Solid Mechanics and Its Applications

Edward B. Magrab

Vibrations of Elastic Systems

With Multiphysics Applications

Second Edition

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For
June Coleman Magrab

Preface

Vibrations occur all around us: in the human body, in mechanical systems and sensors, in buildings and structures, and in vehicles used in the air, on the ground, and in the water. In some cases, these vibrations are undesirable, and attempts are made to avoid them or to minimize them; in other cases, vibrations are controlled and put to beneficial uses. Irrespective of the objective of a vibration analysis, the vibrating systems must be modelled. Therefore, the main goal of the book is to take the large body of material relating to the modeling and analysis of vibrating elastic systems and present it in such a manner that one can select the least complex model to capture the essential features of the system being investigated. The essential features of the system could include such effects as in-plane forces, elastic foundations, in-span attachments and attachments to the boundaries, and such complicating factors as piezoelectric elements, elastic coupling to another system, variable geometry, and fluid loading. To assist in the model selection, a very large number of numerical results has been generated for this book so that one is able compare the various models to determine how changes to boundary conditions, system parameters, and complicating factors affect the natural frequencies and mode shapes and the response to externally applied displacements and forces.

The material presented is reasonably self-contained and employs only a few solution methods to obtain the results. For continuous systems, the governing equations and boundary conditions are derived from the determination of the contributions to the total energy of the system and the application of the extended Hamilton's principle. To make the application of the energy approach more efficient, an appendix, Appendix B, is provided with a summary of a general derivation of the extended Hamilton's principle for systems with one or more dependent variables and the conditions necessary for one to be able to generate orthogonal functions. Since a primary solution method employed in this book is the separable of variables, the generation and use of orthogonal functions is very important. Consequently, the use of energy approach, the application of the extended Hamilton's principle, and the results of Appendix B provide the basis for a consistent approach to deriving the governing equations and boundary conditions and the basis for two very powerful solution techniques: the generation of orthogonal functions and the separation of variables and the Rayleigh-Ritz method. The expression for the total energy of the system is used directly as the starting point for the Rayleigh-Ritz method. Irrespective of the solution method, almost all solutions that are obtained in this book have been numerically evaluated by the author and presented in tables and annotated 2D and 3D graphs.

For this edition, multiphysics applications have been added and include: fluid loading on the exterior of beams, plates, and cylindrical shells; fluid loading on the interiors of beams (pipes) and cylindrical shells; beams with in-span single degree-of-freedom systems and inerters, and layered piezoelectric beams for use as energy harvesters. In addition, a new chapter, Chap. 7, introducing the Mindlin-Reissner plate theory has been added, the material on thin rectangular plates with two opposite edges hinged has been expanded, and the following new topics included: inerters, pre-twisted beams, moving masses on beams, beam with a pendulum, finite-length metamaterial beams, and beams with functionally graded materials.

The book is organized into eight chapters, seven of which describe different vibratory models and their ranges of applicability. In Chap. 2, single and two degree-of-freedom system models are used to obtain a basic understanding of vibration isolators and absorbers. In this regard, new material on inerters, quasi-zero stiffness spring configurations, and bio-inspired designs are introduced.

In Chap. 3, the Euler-Bernoulli beam is presented. The effects on the vibratory response of this model are determined for various factors: in-span and boundary attachments such as a concentrated spring, concentrated mass, single degree of freedom system, or inerter. The latter two are examined for their ability to act as vibration absorbers. The effects of the cross-section properties on the natural frequencies are examined in detail: continuously varying tapers, constant with abrupt changes in properties, extended rigid mass, and pre-twist.

In Chap. 4, the response to forced excitation for various complicating factors for the Euler-Bernoulli beam are considered. The complicating factors include beams with a single degree-of-freedom system or an inerter, the coupling of torsion and bending, fluid loading, beams conveying fluids, and a piezoelectric layered beam used as an energy harvester. The beam with an in-span single degree of freedom system vibration absorber is extended to form a metamaterial and then used to illustrate the band gap phenomenon.

In Chap. 5, the Timoshenko theory is introduced, which gives improved estimates for the natural frequency. One of the objectives of this chapter is to numerically show under what conditions one can use the Euler-Bernoulli beam theory and when one should use the Timoshenko beam theory. Therefore, many of the same systems that are examined in Chap. 3 are re-examined in this chapter and the results from each theory are compared and regions of applicability are identified. Lastly, the natural frequencies are determined for Timoshenko beams with functionally graded materials.

The transverse vibrations of thin rectangular and annular circular plates are presented in Chap. 6 along with extensive numerical results. For solid circular plates, the effects of a concentrated mass and the effects of fluid loading are determined.

In Chap. 7, the Mindlin-Reissner plate theory is introduced, and the natural frequencies and mode shapes are determined for a rectangular plate hinged on all four edges and for a clamped solid circular plate. The results for each of these cases are compared to those obtained in Chap. 6 for the thin plate theory.

In the last chapter, Chap. 8, the Donnell and Flügge theories are introduced for cylindrical shells and used to obtain the natural frequencies and mode shapes for several boundary conditions. The results from these shell theories are compared to each other and to those predicted by the Euler-Bernoulli and Timoshenko beam theories and regions of applicability are identified. The effects of fluid loading when a fluid is in the interior of the shell and when it is on the exterior of the shell are determined.

College Park, USA Edward B. Magrab

Contents

Chapter 1
Introduction

Abstract The importance of vibrations and how vibrating systems are analyzed is presented. This is followed by a statement of the book's goals and a discussion of how they will be met.

1.1 A Brief Historical Perspective

It is likely that the early interest in vibrations was due to the development of musical instruments such as whistles and drums. In was in modern times, starting around 1583, when Galilei Galileo made his observations about the period of a pendulum, that the subject of vibrations attracted scientific scrutiny. In the 1600s, strings were analyzed by Marin Mersenne and John Wallis; in the 1700s, beams were analyzed by Leonhard Euler and Daniel Bernoulli and plates were analyzed by Sophie Germain; in the 1800s, plates were analyzed by Gustav Kirchhoff and Simeon Poisson, and shells by D. Codazzi and A. E. H. Love. A complete historical development of the subject can be found in [1]. Lord Rayleigh's book *Theory of Sound*, which was first published in 1877, is one of the early comprehensive publications on the subject of vibrations. Since the publication of his book, there has been considerable growth in the diversity of devices and systems that are designed with vibrations in mind: mechanical, electromechanical, biomechanical and biomedical, ships and submarines, and civil structures. Along with this explosion of interest in quantifying the vibrations of systems came great advances in the computational and analytical tools available to analyze them.

1.2 Importance of Vibrations

Vibrations occur all around us. In the human body, where there are low-frequency oscillations of the lungs and the heart and high-frequency oscillations of the larynx as one speaks. In man-made systems, where any unbalance in machines with rotating

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parts such as fans, washing machines, centrifugal pumps, rotary presses, and turbines, can cause vibrations. In buildings and structures, where passing vehicular, air, and rail traffic or natural phenomena such as earthquakes and wind can cause oscillations.

In some cases, oscillations are undesirable. In structural systems, the fluctuating stresses due to vibrations can result in fatigue failure. When performing precision measurements such as with an electron microscope externally caused oscillations must be substantially minimized. In air, roadway, and railway vehicles, oscillatory input to the passenger compartments must be reduced. In machinery, vibrations can cause excessive wear or cause situations that make a device difficult to control. Vibrating systems can also produce unwanted audible acoustic energy that is annoying or harmful.

On the other hand, vibrations also have many beneficial uses in such widely diverse applications as vibratory parts feeders, paint mixers, transducers and sensors, ultrasonic devices used in medicine and dentistry, sirens and alarms for warnings, determining fundamental properties of materials, and stimulating bone growth. During the last few decades, there has been an increase in the development of electromechanical devices and systems at the micrometer and nanometer scale. These developments have led to new families of devices and sensors such as vibrating cantilever beam mass sensors, piezoelectric beam energy harvesters, carbon nanotube oscillators, and vibrating cantilever beam sensors for atomic force microscopes.

1.3 Analysis of Vibrating Systems

The analyses of systems subject to vibrations or designed to vibrate have many aspects. Typically, a system is designed to meet a set of vibration performance criteria such as to oscillate at a specific frequency, avoid a system resonance, operate at or below specific amplitude levels, have its response controlled, and be isolated from its surroundings. These criteria may involve the entire system or only specific portions of it. To determine if the performance criteria have been met, experiments are performed to determine the characteristics of the input to the system, the output from the system, and the system itself. Some of the characteristics of interest could be whether the input is harmonic, periodic, transient, or random and its respective frequency content and magnitude. Some of the characteristics of the output of the system could be the magnitude and frequency content of the force, velocity, displacement, acceleration, or stress at one or more locations. Some of the characteristics of the system itself could be its natural frequencies and mode shapes and its response to a specific input quantity.

To design a system to meet its performance criteria, it is often necessary to model the system and then to analyze it in the context of these criteria. The type of model one uses may be a function of its size: the sub micrometer scale, micrometer scale, millimeter scale, or the centimeter scale and greater. The model will also be a function of its shape, the way in which it is expected to oscillate, the way it is supported, and how it is constrained. If shape can be ignored, then the system can be modeled as a spring-mass system. If geometry is important, then one must choose an appropriate representation such as a beam, plate, or shell and decide if the geometry can be treated as a constant geometry or if it must be treated as a system with variable geometry. The system's environment, in conjunction with its size, will determine which type of damping is important and if it must be taken into account. The model may also have to include the effects of any attachments to its interior and to its boundaries and may have to account for externally applied constraints and forces such as an elastic foundation, in-plane forces, and coupling to other elastic systems. Thus, there are many decisions that must be made regarding what should be included in the model so that it adequately represents the actual system.

1.4 About the Book

The main goal of the book is to take the large body of material relating to the modeling and analysis of vibrating elastic systems that include single and two degree-of-freedom spring-mass systems and inerters, Euler–Bernoulli and Timoshenko beams, thin and Mindlin-Reissner rectangular and circular plates, and Donnell and Flügge theories of cylindrical shells and present it in such a manner that one is able to select the least complex model that can be used to capture the essential features of the system being investigated. The essential features of the system could include such effects as in-plane forces, elastic foundations, an appropriate form of damping, in-span attachments and attachments to the boundaries, and such complicating factors as piezoelectric elements, elastic coupling to another system, and fluid loading. To assist in the model selection, a very large number of numerical results has been generated so that one is able compare the various models to determine how changes to boundary conditions, system parameters, and complicating factors affect the natural frequencies and mode shapes and the response to externally applied displacements and forces.

To be able to cover the wide range of models and complicating factors in sufficient detail, an efficient means of presenting the material is required. The approach employed here has been to obtain an expression for the total energy of each model and then to use the extended Hamilton's principle to derive the governing equations and boundary conditions. The expression for the total energy of the system includes the effects of any complicating factors. In addition to providing an efficient and consistent way in which to obtain the governing equations and boundary conditions, the expression for the total energy of the system can be used directly as the starting point for the Rayleigh–Ritz method. Another advantage of the energy approach is that the results given here can be extended to systems that include other effects by modifying the expression for the total energy. A list of the elastic systems, their boundary and in-span attachments, and their additional factors that are considered in this book are given in Table 1.1.

To make the application of the energy approach more efficient, an appendix, Appendix B, is provided with a summary from a general derivation of the extended

Table 1.1 The elastic systems, their boundary and in-span attachments, and their additional factors that are considered in this book

System		Additional factors	Cross section	Boundary attachments	In-Span attachments
Spring-Mass	Single degree-of-freedom	Structural damping Viscous damping: Kelvin model, Maxwell model Quasi-zero stiffness springs Inerters	–	–	–
	Two degree-of-freedom	Inerter absorber Single degree-of-freedom absorber	–	–	–
Beams	Euler–Bernoulli theory	Axial force Elastic foundation Moving mass Fluid loading: internal and external Elastic coupling to another beam Layered piezoelectric beam Coupled torsion and bending: flutter Metamaterial	Constant Continuously variable Constant with abrupt change in properties Pre-twisted	Translation spring Torsion spring Mass Extended mass Pendulum	Translation spring Mass Single and two degree-of-freedom systems Finite-length rigid mass Inerter
	Timoshenko theory	Axial force Elastic foundation Elastic coupling to another beam Functionally graded material Moving mass	Constant Continuously variable Constant with abrupt change in properties	Translation spring Torsion spring Mass	Translation spring Torsion spring Mass Single degree-of-freedom system

(continued)

Table 1.1 (continued)

System		Additional factors	Cross section	Boundary attachments	In-Span attachments
Thin plates	Rectangular	In-plane force Elastic foundation	Constant	Translation spring Torsion spring Mass	Mass Spring
	Circular	In-plane force Elastic foundation Fluid loading	Constant	Translation spring Torsion spring Mass	Mass
Mindlin-Reissner plates	Rectangular	-	Constant	Translation spring Torsion spring	–
	Circular	–	Constant	–	–
Thin cylindrical shells	Donnell’s theory	Internal fluid External fluid	Constant	–	–
	Flügge’s theory	–	Constant	–	–

Hamilton's principle for systems with one or more dependent variables and the conditions necessary for one to be able to generate orthogonal functions. Since a primary solution method employed in this book is the separable of variables, the generation and use of orthogonal functions is very important. Consequently, the use of energy approach, the application of the extended Hamilton's principle, and the results of Appendix B provide the basis for a consistent approach to deriving the governing equations and boundary conditions and the basis for two very powerful solution techniques: the generation of orthogonal functions and the separation of variables and the Rayleigh–Ritz method. It will be seen that a major advantage of the use of the extended Hamilton's principle is that the boundary conditions are a natural consequence of the method. This will prove to be very important when the Timoshenko beam theory, thin plate and Mindlin-Reisner plate theories, and thin cylindrical shell theories of Donnell and Flügge are considered. In these cases, obtaining the boundary conditions can be quite involved if the force balance and moment balance methods are used.

To determine the effects that various parameters and complicating factors have on a system, the following procedure is employed. For each elastic system, a solution for a very general set of boundary conditions and complicating factors as is practical is obtained. Once the general solution has been obtained, many of its special cases are examined in a direct and straightforward manner. This approach, while introducing a little more algebraic complexity at the outset, is a very efficient way of obtaining a solution to a class of systems and greatly reduces the need to re-solve and/or re-derive the equations each time another combination of factors is examined. In most instances, the systems' special cases are listed in tables.

To be able to use the least complex model to represent a system, each subsequent system is compared to its simpler model. For example, the determination of the conditions when the Euler–Bernoulli beam theory can be used instead of the Timoshenko beam theory, when the thin plate theory can be used instead of the Mindlin-Reissner plate theory, and when a beam can be used to model a shell.

An underlying aspect that allows one to present the large amount of material given in this book is the availability of the modern computer environments such as Mathematica® and Matlab®. These programs permit one to devote less space to presenting special numerical solution techniques and more space to the development of the governing equations and boundary conditions, obtaining the general solutions, and presenting and discussing the numerical results. Consequently, virtually all solutions that are derived in this book have been numerically evaluated by the author. This has produced a substantial amount of annotated graphical and tabular results that illustrate the influence that the various system parameters have on their respective responses. In addition, the numerical results are presented in terms of non dimensional quantities making them applicable to a wide range of systems.

Reference

1. Love AEH (1927) A treatise of the mathematical theory of elasticity, 4th edn. Dover, New York, pp 1–31

Chapter 2
Spring-Mass Systems and Inerters

Abstract Several single degree-of-freedom system models are created by arranging their viscous dampers and springs in different configurations. These configurations create the Kelvin model, the Maxwell model, various quasi-zero stiffness models, and bio-inspired designs. For several of these configurations, the response to transient and harmonic force and base excitation are obtained. The definitions of the amplitude and phase response functions and the transmissibility are given. Two degree-of-freedom systems are presented and discussed in their role of vibration absorber and their frequency–response functions are defined. Single and two degree-of-freedom inerters are presented and the role of the inerter as a vibration absorber is indicated.

2.1 Introduction

In determining the response of structural systems to dynamic excitation, one often encounters situations wherein certain aspects of the response are undesirable. In some of these cases, this response can be improved with the attachment of a single degree of freedom system. In other cases, one attempts to minimize the interactions of vibrating object with its support structure using vibration isolation techniques. When the support structure is a flexible one, e.g., a beam or plate, the vibration isolation system is affected by the structure's flexibility and, therefore, must be included in the model. Therefore, in this chapter, we shall analyze one and two degree-of-freedom systems in the absence of the flexible structural aspects; in the subsequent chapters, the structural interactions will be considered.

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2.2 Some Preliminaries

2.2.1 Single Degree-of-Freedom Systems

A single degree-of-freedom system is shown in Fig. 2.1. This model with a spring and viscous damper in parallel is known as the Kelvin-Voigt model. In this figure, m (kg) is the mass, k (N/m) is the linear spring constant, and c (Ns/m) is the coefficient of viscous damping. The static displacement of the mass is δ_{st}. The mass undergoes a displacement $x(t)$ (m) and the base has applied to it a known displacement $y(t)$ (m). Both displacements are with respect to an inertial frame. The mass is subjected to an externally applied time-varying force $f(t)$ (N) and a gravity force mg (N), where $g =$ 9.81 m/s^2 is the acceleration of gravity. Examples of external forces acting on a mass are fluctuating air pressure loading such as that on the wing of an aircraft, fluctuating electromagnetic forces such as in a loudspeaker coil, electrostatic forces that appear in some microelectromechanical devices, forces caused by an unbalanced mass in rotating machinery, and buoyancy forces on floating systems.

We shall derive the equations of motion using Lagrange's equations. To do this, we determine the following. The kinetic energy is of the system is

$$T = \frac{1}{2}m\dot{x}^2 \tag{2.1}$$

where the over dot indicates the derivative with respect to the time t. The potential energy is given by

$$U = \frac{1}{2}k(\delta_{st} + x - y)^2 \tag{2.2}$$

The dissipation function is given by

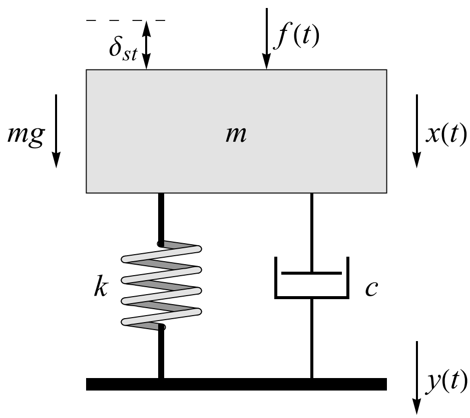

Fig. 2.1 Vertical oscillations of a spring-mass-damper system with moving base

$$D = \frac{1}{2}c(\dot{x} - \dot{y})^2 \tag{2.3}$$

The generalized force is

$$Q = mg + f(t) \tag{2.4}$$

Upon substituting Eqs. (2.1) to (2.4) into Lagrange's equation, which is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial D}{\partial \dot{x}} + \frac{\partial U}{\partial x} = Q \tag{2.5}$$

we obtain,

$$m\frac{d^2x}{dt^2} + c\frac{d}{dt}x + k(x + \delta_{st}) = mg + f(t) + c\frac{dy}{dt} + ky \quad \text{N} \tag{2.6}$$

It is seen from Eq. (2.6) that the time-independent portion of Eq. (2.6) about $x = 0$ gives

$$\delta_{st} = \frac{mg}{k} \quad \text{m}$$

Then, Eq. (2.6) becomes

$$m\frac{d^2x}{dt^2} + c\frac{d}{dt}x + kx = f(t) + c\frac{dy}{dt} + ky \quad \text{N} \tag{2.7}$$

In practice, Eq. (2.7) is used as follows. When $f(t) \neq 0$, $y(t) = 0$ and when $y(t) \neq 0$, $f(t) = 0$. Equation (2.7) represents the motion of the mass about the static equilibrium position $x = 0$.

It is noted from the above operations that $\partial U/\partial x$ is equal to the spring force, denoted $F_s(x)$, applied to the mass by the extension (or contraction) of the spring. The spring constant is determined from

$$k(x) = \frac{dF_s}{dx} \quad \text{N/m} \tag{2.8}$$

In the present case, the force–displacement relation is linear and is of the form $F_s(x) = k(x - y + \delta_{st})$. Then, Eq. (2.8) yields $k(x) = k$. As we shall see in Sect. 2.2.6 there are many situations in which the spring force is not linearly related to x.

The dynamic force on the base is

$$F_b = c\frac{dx}{dt} + kx \tag{2.9}$$

Before proceeding, the following definitions are introduced.

Natural Frequency—ω_n

For translating systems

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}} \quad \text{rad/s} \tag{2.10}$$

where f_n is the natural frequency in Hz and we have used the definition of the static displacement.

Damping Factor—ζ

For translating systems

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} = \frac{c\omega_n}{2k} \tag{2.11}$$

When $0 < \zeta < 1$ the system is called an underdamped system, when $\zeta = 1$ it is a critically damped system, and when $\zeta > 1$ it is an overdamped system. When $\zeta = 0$, the system is undamped.

We return to Eq. (2.7) and introduce Eqs. (2.10) and (2.11) into Eq. (2.7) to arrive at the following governing equation of motion in terms of the natural frequency and damping factor

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{f(t)}{m} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y \tag{2.12}$$

If we let $\tau = \omega_n t$, then Eq. (2.12) becomes

$$\frac{d^2x}{d\tau^2} + 2\zeta\frac{dx}{d\tau} + x = y + 2\zeta\frac{dy}{d\tau} + \frac{f(\tau)}{k} \quad \text{m} \tag{2.13}$$

2.2.2 *General Solution for Harmonically Varying Forcing and Base Excitation*

We examine two cases of Eq. (2.13) with the assumptions that $0 \le \zeta < 1$, the initial conditions are zero, and the forcing and base excitations are undergoing harmonic oscillations of the form

$$\begin{aligned} f(t) &= F_o e^{j\Omega\tau} \\ y(t) &= Y_o e^{j\Omega\tau} \end{aligned} \tag{2.14}$$

where $\Omega = \omega/\omega_n$, F_o is the known magnitude of the applied force, and Y_o is the known magnitude of the base's displacement. It is seen that when $\omega = \omega_n$, $\Omega = 1$. In the first case, we set $y(t) = 0$ and in the second case we set $f(t) = 0$. For both cases, we assume that

$$x(\tau) = X_o e^{j\Omega\tau} \tag{2.15}$$

For case 1 ($Y_o = 0$), we substitute the first of Eqs. (2.14) and (2.15) into Eq. (2.13) to find that

$$X_o = \frac{F_o}{k} H(\Omega) e^{-j\theta(\Omega)} \quad \text{m} \tag{2.16}$$

where

$$\begin{aligned} H(\Omega) &= \frac{1}{\sqrt{\left(1-\Omega^2\right)^2 + (2\zeta\Omega)^2}} \\ \theta(\Omega) &= \tan^{-1}\frac{2\zeta\Omega}{1-\Omega^2} \end{aligned} \tag{2.17}$$

The quantity $H(\Omega)$ is called the amplitude response and the quantity $\theta(\Omega)$ is called the phase response. From Eqs. (2.15) and (2.16), we find that

$$x(\tau) = \frac{F_o}{k} H(\Omega) e^{j(\Omega\tau - \theta(\Omega))} \quad \text{m} \tag{2.18}$$

A plot of $H(\Omega)$ and $\theta(\Omega)$ is shown in Fig. 2.2. When a harmonic force applied to the mass, it is seen from Fig. 2.2 that for viscous damping, the phase angle is 90° when $\Omega = 1$ irrespective of the value of ζ. In addition, there are three distinct frequency regions of $H(\Omega)$. The first region is when $\Omega \ll 1$, where $H(\Omega) \cong 1$ and, from Eq. (2.16), $x(\tau) \sim 1/k$. This region is denoted the stiffness-controlled region and is important in sensor design. The second region is when $\Omega \cong 1$, where $H(\Omega) \sim 1/(2\zeta)$ and from Eqs. (2.16) and (2.11) $x(\tau) \sim 1/c$. This region is called the damping-controlled region and is important in the design of vibration absorbers. The third region is when $\Omega \gg 1$, where $H(\Omega) \sim 1/\Omega^2$ and $x(\tau) \sim 1/m$. This region is called the mass-controlled region and is important in the design of vibration isolators.

For case 2 ($F_o = 0$), we substitute the second of Eq. (2.14) and Eq. (2.15) into Eq. (2.13) to find that

$$X_o = Y_o H_b(\Omega) e^{j\varphi(\Omega)} \quad \text{m} \tag{2.19}$$

where

$$H_b(\Omega) = H(\Omega)\sqrt{1 + (2\zeta\Omega)^2}$$

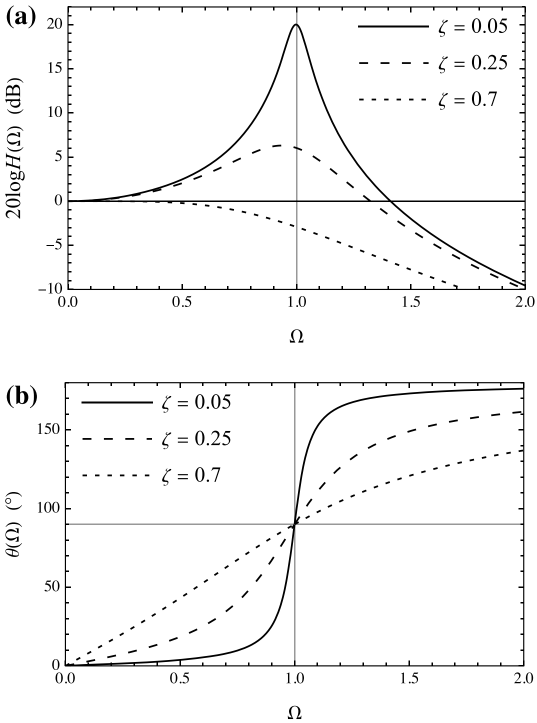

Fig. 2.2 Response of a single degree-of-freedom system with viscous damping when a harmonic force excitation is applied to the mass. **a** Amplitude response. **b** Phase response

$$\varphi(\Omega) = \tan^{-1} \frac{2\zeta\Omega^3}{1 + \Omega^2(4\zeta^2 - 1)} \tag{2.20}$$

A plot of $H_b(\Omega)$ and $\varphi(\Omega)$ is shown in Fig. 2.3.

Transmissibility

When a force is applied to the mass, the transmissibility, denoted *TR*, is defined as the ratio of the magnitude of the force transmitted to the fixed base to the force applied to the mass. To determine the magnitude of the force transmitted to the fixed base, we use Eqs. (2.9) and (2.18) to obtain

$$|F_b| = F_o H_b(\Omega) \tag{2.21}$$

where $H_b(\Omega)$ is given by Eq. (2.20). Then, the transmissibility is

$$TR = \frac{|F_b|}{F_o} = H_b(\Omega) \tag{2.22}$$

When the base of the single degree-of-freedom is subjected to a harmonic displacement, the transmissibility is defined as the ratio of the magnitude of the displacement

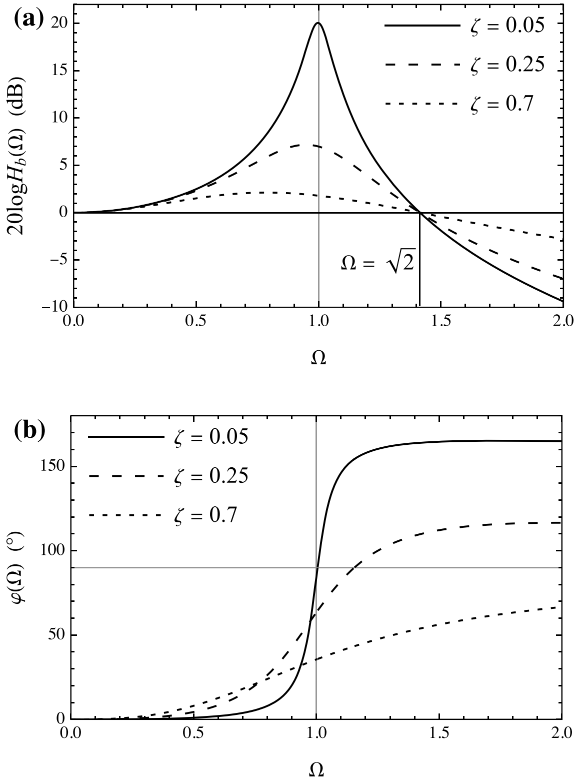

Fig. 2.3 Response of a single degree-of-freedom system with viscous damping when a harmonic displacement excitation is applied to the base. **a** Amplitude response. **b** Phase response

of the mass to the magnitude of the base displacement. In this case, the transmissibility is, from Eq. (2.19),

$$TR = \frac{|X_o|}{Y_o} = H_b(\Omega) \tag{2.23}$$

We see that the expression for *TR* is the same for cases 1 and 2.

Although the TR can be obtained from Fig. 2.3, it has been replotted in Fig. 2.4 for two values of the damping factor. From this figure, it is seen that the transmissibility ratio decreases as Ω increases and as ζ decreases. However, it is seen that to get, say, a 4% *TR*, one can operate at a frequency ratio $\Omega = 5.2$ when $\zeta = 0.03$ whereas for a damping factor of $\zeta = 0.15$ one must operate at a frequency ratio $\Omega = 8.2$. Thus, when one seeks vibration isolation, one of the objectives is to have a natural frequency as low as possible. However, from Eq. (2.10) it is seen that this may require a large static displacement, which can lead to implementation difficulties. In Sect. 2.2.6, several vibration isolation systems to overcome this drawback will be discussed.

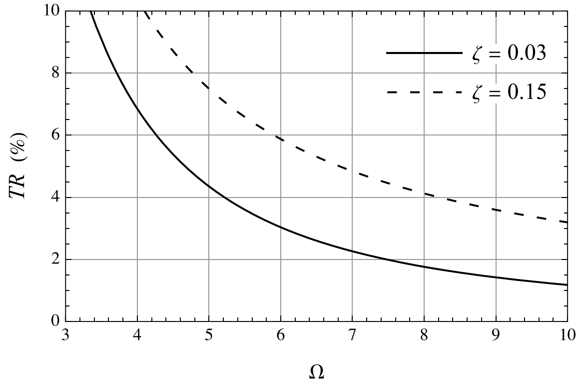

Fig. 2.4 Percentage *TR* for two values of ζ

2.2.3 Structural Damping

Structural damping is a damping model that assumes that the dissipation in the system is due to losses in the material that provides the stiffness for the system. One type of structural damping model is obtained by assuming that the structural damping is independent of frequency. A model that satisfies this criterion is to replace $k(x - y)$ in Eq. (2.7) with

$$k(x-y) \rightarrow k(x-y) + k\frac{2\eta}{\omega}\frac{\partial}{\partial t}(x-y) \quad \text{N} \tag{2.24}$$

where η is an empirically determined constant. This model is restricted to systems undergoing harmonic oscillations at frequency ω. Then, Eq. (2.13) can be written as

$$\frac{d^2x}{d\tau^2} + 2\zeta_{vs}\frac{dx}{d\tau} + x = y + 2\zeta_{vs}\frac{dy}{d\tau} + \frac{f(\tau)}{k} \tag{2.25}$$

where

$$\zeta_{vs} = \left(\zeta + \frac{\eta}{\Omega}\right) \tag{2.26}$$

Thus, we can use results given by Eqs. (2.17) and (2.20) directly by replacing in these results ζ with ζ_{vs}. Then, Eq. (2.18), which is for case 1, becomes

$$x(\tau) = \frac{F_o}{k}H_{vs}(\Omega)e^{j(\Omega\tau - \theta_{vs}(\Omega))} \quad \text{m} \tag{2.27}$$

where

$$H_{vs}(\Omega) = \frac{1}{\sqrt{\left(1-\Omega^2\right)^2 + (2\zeta\Omega + 2\eta)^2}}$$