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Geometry of Submanifolds and Applications

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Springer

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Preface

Beginning with the theory of curves and surfaces in ordinary Euclidean 3-space, the theory of submanifolds as a subarea of differential geometry dates back as far as differential geometry itself. The theory has evolved in the last century into a very important area of mathematics with many applications, and making use of a variety of techniques. Since the late 1950s, the theory has a rapid development which attracted more and more researchers. Consequently, a lot of interesting results have been achieved (see, e.g., [7] for an extensive survey).

In 1956, John F. Nash proved the following famous embedding theorem.

Theorem 1. [17] Every Riemannian manifold can be isometrically embedded in a Euclidean space with sufficiently high codimension.

Nash's embedding theorem is aimed for in the hope that if each Riemannian manifold could always be isometrically embedded as an Euclidean submanifold, then it could yield the opportunity to use help from extrinsic geometry. But this hope was not materialized for many years, as pointing out by M. Gromov in his 1985 survey article [15]. There were several reasons why it is so difficult to apply Nash's theorem. One reason is that it requires in general very large codimension for a Riemannian manifold to admit an isometric embedding in an Euclidean space, On the other hand, submanifolds of higher codimension are difficult to understand. The second reason is that at that time there did not exist general optimal relationships between known intrinsic invariants and main extrinsic invariants for an arbitrary Euclidean submanifold. These difficulties lead to the following fundamental problem in submanifold theory (see, e.g., [4, 5, 6, 8]).

Problem 1. Find simple optimal relationship between extrinsic invariants and intrinsic invariants of a submanifold and find their applications.

In order to give some answers to this fundamental problem, one of the editors of this book introduced in the early 1990s new types of Riemannian invariants, i.e., δ-invariants (also known as Chen invariants). He was also able to provide solutions to Problem 1 by establishing optimal inequalities between his δ-invariants and the squared mean curvature and gave many applications (see, e.g., [7, 8, 9]). A special case of his inequalities is known today as the Chen–Ricci inequality. The main purpose of the first chapter of this book is to provide a comprehensive survey on recent developments on the Chen–Ricci inequality.

Differential geometry has many nice applications to Einstein's general relativity theory. The second chapter discusses perfect fluid spacetimes fulling $f(R; T)$-gravity when Ricci and gradient Ricci solitons, Yamabe and gradient Yamabe solitons, η-Ricci and gradient η-Ricci solitons are its metrics.

A $2n$-dimensional manifold is called symplectic if it admits a non-degenerate closed 2-form Ω. An n-dimensional submanifold M of a symplectic $2n$-manifold $(M; \Omega)$ is called Lagrangian if $\Omega|_{\mathrm{TM}} = 0$. Lagrangian submanifolds appear naturally in mathematical physics. Chapter 3 provides a nice survey on Lagrangian submanifolds of the nearly Kähler 6-sphere.

A hypersurface of a Riemannian or Lorentzian space form is called a Pythagorean hypersurface [2] if its first, second, and third fundamental forms satisfy a Pythagorean-like formula: $\mathrm{I}^2 + \mathrm{II}^2 = \mathrm{III}^2$. In Chap. 4, Pythagorean hypersurfaces in real space forms were extended to Pythagorean submanifolds. The authors of this chapter provide examples and show that Pythagorean submanifolds are isoparametric where the principal curvatures are given in terms of the Golden ratio. They also classify Pythagorean hypersurfaces in real space forms.

The notion of f-structure was introduced by K. Yano in 1961, which generalized both almost-complex and almost-contact structures. Chapter 5 is devoted to the study of the class of manifolds which admit an f-structure with two-dimensional parallelizable kernel.

A submersion is a differentiable map between two manifolds whose differential is everywhere surjective. Riemannian submersions are submersions equipped with compatible Riemannian metrics. Chapter 6 focuses on Kähler submersions which are special case of almost-Hermitian submersions. Among others, the authors of this chapter prove, under certain conditions, that the fibers and the base manifold of such submersions are almost-Yamabe solitons.

For a surface M in a Euclidean 4-space, P. Wintgen [20] proves that the Gauss curvature G and normal curvature K^D of M satisfy $G + |K^D| \leq \|H\|^2$; where $\|H\|^2$ is the squared mean curvature. Chapter 7 studies generalized Wintgen inequality for a contact pseudo-slant submanifold of (ϵ)-para Sasakian space form. The authors describe those submanifolds satisfying the equality case of the inequality. They also derive an inequality for Ricci solitons to discover some connections between intrinsic and extrinsic invariants.

S. Decu, S. Haesen, and L. Verstraelen [13] introduced δ-Casorati curvatures in 2017. Today, there are many interesting results on δ-Casorati curvatures obtained by various authors [10]. Chapter 8 presents two inequalities involving δ-Casorati curvatures of quaternion bi-slant submanifolds in quaternion space forms. The authors of this chapter also characterize those submanifolds satisfying the equality cases of the inequalities.

Chapter 9 provides another application of differential geometry to general relativity. The authors of this chapter study gravity and dark matter in the framework of metric-affine geometry.

Chapter 10 studies and analyzes the properties of metallic structure on manifolds using the metallic ratio which is a generalization of the Golden proportion. The author of this chapter discusses statistical metallic manifolds and statistical submersions.

Chapter 11 studies tangent bundle of three-dimensional quasi-Sasakian manifolds endowed with a quarter-symmetric non-metric ξ-connection. The authors of this chapter prove several special curvature properties for such manifolds with the connection on its tangent bundle.

In Chap. 12, the author introduces and studies a Darboux mate of a given spherical Legendre curve in the Euclidean unit sphere S^2. The authors also investigate a triple sequence of curvatures provided by the higher-order derivatives of the last Frenet equation for the frontal of the Legendre curve.

In the last chapter, the authors investigate Riemannian 3-manifolds which admit conformal η-Ricci–Yamabe solitons or gradient conformal η-Ricci–Yamabe solitons. Several results in this respect are presented.

Given the enormous amount of work on the theory of submanifolds and their applications that has been published over the past few decades, the editors decided it would be reasonable to invite a number of experts to contribute their work in the form of chapters to highlight the current advances in differential geometry with a focus on the geometry of submanifolds. Several geometers responded to our call. The editors wish to thank each and every contribution.

The editors hope that readers will find this book useful as a resource for conducting creative and fruitful research as well as an approachable introduction to the geometry of submanifolds and its applications.

East Lansing, USA — Bang-Yen Chen
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Recent Developments on Chen–Ricci Inequalities in Differential Geometry

Bang-Yen Chen and Adara M. Blaga

1 Introduction

Nash [148] proved in 1956 his famous embedding theorem.

Theorem 1.1 *Every Riemannian n-manifold can be isometrically embedded in a Euclidean m-space with dimension $m = \frac{n}{2}(n+1)(3n+11)$.*

This embedding theorem aimed for in the hope that if each Riemannian manifold could always be regarded as a Euclidean submanifold, then it could yield the opportunity to use help from extrinsic geometry. However, this hope was not materialized for many years, as pointed out in Gromov's 1985 survey article [84].

There were several reasons why it is so difficult to apply Nash's theorem. One reason is that it requires in general very large codimension for a Riemannian manifold to admit an isometric embedding in Euclidean spaces. On the other hand, submanifolds of higher codimension are difficult to understand. Another reason is that at that time there did not exist general optimal relationships between known intrinsic invariants and main extrinsic invariants for arbitrary Euclidean submanifolds (see [53]). This leads to the following fundamental problem in submanifold theory (see [38, 42]).

Problem 1.1 *Find simple relationship between main extrinsic invariants and main intrinsic invariants of a submanifold.*

In order to give some answers to this fundamental problem, the first author introduced in the early 1990s new types of Riemannian invariants, known as δ-invariants

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or Chen invariants, which are very different in nature from the "classical" Ricci and scalar curvatures (see, e.g., [37, 41, 43]). Using his invariants, Chen was able to give solutions to Problem 1.1 by establishing sharp inequalities between his δ-invariants and the squared mean curvature. Using these inequalities, he introduced and investigated the notion of ideal immersions.

Later in 1996, the first author established in [38] a sharp inequality involving Ricci curvature Ric and squared mean curvature $\|H\|^2$ for any submanifold M in a real space form $\widetilde{M}^m(c)$ of constant sectional curvature c, namely,

$$\mathrm{Ric}(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2, \quad n = \dim M,$$

for any unit vector X tangent to M. Since then, this inequality has drawn the attention of many geometers around the world. Consequently, many inequalities of similar type were proved by many geometers for different kinds of submanifolds in various ambient manifolds. The main purpose of this article is thus to present a comprehensive survey on recent developments in this inequality. It is both authors' intention that this survey article will provide a useful reference for graduate students as well as researchers working on this interesting research topic in differential geometry.

2 Preliminaries

Throughout this article, by a manifold we mean a connected differentiable manifold. Manifolds, maps, functions, vector fields, etc. are assumed to be of class C^∞. Let $\mathfrak{X}(M)$ denote the space of all vector fields on a manifold M, $\mathcal{F}(M)$ the space of functions, and T^1M the unit tangent bundle of M. For the basic knowledge on Riemannian manifolds and submanifolds, we refer to the books [54, 55, 154, 167].

2.1 Basic on Riemannian Manifolds

Let (M, g) be an n-dimensional Riemannian manifold. We denote by ∇ the Levi–Civita connection of (M, g). Then the Riemann curvature tensor R is the (0, 3)-tensor field defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$, where [,] is the Lie bracket. For orthonormal vectors X, Y of a 2-plane $\pi \subset T_pM$, the sectional curvature of the plane spanned by X, Y is

$$K(\pi) = g(R(X, Y)Y, X).$$

The *Ricci tensor* at $p \in M$, denoted by Ric_p, is defined by

$$\mathrm{Ric}_p(Y, Z) = \mathrm{Trace}\{X \mapsto R(X, Y)Z\}$$

equivalently,

$$\mathrm{Ric}(X, Y) = \sum_{\ell=1}^{n} \langle R(E_i, X)Y, E_i\rangle,$$

where $\{E_1, \ldots, E_n\}$ is an orthonormal basis of T_pM. A Riemannian manifold is called an *Einstein manifold* if $\mathrm{Ric} = cg$ for some constant c.

For $X \in T^1M$, the *Ricci curvature* Ric at X is defined by $\mathrm{Ric}(X) = \mathrm{Ric}(X, X)$. The *scalar curvature* τ of M is given by $\tau = \sum_{i<j} K(E_i \wedge E_j)$. The Ricci and scalar curvatures are independent of the choice of the orthonormal basis $\{E_1, \ldots, E_n\}$.

The Ricci curvature was extended to k-Ricci curvature as follows [42]: For a unit vector X in a k-plane section L^k, we choose an orthonormal basis $\{E_1, \ldots, E_k\}$ of L^k such that $E_1 = X$. Then the *k-Ricci curvature* Ric_{L^k} of L^k at X is defined by

$$\mathrm{Ric}_{L^k}(X) = \sum_{j=2}^{k} K_{1j}, \quad K_{ij} = K(E_i \wedge E_j). \tag{2.1}$$

The *scalar curvature* $\tau(L^k)$ of L^k is $\tau(L^k) = \sum_{1\le i<j\le k} K_{ij}$. And the *$k$-Ricci curvature invariant* θ_k was defined in [42] to be

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{L^k, X} \mathrm{Ric}_{L^k}(X), \quad p \in M, \tag{2.2}$$

where L^k runs over all k-plane sections $\subset T_pM$ and X runs over unit vectors in L^k.

2.2 Basic on Riemannian Submanifolds

Let M be a submanifold of a Riemannian manifold $\widetilde{M}$. Denote by ∇ and $\widetilde{\nabla}$ the Levi–Civita connections of M and $\widetilde{M}$, respectively. The formulas of Gauss and Weingarten are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for $X, Y \in \mathfrak{X}(M)$ and ξ a normal vector field of M, where h, A, and D are the *second fundamental form*, the *shape operator*, and the *normal connection*.

For each normal vector ξ of M at $p \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM. The shape operator and the second fundamental form are related simply by $\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$.

The *mean curvature vector* of M in $\widetilde{M}$ is given by

$$H = \frac{1}{n}\,\mathrm{Trace}\,h, \quad n = \dim M.$$

A submanifold is said to be *minimal* (respectively, *totally geodesic*) if its mean curvature vector (respectively, its second fundamental form) vanishes identically.

Let R and $\widetilde{R}$ denote the Riemann curvature tensors of M and $\widetilde{M}$, respectively. If $\widetilde{M}$ is of constant curvature c, then the three fundamental equations of Gauss, Codazzi, and Ricci are given, respectively, by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = {} & \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle \\ & + c\,(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle),\end{aligned}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z),$$

$$\langle R^{\perp}(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ and ξ, η normal vector fields of M, where $R^{\perp}$ denotes the normal curvature tensor associated with D and $\overline{\nabla}h$ is given by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.3 Submanifolds of Kähler Manifolds

There exist three important classes of submanifolds M in an almost Hermitian manifold $(\widetilde{M}, J, g)$; namely, complex, totally real, and slant submanifolds based on the action of the almost complex structure J of $\widetilde{M}$ acting on the tangent bundle TM defined as follows: A submanifold M of $\widetilde{M}$ is called *complex* if TM is invariant under the action of J, i.e., $J(T_pM) = T_pM$ for any $p \in M$; M is called *totally real* if J maps each T_pM into the corresponding normal space, i.e., $J(T_pM) \subseteq T_p^{\perp}M$ for any $p \in M$. A totally real submanifold M of $\widetilde{M}$ is called *Lagrangian* if $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \widetilde{M}$. We refer to [46] for general results on Lagrangian submanifolds.

For a vector $0 \neq X \in T_pM$, the angle $\theta(X)$ between JX and T_pM is called the *Wirtinger angle* of X. A submanifold M of $\widetilde{M}$ is called *slant* or (θ-*slant*) [35, 36] if $\theta(X)$ is independent of the choice of $X \in T_pM$ and of $p \in M$. In this case, the angle θ is called the *slant angle*. Clearly, complex or totally real submanifolds are slant submanifolds with slant angle $\theta = 0$ or $\theta = \frac{\pi}{2}$, respectively. A slant submanifold is called *proper* if it is neither complex nor totally real. From J-action points of view, slant submanifolds are the most natural submanifolds of almost Hermitian manifolds.

For any vector X tangent to a submanifold M of an almost Hermitian manifold, we put

$$JX = PX + FX,$$

where PX and FX denote the tangential component and normal component of JX, respectively. A proper slant submanifold of a Kähler manifold $\widetilde{M}$ is called *Kählerian slant* [36] if the endomorphism P is parallel with respect to the Levi–Civita connection ∇, i.e., $\nabla P = 0$.

2.4 *H*-Umbilical Submanifolds

The notion of H-umbilical submanifold was introduced in [39, 40] as follows: An n-dimensional Lagrangian submanifold M of an almost Hermitian manifold is called *H-umbilical* if its second fundamental form h takes the form:

$$\begin{aligned} &h(E_1, E_1) = \lambda J E_1,\ h(E_2, E_2) = \cdots = h(E_n, E_n) = \mu J E_1, \\ &h(E_1, E_j) = \mu J E_j,\ h(E_j, E_k) = 0,\ 2 \le j \ne k \le n, \end{aligned}$$

for some suitable functions λ and μ with respect to an orthonormal local frame $\{E_1, \dots, E_n\}$. The ratio $\lambda : \mu$ is called the *ratio* of the H-umbilical submanifold M.

3 Chen–Ricci Inequalities for Submanifolds in Riemannian Manifolds

The *relative null subspace* $\mathcal{N}_p$ of a submanifold M in a Riemannian manifold at a point $p \in M$ is defined by (see [112])

$$\mathcal{N}_p = \{X \in T_pM : h(X, Y) = 0 \ \ \forall Y \in T_pM\},$$

where h is the second fundamental form of M.

In 1996, the first author established a pointwise sharp relationship between Ricci curvature and the squared mean curvature for submanifolds in a real space form.

Theorem 3.1 ([38]) *If M is an n-dimensional submanifold of a real space form $R^m(c)$, then*

(1) *For any $X \in T_p^1 M$, we have:*

$$\mathrm{Ric}(X) \le \frac{n^2}{4} \|H\|^2 + (n-1)c. \tag{3.1}$$

(2) *If* $H(p) = 0$*, then a unit tangent vector* $X \in T_p^1 M$ *satisfies the equality case of* (3.1) *if and only if* X *lies in the relative null subspace* $\mathcal{N}_p$.
(3) *The equality case of* (3.1) *holds identically for all unit vectors in* $T_p^1 M$ *if and only if either* p *is a totally geodesic point, or* $n = 2$ *and* p *is a totally umbilical point.*

To prove answers to Problem 1, the first author introduced his δ-invariants, $\delta(n_1, \ldots, n_k)$, and proved the following optimal inequality (see also [37, 41, 44, 52]).

Theorem 3.2 ([48]) *Let* $\phi : M \to \widetilde{M}$ *be an isometric immersion between Riemannian manifolds. Then we have:*

$$\delta(n_1, \ldots, n_k) \le c(n_1, \ldots, n_k)\|H\|^2 + b(n_1, \ldots, n_k) \max \widetilde{K},$$

where $\max \widetilde{K}_p$ *denotes the maximum sectional curvature of* $\widetilde{M}$ *restricted to 2-plane sections of the tangent space* $T_p M$*, and*

$$c(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)},$$
$$b(n_1, \ldots, n_k) = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1).$$

If we denote the maximum of the Ricci curvature of M by $\widehat{\mathrm{Ric}}$, then, due to

$$\delta(n-1) = \widehat{\mathrm{Ric}}, \quad b(n-1) = n-1, \quad c(n-1) = \frac{n^2}{4},$$

we obtain immediately the following result from Theorem 3.2.

Proposition 3.3 *Let* M *be an* n*-dimensional submanifold of a Riemannian manifold* $\widetilde{M}$*. Then, for any* $X \in T^1 M$*, we have:*

$$\mathrm{Ric}(X) \le \frac{n^2}{4}\|H\|^2 + (n-1)\max \widetilde{K}. \tag{3.2}$$

For submanifolds of complex space forms, (3.2) implies the following.

Corollary 3.4 *If* M *is an* n*-dimensional submanifold of a complex space form* $\widetilde{M}(c)$*, then, for any* $X \in T^1 M$*, we have:*

$$\widehat{\mathrm{Ric}} \le \frac{n^2}{4}\|H\|^2 + (n-1)c \quad \text{for} \quad c \ge 0, \tag{3.3}$$

$$\widehat{\mathrm{Ric}} \le \frac{n^2}{4}\|H\|^2 + \frac{n-1}{4}c \quad \text{for} \quad c < 0. \tag{3.4}$$

Remark 3.5 Inequality (3.4) was also obtained by Sasahara [168].

In 1999, B. D. Suceavă proved the following.

Theorem 3.6 ([186]) *Let M be an n-dimensional submanifold of a Riemannian manifold $\widetilde{M}$. Then the scalar curvature τ of M satisfies*

$$\tau \leq \frac{n(n-1)}{2}\|H\|^2 + \sum_{1\leq i<j\leq n} \widetilde{K}(e_i, e_j), \tag{3.5}$$

where $\widetilde{K}(e_i, e_j)$ denotes the sectional curvature of $\widetilde{M}$ restricted to the plane section spanned by $\{e_i, e_j\}$ for an orthonormal basis $\{e_1, \ldots, e_n\}$.

Applying the method used in Theorem 3.6, Tripathi refined Proposition 3.3 and Theorem 3.1 in 2005 to the following.

Theorem 3.7 ([101, 192]) *Let M be a submanifold of a Riemannian manifold $\widetilde{M}$. Then*

(1) *For any $X \in T_p^1 M$, we have:*

$$\mathrm{Ric}(X) \leq \frac{n^2}{4}\|H\|^2 + (n-1)\,\widetilde{\mathrm{Ric}}(X), \quad n = \dim M, \tag{3.6}$$

where $\widetilde{\mathrm{Ric}}_{T_pM}(X)$ is the n-Ricci curvature of $\widetilde{M}$ restricted to T_pM (see (2.1)).

(2) *If $H(p) = 0$, then the equality case of (3.6) is satisfied for $X \in T_p^1 M$ if and only if $X \in \mathcal{N}_p$.*

(3) *The equality case of (3.6) holds for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.*

Remark 3.8 There exists no direct relationship between the Ricci curvature $\widetilde{\mathrm{Ric}}(X)$ and the n-Ricci curvature $\widetilde{\mathrm{Ric}}_{TM}(X)$ of $\widetilde{M}$ in general.

In [98], Hineva obtained the lower estimate of the Ricci curvature of a submanifold M of an arbitrary Riemannian manifold $\widetilde{M}$:

$$\begin{aligned}\mathrm{Ric}(X) \geq\ & \widetilde{\mathrm{Ric}}(X) \\ & + \frac{n-1}{n}\left[2n\|H\|^2 - S - \frac{n-2}{\sqrt{n-1}}\sqrt{n\|H\|^2(S - n\|H\|^2)}\right],\end{aligned} \tag{3.7}$$

where S is the squared norm of the second fundamental form $\|h\|^2$.

For a non-totally geodesic submanifold M, the equality in the lower estimate (3.7) is achieved only when M is quasi-umbilical with flat normal bundle, and for codimensions p with $1 \leq p \leq n-3$, the equality in (3.7) holds only when M is conformally flat with a flat normal bundle.

In [42], the first author extended Theorem 3.1 to the following.

Theorem 3.9 ([42]) *Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then, for any integer k, $2 \leq k \leq n$, we have:*

$$\theta_k \leq \frac{n^2(k-1)}{4(n-1)}\|H\|^2 + (k-1)c, \tag{3.8}$$

where θ_k is the k-Ricci curvature invariant defined by (2.2).

The equality case of (3.8) holds identically at a point $p \in M$ if and only if either p is a totally geodesic point, or $n = k = 2$ and p is a totally umbilical point.

In [42], the first author also established the following sharp relationship between the shape operator A_H and θ_k for any submanifold in real space form $R^m(c)$:

$$\begin{cases} A_H > \frac{n-1}{n}(\theta_k(p) - c)I & \text{if } \theta_k(p) \neq c; \\ A_H \geq 0 & \text{if } \theta_k(p) = c. \end{cases}$$

4 Chen–Ricci Inequalities for Submanifolds in Manifolds of Quasi-constant Curvature

Riemannian manifolds of quasi-constant curvature were introduced by the first author and Yano in [62] as follows: A Riemannian m-manifold $(\widetilde{M}, g)$ is said to be of *quasi-constant curvature* if there exist a unit vector field ξ, called the *generator*, and functions κ, μ such that the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ satisfies

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) = \kappa[&g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &+ \mu[g(X, W)\zeta(Y)\zeta(Z) - g(X, Z)\zeta(Y)\zeta(W) \\ &+ g(Y, Z)\zeta(X)\zeta(W) - g(Y, W)\zeta(X)\zeta(Z)], \end{aligned}$$

where ζ is the 1-form dual to a unit vector field ξ. Denote such manifold by $\widetilde{M}^m_{\kappa,\mu}(\zeta)$.

Özgür [156] studied the Chen–Ricci inequality for submanifolds in $\widetilde{M}^m_{\kappa,\mu}(\zeta)$. He proved the following.

Theorem 4.1 *Let M be an n-dimensional ($n \geq 3$) submanifold in $\widetilde{M}^m_{\kappa,\mu}(\zeta)$. Then, for any integer k, $2 \leq k \leq n$, we have:*

$$\|H\|^2 \geq \theta_k - \kappa - \frac{2\mu}{n}\|\zeta^T\|^2,$$

where ζ^T denotes the tangential component of ζ.

As a generalization of Theorem 3.9, Zhang et al. [217] proved the following.

Theorem 4.2 *Let M be an n-dimensional ($n \geq 3$) in $\widetilde{M}^m_{\kappa,\mu}(\zeta)$ with a semi-symmetric metric connection. Then, for any integer k, $2 \leq k \leq n$, we have:*

$$\|H\|^2 \geq \theta_k - \kappa - \frac{2\mu}{n}\|\zeta^T\|^2 + \frac{2}{n}\lambda,$$

where λ is the trace of the $(0, 2)$-tensor field s.

In [217], Zhang et al. also established several extensions of Theorem 3.1.

In 2009, De and Gazi [67] defined a Riemannian manifold of *nearly quasi-constant curvature* as a Riemannian manifold whose curvature tensor satisfies

$$\begin{aligned}\widetilde{R}(X, Y, Z, W) = \kappa[&g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &- \mu[g(X, W)B(Y, Z) - g(X, Z)B(Y, W) \\ &\quad + g(Y, Z)B(X, W) - g(Y, W)B(X, Z)],\end{aligned}$$

where κ, μ are functions and B is a non-zero symmetric tensor of type $(0, 2)$.

In [215], Zhang proved the following result.

Theorem 4.3 *Let M be an n-dimensional ($n \geq 3$) in a Riemannian manifold of nearly quasi-constant curvature. Then, for any $X \in T^1M$, we have:*

$$\mathrm{Ric}(X) \leq (n-1)\kappa + \mu[(n-2)B(X, X) + \lambda] + \frac{n^2}{4}\|H\|^2, \tag{4.1}$$

where λ is the trace of B.

The equality case of (4.1) holds identically for all unit vectors in T^1_pM if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.

For further results in this direction, see also [89, 216].

5 Chen–Ricci Inequalities for Real Space Forms with Non-symmetric Connections

Hayden [93] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. Since then, non-symmetric metric connections have been studied by many geometers. Let (M, g) be a Riemannian manifold. A linear connection $\widetilde{\nabla}$ on M is called a *semi-symmetric connection* if the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ given by $\widetilde{T}(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]$ satisfies

$$\widetilde{T}(X, Y) = \omega(Y)X - \omega(X)Y, \tag{5.1}$$

for a 1-form ω. Let $\widetilde{P}$ be the vector field dual to ω, i.e., $\omega(X) = g(X, \widetilde{P})$. A semi-symmetric connection $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* if $\widetilde{\nabla}g = 0$

holds identically. Associated with ∇ and $\widetilde{P}$, there exists a unique semi-symmetric metric connection $\widetilde{\nabla}$ satisfying

$$\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)\widetilde{P}.$$

We put

$$\alpha(X, Y) = (\tilde{\tilde{\nabla}}_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(\widetilde{P})g(X, Y),$$

where $\tilde{\tilde{\nabla}}$ is the Levi–Civita connection of $(\widetilde{M}, g)$.

Yano [209] proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically.

Now, let us consider a Riemannian manifold $(\widetilde{M}, g)$ endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$ and we denote the Levi–Civita connection of $(\widetilde{M}, g)$ by $\tilde{\tilde{\nabla}}$.

If $(\widetilde{M}, g)$ is a real space form $R^m(c)$, then the curvature tensor $\tilde{\tilde{R}}$ with respect to $\tilde{\tilde{\nabla}}$ satisfies

$$\tilde{\tilde{R}}(X, Y, Z, W) = c[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)].$$

Let M be a submanifold of the real space form $R^m(c)$. Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ can be written as

$$\widetilde{R}(X, Y, Z, W) = \tilde{\tilde{R}}(X, Y, Z, W) + s(X, Z)g(Y, W) - s(Y, Z)g(X, W),$$

where s is a $(0, 2)$-tensor field defined by

$$s(X, Y) = (\tilde{\tilde{\nabla}}_X \omega)Y - \omega(X)\omega(Y). \tag{5.2}$$

The next result was proved by Özgür and Mihai [158] and also by Zhang and Zhang [214].

Theorem 5.1 *Let M be an n-dimensional ($n \geq 3$) submanifold of a real space form $R^m(c)$ equipped with a semi-symmetric non-metric connection. Then*

(1) *For any $X \in T_p^1 M$, we have:*

$$\|H\|^2 \geq \frac{4}{n^2}\big[\mathrm{Ric}(X) - (n-1)(c - \Omega)\big], \tag{5.3}$$

where $\Omega = s(X, X) + g(\widetilde{P}^{\perp}, h(X, X))$.

(2) *If $H(p) = 0$, then a unit tangent vector $X \in T_p^1 M$ satisfies the equality case of (5.3) if and only if X lies in the relative null subspace $\mathcal{N}_p$.*

(3) *The equality case of* (5.3) *holds identically for all unit vectors in* $T_p^1 M$ *if and only if either* p *is a totally geodesic point, or* $n = 2$ *and* p *is a totally umbilical point.*

Remark 5.2 A further result in this direction was also obtained by Wang [206].

In [135], Mihai and Özgür also proved the following.

Theorem 5.3 *Let* M *be an* n*-dimensional* ($n \geq 3$) *submanifold of a real space form* $R^m(c)$ *with a semi-symmetric non-metric connection such that the vector field* $\widetilde{P}$ *is tangent to* M*. Then, for any integer* k, $2 \leq k \leq n$*, we have:*

$$\theta_k \leq \|H\|^2 + c - \frac{2}{n}\lambda,$$

where λ *is the trace of the* $(0, 2)$*-tensor field* s.

A linear connection $\widetilde{\nabla}$ on a real space form is called a *Ricci quarter-symmetric connection* if its torsion tensor $\widetilde{T}$ satisfies

$$\widetilde{T}(X, Y) = \omega(Y)LX - \omega(X)LY,$$

for a 1-form ω, where L is the Ricci operator given by $g(LX, Y) = \mathrm{Ric}(X, Y)$.

Poyraz and Yoldaş [166] studied the Ricci curvature of submanifolds of real space forms endowed with a Ricci quarter-symmetric metric connection. They extended Theorem 3.9 to the following.

Theorem 5.4 *Let* M *be an* n*-dimensional* ($n \geq 3$) *submanifold of a real space form* $R^m(c)$ *endowed with a Ricci quarter-symmetric metric connection. Then, for any integer* k, $2 \leq k \leq n$*, we have:*

$$\theta_k \leq \|H\|^2 + c\left[2 - \frac{4(n-1)m}{n}\right].$$

6 Chen–Ricci Inequalities for Totally Real and Lagrangian Submanifolds in Complex Space Forms

The curvature tensor of a complex space form $\widetilde{M}(c)$ of constant holomorphic sectional curvature c satisfies

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ].$$

6.1 Totally Real Submanifolds of Complex Space Forms

For totally real submanifolds of a complex space form $\widetilde{M}(c)$, the first author proved the following.

Theorem 6.1 ([43]) *Let M be an n-dimensional totally real submanifold of a complex space form $\widetilde{M}(c)$. Then we have:*

$$\widehat{\mathrm{Ric}} \leq \frac{n^2}{4}\|H\|^2 + (n-1)c. \tag{6.1}$$

The equality case of (6.1) holds identically if and only if either M is totally geodesic Lagrangian submanifold, or $n = 2$ and M is totally umbilical.

In particular, for a Lagrangian submanifold M in $\widetilde{M}(c)$, we obtain the following.

Theorem 6.2 ([43]) *If M is an n-dimensional Lagrangian submanifold of a complex space form $\widetilde{M}(c)$, then we have:*

$$\widehat{\mathrm{Ric}} \leq \frac{n^2}{4}\|H\|^2 + (n-1)c. \tag{6.2}$$

If the equality case of (6.2) holds identically, then M is minimal.

6.2 Lagrangian Submanifolds of Complex Space Forms

Using an optimization technique, T. Oprea improved inequality (6.2) to the following.

Theorem 6.3 ([153]) *Let M be an n-dimensional Lagrangian submanifold of a complex space form $\widetilde{M}(c)$. Then we have:*

$$\widehat{\mathrm{Ric}} \leq \frac{n-1}{4}\big(n\|H\|^2 + c\big). \tag{6.3}$$

In [68], S. Deng provided an algebraic proof of the inequality (6.2). Furthermore, Deng also characterized Lagrangian submanifolds satisfying (6.3). In fact, he proved the following.

Theorem 6.4 ([68]) *Let M be an n-dimensional Lagrangian submanifold of a complex space form $\widetilde{M}(c)$. Then we have:*

$$\widehat{\mathrm{Ric}} = \frac{n-1}{4}\big(n\|H\|^2 + c\big). \tag{6.4}$$

The equality case of (6.4) holds at $p \in M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a H-umbilical point with ratio 3.