds.

Although its origins are in the classical framework of real functions and manifolds, the notion of foliation is also very useful in the holomorphic world. Indeed, it has ancient origins in the study of complex differential equations. From these first problems, the introduction of singularities as an object of study is a natural step. We mention the works of P. Painlevé ([58, 59]) and Malmquist ([49]). With P. Painlevé the study of rational complex differential equations of the form $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$ has its first more specific methods and results. After Painlevé many

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authors have contributed for the initial push up of the theory, among them are E. Picard, G. Darboux, H. Poincaré, H. Dulac, Briot and Bouquet.

Complex differential equations appear naturally in mathematics and in natural sciences ([2, 3, 35]). For instance, we mention the theory of electrical circuits, valves and electromagnetic waves ([60]). Another motivation is the search for and study of new (classes of) transcendent functions, as the liouvillian functions ([69, 71]).

With the advent of the geometric theory of foliations and the modern results of Cartan, Oka, Nishino ([57]), Suzuki ([74–76]) and others, on the theory of analytic functions of several complex variables and some from algebraic and analytic geometry, this field of research became quite active again. To these days it is one of the active branches of modern research in mathematics. It also includes important connections with algebra and algebraic geometry ([30, 41, 47, 71]). We refer to [22, 38, 46] for accounts on the subject; see also [6, 8, 12, 14, 17, 20, 39, 40, 48, 56] for some other aspects not mentioned in the text.

1.2 The Notion of Holomorphic Foliation

This section introduces classical notions of foliations in the complex analytic framework. The reader which is already familiar with these notions in the real smooth case, may skip to the next section. We refer to [11, 27, 34] or [54] for more complete accounts on the theory of real foliations.

1.2.1 Motivation

There are some ways of motivating the concept of foliation. Probably, the very first is given by a holomorphic submersion $f: M \to N$ from a complex manifold M into a complex manifold N. By the complex analytic version of the local form of submersions, the level sets $f^{-1}(y)$, $y \in N$ are embedded complex submanifolds of M. These fibers are *locally* organized as the fibers of a projection $(x, y) \mapsto y$. This local picture is not necessarily global, and the fibers may be disconnected.

A second important example is given by a non-singular closed holomorphic 1-form ω on a complex manifold M. By the complex version of the integration lemma of Poincaré we can write *locally* $\omega = df$ for a holomorphic submersion map f taking values on $\mathbb{C}$. Any local function f as above, defined in an open subset in M, is called a *first integral* for ω.
Notice that two local *first integrals* f and $\widetilde{f}$ for ω in a same connected subset of M are related by $\widetilde{f} = f + c$ for some constant $c \in \mathbb{C}$. Therefore, f and $\tilde{f}$ do share level sets, these local sets can therefore be globalized as immersed (locally closed) complex submanifolds of M, again locally organized as fibers of a projection.

The third and last basic example we shall mention is the one provided by a holomorphic (complex analytic) vector field X on a complex manifold M. Given a point $p \in M$ which is not a singular point of X, the complex flow-box theorem gives a conjugation between (X, U), where $p \in U \subset M$ is an open neighborhood, and a nonzero constant complex vector field on $\mathbb{C}^m$, where $m = \dim M$ is the complex dimension of M. The orbits of X in U then follow the same geometrical condition of the above examples.

The above examples motivate the classical definition of foliation below.

1.2.2 Definition of Holomorphic Foliation

The purpose of this section is to introduce, in a formal way, the concept of holomorphic foliation. In fact, a holomorphic foliation is, in particular, a foliation in the classical sense.

Definition 1.2.1 Let M be a complex manifold of (complex) dimension n. A *holomorphic foliation of M, of dimension* k, or *codimension* $n-k$, $1 \leq k \leq n-1$, is a decomposition $\mathcal{F}$ of M in pairwise disjoint immersed complex submanifolds (called *leaves* of the foliation $\mathcal{F}$) of dimension (complex) k, and having the following properties:

- (i) $\forall p \in M$ there exists a unique submanifold L_p of the decomposition that passes by p (called the *leaf through* p).
- (ii) $\forall p \in M$, there exists a holomorphic chart of M (called *distinguished chart of* $\mathcal{F}$), (φ, U), $p \in U$, $\varphi\colon U \to \varphi(U) \subset \mathbb{C}^n$, such that $\varphi(U) = P \times Q$, where P and Q are open polydiscs in $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$ respectively.
- (iii) If L is a leaf of $\mathcal{F}$ such that $L \cap U \neq \emptyset$, then $L \cap U = \bigcup_{q \in D_{L,U}} \varphi^{-1}(P \times \{q\})$, where $D_{L,U}$ is a countable subset of Q.

  The subsets of U of the form $\varphi^{-1}(P \times \{q\})$ are called *plaques* of the distinguished chart (φ, U).

A foliation of dimension one is also called *foliation by curves*. In this case, the leaves are Riemann surfaces.

Observe that (iii) also implies that the leaves are immersed submanifolds immersed in M. Indeed, the intersection of a leaf with a distinguished chart is a union of disjoint plaques.

1.2.3 Other Definitions of Foliation

There are essentially three ways to define foliations (cf. [11, 27, 34]). In addition to the one we just have given in Definition 1.2.1 above, we have the following.

Definition 1.2.2 Given a complex manifold M of dimension m; by a *codimension* $0 \leq n \leq m$ *holomorphic (complex analytic) foliation of* M, we mean an atlas $\mathcal{F} = \{(U_j, \varphi_j)\}_{j \in J}$ of M, where each coordinate chart $\varphi_j\colon U_j \subset M \to \varphi_j(U_j) \subset \mathbb{C}^{m-n} \times \mathbb{C}^n$ is holomorphic and we have the following compatibility condition:

For each non-empty intersection $U_i \cap U_j \neq \phi$, the corresponding change of coordinates

$$\varphi_j \circ \varphi_i^{-1}\big|_{\varphi_i(U_i \cap U_j)}\colon \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

preserves the natural horizontal fibration $y = y_0$, for some constant $y_0 \in \mathbb{C}$, of $\mathbb{C}^{m-n} \times \mathbb{C}^n \ni (x, y)$.

This is equivalent to say that, in coordinates $(x, y) \in \mathbb{C}^{m-n} \times \mathbb{C}^n$, we have

$$\varphi_j \circ \varphi_i^{-1}(x, y) = \big(h_{ij}(x, y), g_{ij}(y)\big) \in \mathbb{C}^{m-n} \times \mathbb{C}^n.$$

The charts $\varphi_j : U_j \to \varphi_j(U_j)$ are called *foliation charts, trivializing charts* or *distinguished charts* of $\mathcal{F}$. The local *plaques* of $\mathcal{F}$ are the fibers of a foliation chart in $\mathcal{F}$. Given a holomorphic diffeomorphism $\varphi : U \hookrightarrow \varphi(U) \subset \mathbb{C}^m = \mathbb{C}^{m-n} \times \mathbb{C}^n$, we say that φ is *compatible* with the foliation $\mathcal{F}$ if for any $j \in J$ such that $U_j \cap U \neq \emptyset$, we also have

$$\varphi_j \circ \varphi^{-1}(x, y) = \big(h(x, y), g(y)\big) \in \mathbb{C}^{m-n} \times \mathbb{C}^n.$$

In short, this is equivalent to say that $\mathcal{F} \cup \{(U, \varphi)\}$ is still a foliation. Using this and Zorn's lemma, we may consider the foliation atlas $\mathcal{F}$ as *maximal*, in the sense that it contains all the compatible charts of M.

In M we consider the equivalence relation induced by the connected finite union of local plaques. This means that two points $x, y \in M$ are equivalent $x \sim y$ if and only if x and y lie in the same plaque of $\mathcal{F}$ or there is a finite number of plaques $P_1, \dots, P_r$, $r \geq 2$; of $\mathcal{F}$ such that $x \in P_1$, $y \in P_r$ and $P_i \cap P_{i+1} \neq \emptyset$ for all $i = 1, \dots, r-1$. Given a point $x \in M$ we call the corresponding equivalence class $[x] \subset M$ is *the leaf of $\mathcal{F}$ through x*. Usually we denote this leaf by $\mathcal{F}_x$ or by L_x. The leaf $L_x \subset M$ is an immersed complex submanifold, but not necessarily embedded. These leaves then decompose M into disjoint immersed complex submanifolds. Each leaf has dimension $m - n$ and meets a foliation chart domain along plaques of the foliation. For instance, in the case of a submersion $f : M \to N$, the leaves of the corresponding foliation are the connected components of the level sets $f^{-1}(y)$, $y \in N$. The quotient space $M/\sim$ is the *leaf space* of $\mathcal{F}$, also denoted by $M/\mathcal{F}$.

The third definition of foliation uses the notion of distinguished maps. Let $\mathcal{F} = \{(U_j, \varphi_j), j \in J\}$ be a foliation of a manifold M in the sense of Definition 1.2.2. Then $\forall i, j$ the transition map $\varphi_j \circ (\varphi_i)^{-1}$ has the form

$$\varphi_j \circ (\varphi_i)^{-1}(x, y) = (f_{i,j}(x, y), g_{i,j}(y)).$$

The map $g_{i,j}$ is a local diffeomorphism in its domain of definition. This follows from the fact that the derivative of the transition map is given by $D(\varphi_j \circ (\varphi_i)^{-1})(x, y) \cdot (v, w) = (\partial_x f_{i,j}(x, y) \cdot v, Dg_{i,j}(y) \cdot w)$, $(v, w) \in \mathbb{C}^{m-n} \times \mathbb{C}^n$. We define for all i the map $g_i = \Pi_2 \circ \varphi_i$, where Π_2 is the projection onto the second coordinate: $\Pi_2 : \mathbb{C}^{m-n} \times \mathbb{C}^n \to \mathbb{C}^n$, $(x, y) \mapsto y$. We claim that $g_j = g_{i,j} \circ g_i$. Indeed, we have $g_{i,j} \circ g_i = g_{i,j} \circ \Pi_2^i \circ \varphi_i = \Pi_2^j \circ (\varphi_j \circ \varphi_i^{-1}) \circ \varphi_i = \Pi_j^j \circ \varphi_j = g_j$. Therefore, a holomorphic foliation $\mathcal{F}$ of codimension n of a manifold M^m is equipped with an open cover $\{U_i\}_{i \in I}$ of M and holomorphic submersions $g_i : U_i \to$

$\mathbb{C}^n$ such that for all i, j there is a local diffeomorphism $g_{i,j} : V_i \subset \mathbb{C}^n \to V_j \subset \mathbb{C}^n$ satisfying the cocycle relations

$$g_j = g_{i,j} \circ g_i, \quad g_{i,i} = Id.$$

The g_i's are the *distinguished maps* of $\mathcal{F}$.

Conversely, suppose that M^m admits an open cover $M = \bigcup_{i \in I} U_i$ such that for each $i \in I$ there is a holomorphic submersion $g_i : U_i \to \mathbb{C}^n$ such that for all i, j there is a diffeomorphism $g_{i,j} : V_i \subset \mathbb{C}^n \to V_j \subset \mathbb{C}^n$ satisfying the cocycle relations above. By the local form of the submersions we can assume that for each $i \in I$ there is a holomorphic diffeomorphism $\varphi_i : U_i \to \mathbb{C}^{m-n} \times \mathbb{C}^n$ such that

$$g_i = \Pi_2 \circ \varphi_i.$$

since

$$\Pi_2 \circ (\varphi_j \circ (\varphi_i)^{-1}) = g_j \circ (\varphi_i)^{-1} = g_{i,j} \circ g_i \circ (\varphi_i)^{-1} = g_{i,j} \circ \Pi_2,$$

we have that the atlas

$$\mathcal{F} = \{(U_i, \varphi_i)\}_{i \in I}$$

defines a holomorphic foliation of codimension n of M. The above suggests the following equivalent definition of foliation.

Definition 1.2.3 A holomorphic *foliation* of M^m of codimension n, is given by the following:

1. An open cover $\{U_i : i \in I\}$ of M.
2. A family of holomorphic submersions $g_i : U_i \to D^n, \forall i \in I$; with the following compatibility property: $\forall i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there is a local diffeomorphism $g_{i,j} : V_i \subset \mathbb{C}^n \to V_j \subset \mathbb{C}^n$ satisfying the cocycle relations

$$g_j = g_{i,j} \circ g_i, \quad g_{i,i} = Id.$$

The submersions g_i's are the *distinguished maps* of the foliation $\mathcal{F}$.

This last definition leads to several interesting definitions. For instance, a foliation $\mathcal{F}$ of M is said to be *transversely holomorphic* or *transversely affine* depending on whether, for some convenient choice, its distinguished maps $g_{i,j}$ are holomorphic or affine maps. We shall resume this subject later on. In order to distinguish foliations, we shall use the following definition.

Definition 1.2.4 Two holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ of manifolds M and M' respectively are *holomorphically equivalent* if there is a holomorphic diffeomorphism $h : M \to M'$ sending leaves of $\mathcal{F}$ into leaves of $\mathcal{F}'$. In other words, if $\mathcal{F}_x$