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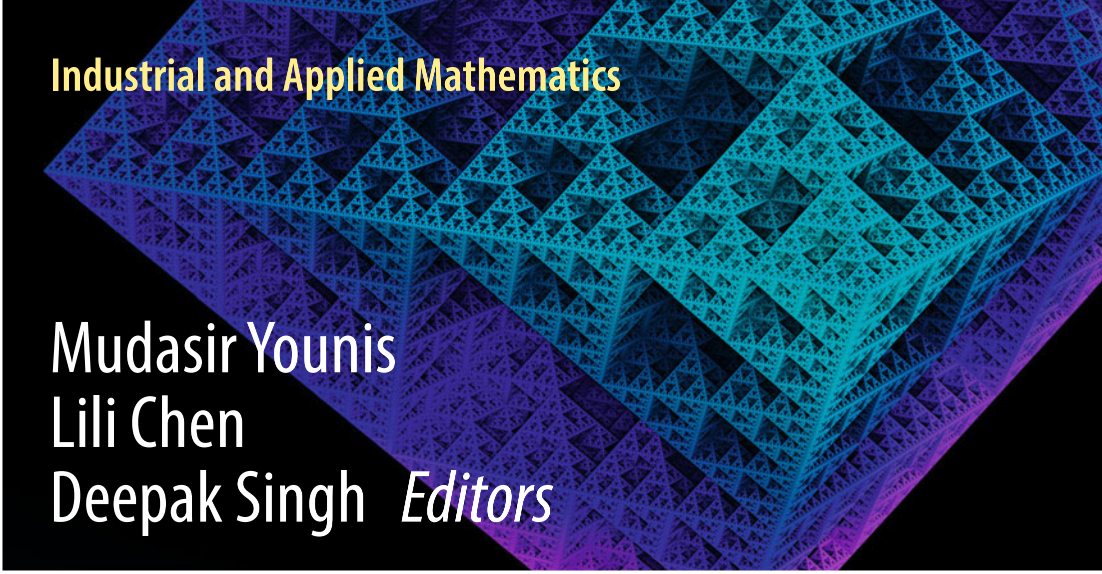

Mudasir Younis
Lili Chen
Deepak Singh *Editors*

Recent Developments in Fixed-Point Theory

Theoretical Foundations and Real-World Applications

Springer

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Preface

Nonlinear functional analysis includes the essential and comprehensive field of fixed-point theory, which is used as a method to solve numerous nonlinear problems in the science and engineering. Since a couple of years ago, experts in fixed-point theory have been focusing on how to apply their theories to a variety of physical-related engineering challenges.

Fixed-point theory has entered a new phase that is inextricably linked to measurements, abstract language, space analysis, and the mining of empirical studies in engineering. By incorporating the metric fixed-point theory into a plethora of literature from the fields of computational engineering, quantum dynamics, and medical research, this was frequently maintained. In the analysis of metric spaces, fixed-point theory has briefly been mentioned as independent literature while referencing numerous other mathematical groups. The definition and generalization of the various metric spaces and the concept of contractions are common applications of metric fixed-point theory. The expected outcome of these extensions is also a deeper understanding of the geometric characteristics of generalized metric spaces, set theory, and inexpensive mappings.

This book is intended for scholars, doctoral scholars, and educators intrigued by fixed-point theory. The book will also benefit the mathematical community, professionals, and scientists. Learners of this book will need at least a basic understanding of functional analysis and topology. This book contains chapters written by several notable modern academics in fixed-point theory from around the world. Readers will discover various valuable tools and approaches to help them advance their knowledge and abilities in modern fixed-point theory. The book includes adequate theory and applications of fixed points in various fields. The book provides an overview of existing knowledge and recent cutting-edge advancement by offering fresh novel contributions by world-renowned experts of fixed-point theory.

Following are salient features of the book:

- Provides a comprehensive and up-to-date assessment of the most recent research in fixed-point theory, including major breakthroughs and applications within the environment of various types of generalized metric spaces

- Offers a unique combination of fixed-point theorems and real-world applications that benefits students and researchers interested in developing an integrated research methodology
- Emphasizes the mathematical modeling of nonlinear situation, utilizing the advantages of technology as appropriate
- Takes a fresh look at fixed-point theory, highlighting linkages and applications to fields as diverse as optimization, graph theory, and differential equations
- Stresses on the practical applications of fixed-point theory in diverse fields of research and engineering, making it ideal for scientists researching in engineering sciences
- Offers thorough descriptions of numerical methods and algorithms for addressing fixed-point problems, making it a useful resource for scholars and practitioners
- Offers a variety of examples meant to assist readers deepen their comprehension of the content and apply it to new challenges
- Emphasizes recent advances in fixed-point theory and identifies significant open problems, making it a valuable resource for academics wishing to contribute to this dynamic area of research

Chapter Organization

This book is organized into 17 chapters. Chapter "A Careful Retrospection of Metric Spaces and Contraction Mappings with Computer Simulation" gives a comprehensive outlook of the famous results based on metric fixed-point theory. Illustrative non-trivial examples, along with graphical representation, are enunciated to understand the nature of inequalities of the historical results in the context of metric spaces.

Chapter "Fixed Point Theory for Multi-valued Feng–Liu Operators in Vector–Valued Metric Spaces " extends Feng and Liu's fixed-point result for the case of a set X equipped with a vector-valued metric in the sense of Perov. The results describe the existence, localization, data dependency, and stability features of a fixed-point inclusion with a generalized multi-valued Feng–Liu operator. An extension of the Feng–Liu–Subrahmanyan type of multi-valued contractions is also examined.

In Chapter "Algorithms and Applications for Split Equality Problem with Related Problems ", the authors propose a novel study on "Algorithms and applications for split equality problem with related problems." The chapter's goal is to offer a variety of novel and well-known numerical strategies for addressing the split feasibility problem, the split equality problem, and other related problems, such as projection, inertial techniques, relaxed techniques, and so on. The algorithms' weak (strong) convergence is presented, and the linear convergence of the methods is emphasized. An algorithm is enunciated as an application to tackle signal processing and picture recovery problems.

Chapter "Some Fixed Point Theorems of Generalized Contractions with Application to Boundary Value Problem" investigates the development of an exclusive rational type generalized contraction involving three self-maps. Furthermore, several fixed-point findings matching the requirements of $a(\eta,\pi,\theta)$-generalized rational contraction are proposed within the setting of complete metric spaces with a partial order. These findings not only generalize but expand numerous well-known findings from previous research.

Chapter "Fixed Points of Coset and Orbit Space Actions: An Application of Semihypergroup Theory" is devoted to presenting certain families of left/right coset, double coset, and orbit spaces that arise from the category of locally compact groups. The study is concerned with looking at their behavior on compact subsets of generic locally convex spaces and specific Banach spaces. Utilizing recent discoveries in abstract harmonic analysis and the theory of semihypergroups, in particular, an overview of possible characterizations for the existence of common fixed points of such actions in terms of the amenability of the underlying spaces is presented.

A counterpart of Meir–Keeler's fixed-point result in suprametric space is proved in Chapter "Strange Chaotic Attractors and Existence Results via Nonlinear Fractional Order Systems and Fixed Points", and an application to strange attractors in the context of the Atangana–Baleanu derivative is examined.

The existence findings for the n-product of fractional nonlinear equations in Orlicz spaces are studied in Chapter "On L_ϕ-Solutions for n-Product of Fractional Integral Operators". A wide range of boundedness and continuity assumptions of the researched operators in Orlicz space are explored by using different growth conditions. Also, distinct existence theorems on the product of different n-Orlicz spaces related to the generating N-functions are investigated. The main tools for achieving the results are the fixed-point hypothesis and the measure of noncompactness.

In the context of complete b-metric spaces, Chapter "New Fixed Point Results of Multivalued Contraction Mappings in b-Metric Spaces" examines recent findings for generalized contractive type multi-valued mappings. The results add to and enhance certain recent breakthroughs announced by numerous others with fewer assumptions.

In the framework of controlled type metric spaces, Chapter "Analysis of Fixed Points in Controlled Metric Type Spaces with Application" explores the presence and uniqueness of fixed points using the Ciric-type and Reich-type contractions. The work intends to offer a more thorough knowledge of fixed-point findings by including graph theory in these contraction mappings, a contemporary technique in the current state of the art. The outcomes provided in this chapter demonstrate the versatility of contraction mappings in nonlinear differential equations and their value in mathematics.

New classes of $(G,\alpha,\phi,)$-contractions are suggested in Chapter "A Study of Fixed Point Results in G-Metric Space via New Contractions with Applications", and appropriate FP theorems are demonstrated. The uniqueness of these new contractions resides in the fact that, depending on the selection of parameters, they may be specialized in a variety of ways. In-depth comparative examples are produced to verify the presumptions behind the conclusions. In addition, the existence conditions for solving a boundary value issue utilizing one of the findings are investigated.

Chapter "A Note on the Existence of Fixed Points for Rational Type Contraction Map on Orthogonal Metric Spaces" focuses on instigating the existence of fixed points for rational type contraction maps on orthogonal metric space, a more general metric space in modern literature. Some examples from this research substantiate the presented outcomes.

Chapter "Fixed Point Results in Graphical Convex Extended b-metric Spaces" provides a historical overview of F-contractions. The chapter presents a historical overview of the new fixed-point approaches for single and multi-valued mappings in various spaces. Furthermore, certain enhancements, notably the additional conditions applied to the function F of the contractive condition, are investigated. Researchers who are interested in F-contractions will find this chapter informative.

Chapter "Existence and Computational Approximation of Fixed Points of Generalized Multivalued Mappings in Banach Space" aims to define graphical convex extended b-metric space and examine certain vital aspects of the convex structure. Iterative sequences G-contraction, T-Mann, and T-Agrawal have been addressed in the context of the instigated space. These iterative methods are used to investigate the existence of strong fixed-point theorems. Combining the convex structure of metric spaces with graphs makes the article innovative and appealing to academics interested in locating fixed points inside the graph structure.

Chapter "Common Fixed Point Results in Soft b-Metric Spaces with Application" focuses on the presence and computational approximation of fixed points of generalized multi-valued mappings in Banach spaces. This chapter introduces multi-valued generalized-nonexpansive mappings and associated results for fixed-point existence and approximation. Furthermore, the Picard–Thakur hybrid iterative scheme convergence findings are reviewed and contrasted. It is demonstrated mathematically and visually that the Picard–Thakur hybrid iterative strategy converges to the fixed-point quicker than other schemes presented in the literature. The final part provides an application to integral equations to validate the offered conclusions.

Chapter "Revisiting Darbo's Fixed Point Theory with Application to a Class of Fractional Integral Equations" discusses the novel fixed-point findings in soft b-metric spaces and an application to Volterra integral inclusions. The results of the study can be expanded to other areas in the future, and new results can be drawn from them.

In Chapter "New Topologies on Partial Metric Spaces and M-Metric Spaces", new definitions for partial metric spaces and M-metric spaces are presented. Various topological features of the aforementioned metric spaces are addressed. For partial metric spaces and M-metric spaces, the new topology is demonstrated to be weaker than the previously established topology.

Chapter "Some Recent Fixed Point Results in S_b-Metric Spaces and Applications" defines a generalized proportional (k, ρ)-fractional integral operator with a nonsingular kernel. It is a more generalized version of previously known fractional integral operators such as the Riemann–Liouville fractional integral operator, the Hadamard fractional integral operator, the Katugampola fractional integral operator, etc. Using Darbo's fixed-point theorem, it is proved that there is a solution to the generalized proportional (k, ρ)-fractional integral equation.

Finally, a relevant example is built to validate the acquired findings.

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Contents

About the Editors

Mudasir Younis is a postdoctoral research fellow at the Department of Mathematics and Statistics, Indian Institute of Technology Kanpur (IIT Kanpur). He researched diverse fixed point theorems within the graph structure of metric spaces throughout the last five years of his considerable research, which is a relatively fresh addition to the relevant topic. In this context, he obtained several novel results and worked on determining the existence of solutions to various real-world engineering science and physics problems, such as a damped Spring–Mass system, deformation of an elastic beam, vibrations of a vertical heavy hanging cable, ascending motion of a rocket, tuning circuit problem, and so on. He has received fellowships at both the national and international levels. He has more than twenty-five research papers published in SCI/ESCI/Scopus listed journals and seven other papers communicated to prestigious international publications. He recently earned the International Mathematical Union's Abel Visiting Fellowship for 2022–2023. He has served on the editorial boards of various peer-reviewed journals and presented his work at several international conferences. He is also the author of a book *Fourier Analysis*, published by the University of Kashmir Press.

Lili Chen is an associate professor and doctorate supervisor at the college of Mathematics and System Science at Shandong University of Science and Technology, Qingdao, China. In recent years, she has been primarily involved in research on geometry theory of banach spaces, and fixed point theory. She is a commentator of the American Mathematical Review, a member of the Chinese Mathematical Society, and a member of the American Mathematical Society. She is a reviewer for several international SCI academic journals and has also supervised 8 projects. She has over 30 papers published in peer-reviewed SCI journals. She has so far supervised 14 graduate students and two doctoral students.

Deepak Singh is an associate professor of mathematics, Department of Applied Science, National Institute of Technical Teachers' Training and Research (NITTTR), Bhopal India. He obtained M.Sc. and Ph.D. degrees in mathematics from Barkatullah University, Bhopal. During his teaching career of about 22 years, he has taught various courses in pure mathematics, applied mathematics, and computer applications, at under-graduate and post-graduate level. He has more than 60 research papers to his credit, many of them are published in various SCI/ISI/Scopus/ESCI, Springer, and Elsevier journals. So far, he has supervised 15 Ph.D. candidates. He has also supervised 21 M.Phil. projects. Besides, he keeps participating in various scientific meetings/conferences in India and abroad. So far, he has delivered over 40 contributed/invited talks including 17 in European and Asian countries. He is a reviewer of the American Mathematical Society and also the referee of many SCI/Springer/Elsevier journals. He has been a resource person in refreshers courses for various universities/'IIT's where he delivered a series of lectures. He has also been appointed as an external expert for Ph.D. viva and thesis valuation from various central universities. He has also coordinated/contributed for more than 130 training programs of one week/two weeks duration for technical teachers.

A Careful Retrospection of Metric Spaces and Contraction Mappings with Computer Simulation

Mudasir Younis, Deepak Singh, and Lili Chen

Mathematics Subject Classification (2010) 47H09 · 47H10

1 An Introduction

Fixed-point theory is the fascinating emerging field of the twenty-first century characterized by a remarkable mixture of nonlinear functional analysis, nonlinear operator theory, topology, mathematical modeling, and applications. It is one of the major research areas in nonlinear analysis. Because of the fact that in many real-world problems, the fixed-point theory is the fundamental mathematical tool employed to ascertain the existence of solutions to problems that arise naturally in applications. As a consequence, the fixed-point theory is an essential area of study in pure and applied mathematics, and it is a blooming area of research. Its scope of inquiries not only encompasses the geometric theory of infinite dimensional function spaces and operator-theoretic real-world problems but also widens the range of interdisciplinary fields ranging from engineering to space science, hydromechanics to astrophysics, chemistry to biology, theoretical mechanics to biomechanics, and economics to stochastic game theory. The deep-rooted concepts and techniques pro-

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vide the tools for developing more realistic and accurate models for a variety of phenomena encountered in various applied fields.

Fixed-point theory sheds light on the methodologies for finding a solution to nonlinear equations of the type $Jx = x$, where J is a self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space, or some suitable space.

In the subsequent part, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of natural numbers, the set of all non-negative real numbers and the set of real numbers, respectively.

Definition 1 (*Fixed Point*) Suppose J is a mapping that takes a set Y into itself. A fixed point of J is just a point $x \in Y$ with $J(x) = x$.

A map J can have many fixed points (example: the identity map on a set with many elements) or no fixed points (example: the mapping of "translation-by-one" $x \to x + 1$ on the real line). The fixed points of a function, mapping a real interval into itself, can be visualized as the x-coordinates of the points at which function's graph intersects the line $y = x$. We enunciate this idea by plotting the graph for some nontrivial functions $J_i, i = 1, 2 : \mathbb{R} \to \mathbb{R}$:

$$(i)\quad J_1(x) = \sqrt{\log(\sin x^{\frac{1}{4}} + e^{x+6})} + \sin(\tfrac{1}{2} + x)$$
$$(ii)\quad J_2(x) = e^{\frac{2x+3}{1+x^2}\log(x^3+2)}$$

We determine the fixed points of functions $J_1(x)$ and $J_2(x)$ by visualizing them on graph as follows (Figs. 1 and 2):

Definition 2 (*Common Fixed Point*) Let f and g be two self-mappings on a space Y. We say that $x \in Y$ is a point of coincidence of f and g if $fx = gx$, and say that it is a common fixed point of f and g if $fx = gx = x$.

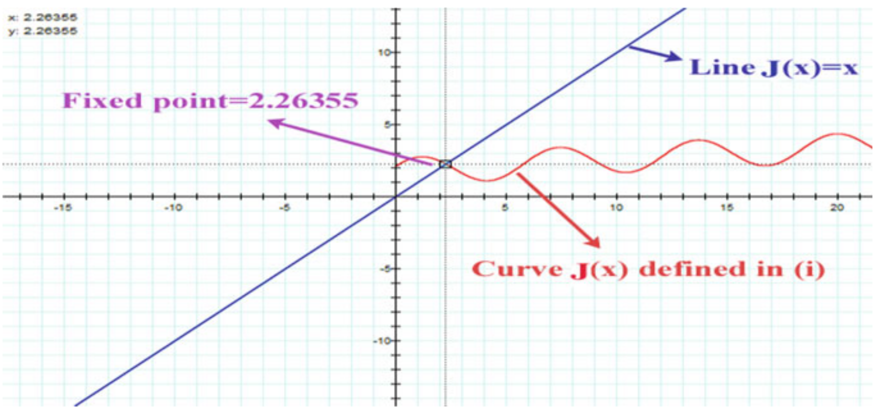

Fig. 1 Fixed point of $J_1(x) = \sqrt{\log(\sin x^{\frac{1}{4}} + e^{x+6})} + \sin(\frac{1}{2} + x)$

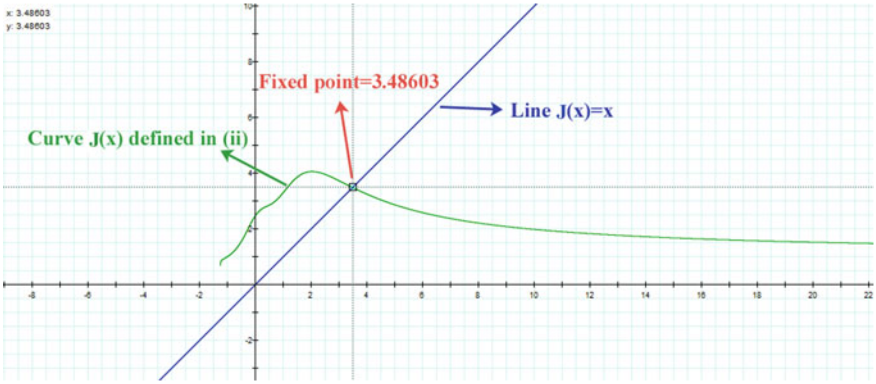

Fig. 2 Fixed point of $J_2(x) = e^{\frac{2x+3}{1+x^2}\log(x^3+2)}$

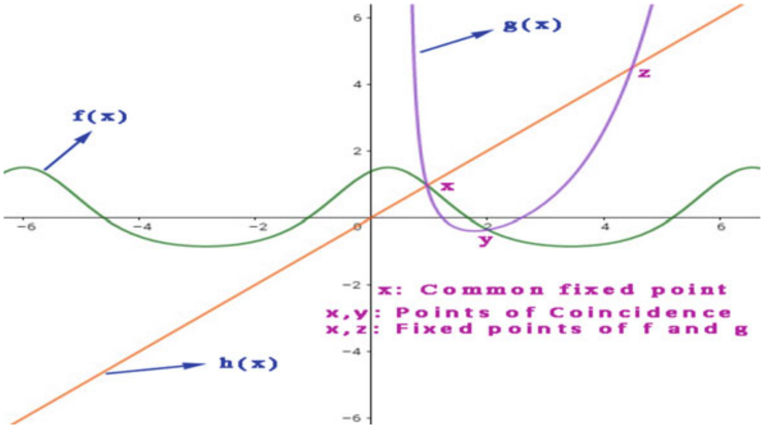

Fig. 3 Common fixed point of mappings $f(x)$ and $g(x)$

Example 1 Let $f(x) = e^{\sin(x-5)} - 1.22$ and $g(x) = \frac{\cosh(x-0.58)}{e^{\ln(x-0.58)}} - 1.9$ for all $x \in [0, \infty)$. Then $x = 0.95518$ is the common fixed point of mappings f and g (Fig. 3).

Some very important examples of fixed points, we generally go through in our study, are as follows:

Example 2 (*Initial-Value Problems*) From a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$, we can generate an initial-value problem

$$y' = g(x, y),\ y(x_0) = y_0. \tag{1}$$

Geometrically, initial-value problem (1) demands a differentiable function y whose graph is a smooth "solution curve" in the plane possessing the following properties:

(i) At each of its points (x, y) the curve has slope $g(x, y)$,
(ii) The curve encompasses the point (x_0, y_0).
As a first attempt to solve the differential equation $y' = g(x, y)$, we might try integrating both sides with respect to x. This results in the following integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} g(t, y(t))dt, \tag{2}$$

which is implied by initial-value problem (1) in the sense that each function y satisfying (1) for some interval of x's containing x_0, also satisfies integral Eq. (2) for that same interval. To make a concern with fixed points, let $C(\mathbb{R})$ denote the vector space of continuous, real-valued functions on $\mathbb{R}$, and consider the integral transform $J : C(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$(Jy)(x) = y_0 + \int_{t=x_0}^{x} g(t, y(t))dt, \quad x \in \mathbb{R}.$$

Equation (2) can thus be rewritten as $Jy = y$, so to say $y \in C(\mathbb{R})$ satisfying (1) turns out to be the same as saying: y is a fixed point of the mapping J.

Example 3 (*Newton's Method*) Let us suppose that J is a differentiable function $\mathbb{R} \to \mathbb{R}$, with continuous and never vanishing derivative J' on $\mathbb{R}$. Consider for J its "Newton" function N, defined by

$$x - \frac{J(x)}{J'(x)} = N(x), \quad x \in \mathbb{R}.$$

One can think of $N(x)$ as the horizontal coordinate of the point at which the line tangent to the graph of J at the point $(x, J(x))$ intersects the horizontal axis. Since J' doesn't vanish, N is a continuous mapping taking $\mathbb{R}$ into itself. Fixed points of N are explicitly the roots of J (those $x \in \mathbb{R}$ such that $J(x) = 0$). Newton's method is concerned with the iteration of the Newton function in anticipation of generating approximations to the roots of J. One starts with an initial guess x_0, sets $x_1 = N(x_0)$, $x_2 = N(x_1), \ldots$, and hopes that the resulting sequence of "Newton iterates" converges to a fixed point of N. Geometrically it appears evident that if the Newton iterate sequence converges then it must converge to a root of J.

Example 4 (*The PageRank algorithm*) Google's prosperity as a web crawler comes from its calculation: the PageRank calculation algorithm. In this calculation, one processes a fixed point of a linear map on R^n, which is itself a contraction, and this fixed point (which is, in fact, a vector) yields the requesting of the pages. Further amalgamation and illustrative insights about "the PageRank calculation algorithm" can be found in the vignette on *How Google works*?.

2 Contraction Mappings: History and Development

A natural question arises that under what conditions on the set Y and the self mapping J, a fixed point exists?. Theorems which establish the existence and uniqueness of such points are called **Fixed-Point Theorems**.

There are three major branches of fixed-point theory in functional analysis and each branch has its celebrated theorems.

– Metric Fixed-Point Theory,
– Topological Fixed-Point Theory,
– Discrete Fixed-Point Theory.

Historically, the above approaches were initiated by the discovery of three major theorems:

– Banach's contraction principle [19],
– Brouwer's fixed-point theorem [33],
– Tarski's fixed-point theorem [177].

Historically, the study of fixed-point theory began in 1912 with a theorem given by famous Dutch mathematician Brouwer [33]. This is the most famous and important theorem on the topological fixed-point property. It can be formulated as:

★ The closed unit ball $\mathscr{B}^n \in \mathbb{R}^n$ has the topological fixed-point property.

He also proved the fixed-point theorems for a square, a sphere and their n-dimensional counterparts which was further extended by Kakutani [90]. Brouwer's theorem has many applications in analysis, differential equation and generally in proving all kinds of so-called existence theorems for many types of equations.

An important generalization of Brouwer's theorem was discovered in 1930 by Schauder [157] it may be stated as follows:

Theorem 1 *Let Y be a Banach Space and $\mathscr{B}$ is a compact, convex subset of Y. If the self map $J : \mathscr{B} \to \mathscr{B}$ is a continuous, then J admits a fixed point.*

The Schauder fixed-point hypothesis has various applications in scientific theory, approximation theory, and different scientific areas like the improvement theory, social science, and engineering. The compactness condition on $\mathscr{B}$ is a solid one, and most of the analysis results don't possess a compact setting. It's natural to prove the results by relaxing the condition of compactness. Schauder demonstrated the subsequent theorem:

Theorem 2 *Let Y be a Banach space and $\mathscr{B}$ be its closed and bounded convex subset. If $J(\mathscr{B})$ is compact, where $J : \mathscr{B} \to \mathscr{B}$ is a continuous map, then J admits a fixed point.*

Meanwhile, the Banach principle came into existence, considered to be one of the elementary principles in nonlinear analysis. In 1922, Banach [19] proved that a contraction mapping in the context of a complete metric space admits a unique fixed point.

In the subsequent part, we focus mainly on the first area, that is, ***metric fixed-point theory*** based on Banach's contraction.

The term *metric fixed-point theory* refers to those fixed point theoretical results in which geometric conditions on the underlying spaces and/or mappings play a crucial role. The first-ever fixed-point theorem in metric space appeared in explicit form in Banach's thesis [19], known as the "Banach's Contraction Principle", used to establish the existence of a solution to an integral equation. Before presenting this remarkable result, formal definition of the distance introduced in 1905 by Frèchet [60] is worth mentioning.

Definition 3 Let Y be a nonempty set, and let $d : Y \times Y \to [0, \infty)$ be a given mapping. We say that d is a metric on Y, if for all $x, y, z \in Y$, the following conditions are fulfilled:

$(d1)$ $d(x, y) = 0$ if and only if $x = y$;

$(d2)$ $d(x, y) = d(y, x)$;

$(d3)$ $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (Y, d) is called a metric space.

Banach [19] published his fixed-point theorem, also known as "Banach's Contraction Principle" which uses the concept of Lipschitz mappings.

Definition 4 Let (Y, d) be a metric space. The self mapping $J : Y \to Y$ is said to be Lipschitzian if there exists a constant $\lambda > 0$ (called Lipschitz constant) such that

$$d(Jx, Jy) \leq \lambda\, d(x, y), \quad \text{for all } x, y \in Y. \tag{3}$$

A Lipschitzian mapping with a Lipschitz constant $\lambda < 1$ is called **contraction**.

Theorem 3 (Banach's Contraction Principle) *Let (Y, d) be a complete metric space and $J : Y \to Y$ be a contraction mapping with Lipschitz constant $\lambda < 1$. Then J has a unique fixed point $x^* \in Y$ and $\lim_{n\to+\infty} J^n x = x^*$ for all $x \in Y$.*

We verify Banach's contraction principle with an illustrative example along with its graphical representation, where the surface representing the right-hand side of the contraction mapping dominates the surface constituting the left-hand side of the contraction mapping.

Example 5 Let $Y = [0, 10]$ be a set. Define a function $d : Y \times Y \to [0, 10]$ by $d(x, y) = |x - y|$, then clearly (Y, d) is a complete metric space. Let J be a self-map on Y defined as

$$J(x) = \frac{\log(2 + e^{x+x^2})}{2 + e^x + \sin x}.$$

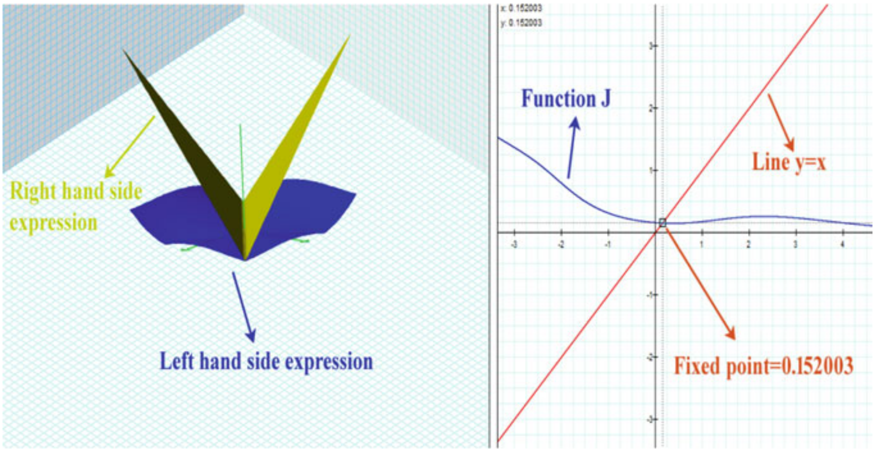

Fig. 4 Visual verification of Banach's contraction principle with fixed point

The following figure shows that surface representing the right-hand side is dominating the surface constituting the left-hand side of (3) with $J(x) = \frac{\log(2+e^{x+x^2})}{2+e^x+\sin x}$ and $\lambda = \frac{11}{12}$.
This demonstrates the validity of Banach's contraction principle, and so, J has a unique fixed point $x = 0.152003$ shown by Fig. 4.

After the establishment of this significant result, it was generalized and extended in different ways by several authors (see, e.g., [2, 4, 38, 53, 63, 64, 77, 92, 95, 96, 114, 127, 178]). Authors obtained numerous fixed-point theorems on the following two approaches.

1. Extending the contraction condition (3) to more general contraction conditions, and
2. Replacing the complete metric space (Y, d) by specific generalized metric spaces.

In the first-mentioned direction, the results due to Edelstein [54, 55] and Rakotch [141] created a new milestone in the literature of fixed-point theory. Rakotch [141] generalized Banach contraction principle in the following way.

Theorem 4 ([141]) *Let Y be a complete metric space and suppose $J : Y \to Y$ satisfies*

$$d(J(x), J(y)) \leq \alpha(d(x, y))d(x, y), \text{ for all } x, y \in Y,$$

where $\alpha : \mathbb{R}^+ \to [0, 1)$ is monotonically decreasing. Then J has a unique fixed point x^* and $\{J^n(x)\}$ converges to x^* for each $x \in Y$.

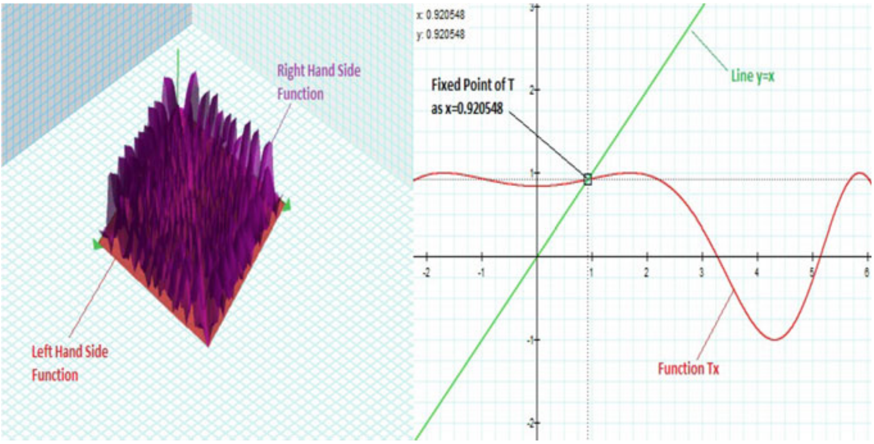

Fig. 5 Visual verification of Rakotch's theorem and the corresponding fixed point

Example 6 Let $Y = [0, \infty)$ be a set. Define a function $d : Y \times Y \to [0, \infty)$ by $d(x, y) = |x - y|$ then clearly (Y, d) is a complete metric space. Let T be a self-map on Y defined as $T(x) = \sin(1 + \frac{x^2}{5})$ with $\alpha(x) = \frac{1}{(x^2+5)^2}$.
The following figure demonstrates that on invoking function $T(x) = \sin(1 + \frac{x^2}{5})$ on (4), the surface representing the right-hand side is superimposing the surface representing the left-hand side. This demonstrates the validity of Rakotch's theorem and so x has a unique fixed point $x = 0.920548$ shown by adjacent figure (Fig. 5).

In 1968, Kannan [91] generalized Banach's contraction principle in some different way, where the map involved does not need to be continuous. This theorem gained quite more attention in the field of metric fixed-point theory.

Theorem 5 ([91]) *Let (Y, d) be a complete metric space and J be a self-map on Y such that, for all $x, y \in Y$, the following condition is satisfied*

$$d(Jx, Jy) \leq \mathscr{K}\,[d(x, Jx) + d(y, Jy)], \tag{4}$$

where $\mathscr{K} \in [0, \frac{1}{2})$. Then J has a unique fixed point $x^* \in Y$.

The following example is worked out to verify Kannan's theorem along with its visualization.

Example 7 Let $Y =]-\infty, 0]$ be a set. A function $d : Y \times Y \to [0, \infty)$ is defined as $d(x, y) = |x - y|$, then clearly (Y, d) is a complete metric space. Let $J : Y \to Y$ be defined by the following

$$J(x) = \frac{1}{2}\log(2\cos x + 1).$$

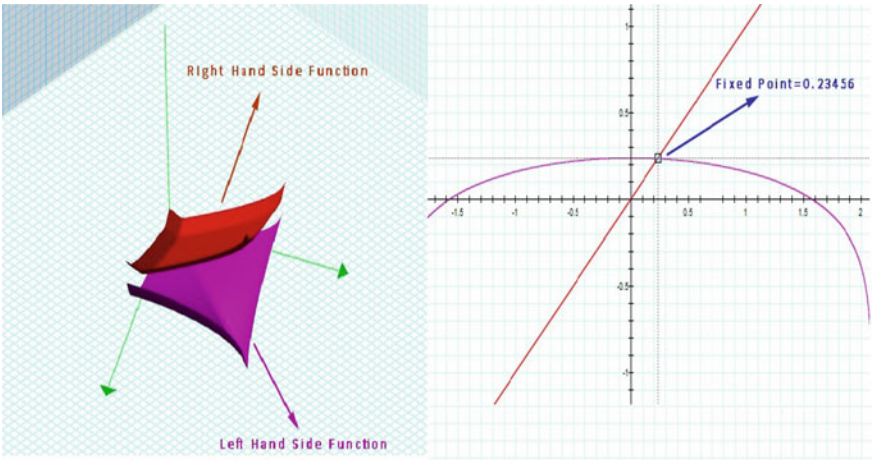

Fig. 6 Pictorial verification of Kannan's theorem and related fixed point of J

Figure 6 authenticates the domination of the right-hand side of the inequality (4) over its left-hand side by invoking the function $J(x) = \frac{1}{2}\log(2\cos x + 1)$ with $\mathscr{K} = \frac{2}{5}$. Hence, all the conditions of Kannan's result are contended and J has a unique fixed point $x = 0.23456$ shown by Fig. 6.

Kannan's theorem is also fundamental because Subrahmanyam [176] proved that Kannan's theory characterizes the metric completeness, that is, a metric space (Y, d) is complete if and only if every Kannan mapping on Y has a fixed point. However, contractions (in the sense of Banach) do not have this property.

The following examples compare Banach's and Kannan's contraction conditions.

Example 8 Let $Y = \mathbb{R}$ be a usual metric space and $J : Y \to Y$ be a mapping defined by

$$J(x) = \begin{cases} 0, & \text{if } \ x \in]-\infty, 2], \\ \frac{1}{2}, & \text{if } \ x \in]2, +\infty[. \end{cases}$$

In this example, J is not continuous on $\mathbb{R}$; therefore, Banach's contraction condition is not satisfied but it satisfies Kannan's contraction with $\mathscr{K} = \frac{1}{5}$.

Example 9 Let $Y = [0, 1]$ and J be the self mapping of non-empty set Y such that, $J(x) = \frac{x}{3}$ for $x \in [0, 1]$.

In this example, Banach's contraction condition is satisfied but, for $x = \frac{1}{3}$ and $\mathscr{K} = 0$, it does not satisfy Kannan's contractive condition.

Later, in 1971, Reich [143] established more general and innovate extension of Banach's principle for single-valued as well as multi-valued mappings. Since then Reich-type mappings have been the center of intensive research for many authors.

Theorem 6 ([143]) *Let* (Y, d) *be a complete metric space and* J *be a self-map on* Y. *There exist some real constants* $\alpha, \beta, \gamma \in \mathbb{R}^+$ *with* $\alpha + \beta + \gamma < 1$, *such that*

$$d(Jx, Jy) \leq \alpha\, d(x, y) + \beta\, d(x, Jx) + \gamma\, d(y, Jy), \tag{5}$$

for all $x, y \in Y$. *Then* J *has a unique fixed point* $x^* \in Y$.

The following example validates Reich's fixed-point theorem.

Example 10 Let $Y = [0, 1]$ be a set. A function $d : Y \times Y \to [0, \infty)$ is defined by $d(x, y) = |x - y|$, then obviously (Y, d) is a complete metric space. Let J be a self-map on Y defined by the following

$$J(x) = \frac{e^{2x^2-1} + \sqrt{x+2}}{7 + \log(x + 50)}.$$

Calculating the right-hand side and the left-hand side of the inequality (5), it follows that the inequality (5) of Reich's theorem is satisfied for $\alpha = 0.39$, $\beta = 0.29$, $\gamma = 0.27$ by invoking the function $J(x) = \frac{e^{2x^2-1}+\sqrt{x+2}}{7+\log(x+50)}$.

Figure 7 authenticates the validity of inequality (5), where the surface corresponding to the right-hand side is overlaying the surface corresponding to the left-hand side, and $x = 0.217634$ is the corresponding unique fixed point.

After that, several authors have introduced a variety of contraction-type conditions and established fixed-point theorems in the framework of complete metric spaces. Bianchini [29], Chatterjea [37], Geraghty [65] and Hardy–Roger [68] also extended Banach's result in their manner.

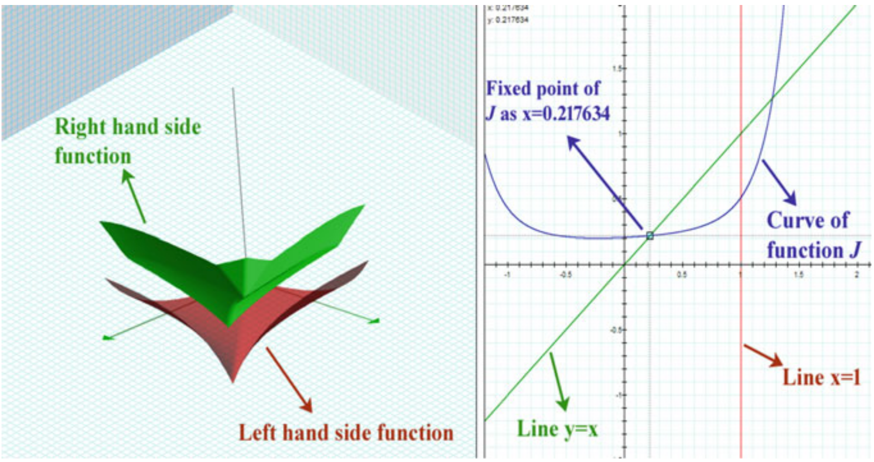

Fig. 7 Visual verification of Reich's with fixed point

In 1973, Hardy–Roger [68] gave an innovative extension of Banach's principle as follows:

Theorem 7 ([68]) *Let* (Y, d) *be a complete metric space and* J *be a self-map on* Y*. There exist some real constants* $\alpha, \beta, \gamma, \delta, \xi \in \mathbb{R}^+$ *with* $\alpha + \beta + \gamma + \delta + \xi$*, such that*

$$d(Jx, Jy) \leq \alpha d(x, Jx) + \beta d(y, Jy) + \gamma d(x, Jy) + \delta d(y, Jx) + \xi d(x, y), \quad (6)$$

for all $x, y \in Y$*. Then* J *has a unique fixed point* $x^* \in J$*.*

The following example validates Hardy–Rogers's theorem.

Example 11 Let $Y = [0, 2]$ be a set. A function $d : Y \times Y \to [0, \infty[$ is defined by $d(x, y) = |x - y|$ then clearly (Y, d) is a complete metric space. Let J be a self-map on Y defined as $J(x) = \frac{e^{x-1}}{1+\log(1+x)}$.
Subsequent figure makes evident that on employing function $J(x) = \frac{e^{x-1}}{1+\log(1+x)}$ on (6) with $\alpha = 0.15$, $\beta = 0.15$, $\gamma = 0.2$, $\delta = 0.1$, $\xi = 0.3$, the surface corresponding to the right-hand side is overlaying the surface corresponding to the left-hand side. This substantiates Hardy–Rogers's theorem and so Jx has a unique fixed point $x = 0.52594$ expressed by Fig. 8.

Later, in 1974, Ćirić [40] presented a new version of Banach's contraction in a quite different way. Ćirić [40] involved all distances $d(Jx, Jy)$, $d(x, y)$, $d(x, Jx)$, $d(y, fJy)$, $d(x, Jy)$, $d(y, Jx)$ in his contraction in a linear way, while Banach [19] used only the first two distances. More precisely, the renowned Ćirić [40] for a single-valued map is the following:

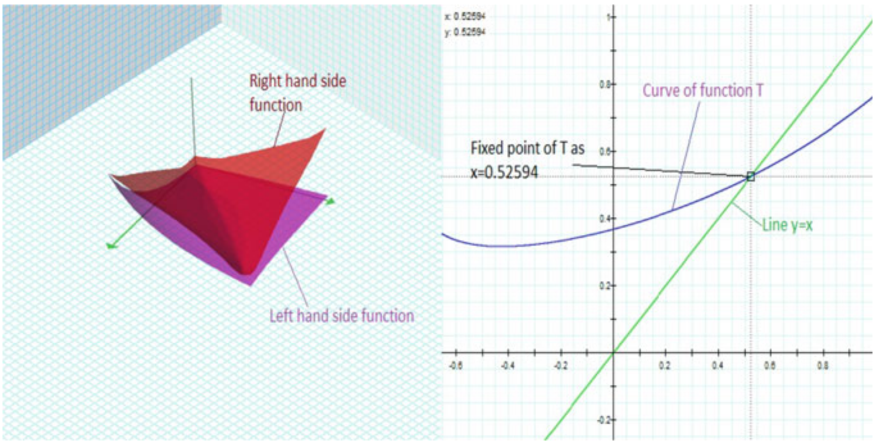

Fig. 8 Visual verification of Hardy–Rogers's result and the corresponding fixed point

Theorem 8 ([40]) *Let* (Y, d) *be a complete metric space and* J *be a self-map on* Y*. There exists a constant* $\lambda \in [0, 1)$*, such that*

$$d(Jx, Jy) \leq \lambda \mathscr{M}(x, y), \tag{7}$$

for all $x, y \in Y$*, where,*

$$\mathscr{M}(x, y) = max\{d(x, Jx), d(y, Jy), d(x, Jy), d(y, Jx), d(x, y)\}.$$

Then J *admits a unique fixed point* $x^* \in J$*.*

The validation of Ćirić's result is endorsed by the subsequent example.

Example 12 Let $Y = [0, \frac{1}{2}]$ be a set. A function $d : Y \times Y \to [0, \infty)$ is defined by $d(x, y) = |x - y|$. Then (Y, d) is a complete metric space. Now, define a self mapping $J : Y \to Y$ by the following

$$J(x) = \frac{e^{2x^2-1}}{\sqrt{1 + \sin(1 + x)}}.$$

The following figure shows that on applying the value of function $J(x) = \frac{e^{2x^2-1}}{\sqrt{1+\sin(1+x)}}$ in (7) with $\lambda = 0.91$, the surface corresponding to the right-hand side dominates the surface corresponding to the left-hand side. This substantiates Ćirić's theorem with unique fixed point $x = 0.322937$ expressed by Fig. 9.

In 1969, Boyd and Wong [30] obtained a more general result. In this result, the authors assumed that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous from the right that is $r_n \downarrow r > 0 \Rightarrow \lim_{n\to\infty} \sup \phi(r_n) \leq \phi(r)$. In the same paper, author showed that, if the space Y is metrically convex, then the upper semi-continuity assumption on ϕ can be

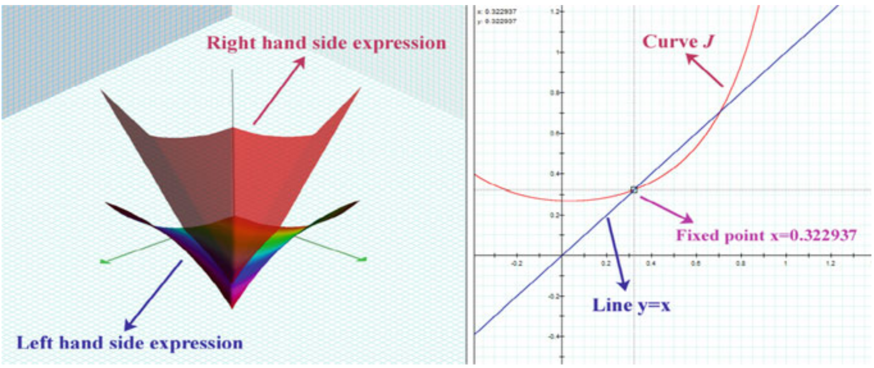

Fig. 9 Computer simulation of Ćirić's result with fixed point

dropped. On the other hand, Matkowski in [110] extended the above result claiming that, if ϕ is assumed to be continuous at 0, then there exists a sequence $t_n \downarrow 0$ for which $\phi(t_n) < t_n$.

Some other generalized results were collected and compared in the well-known article of Rhoades [145], and later one of Collaço and Silva [43]. Further escalations in the level of complexity can be found in a noteworthy paper by van An et al. [181]. In 2003, Berinde [24] introduced the notion of weak contraction (also known as almost contraction) and generalized Banach's result. He enhanced that Banach's and Kannan's mappings are weak contractions. The merit of weak contractions is that they unify large classes of contractive-type operators including quasi-contractions whose fixed points can be obtained employing the Picard iteration and for which both a priori and posteriori estimates are also available.

Berinde [24] adorned the concept of nonlinear-type weak contraction operating a comparison function in a metric space as follows:

Theorem 9 ([24]) *Let (Y, d) be a complete metric space and $J : Y \to Y$ be a self mapping such that there exist $\lambda \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Jx, Jy) \leq \lambda\, d(x, y) + L\, d(y, Jx), \tag{8}$$

for all $x, y \in Y$. Then J has one and only one fixed point $x^* \in Y$.

Due to symmetry of the distance, the weak contraction condition (8) implicitly includes the following dual condition:

$$d(Jx, Jy) \leq \lambda\, d(x, y) + L\, d(x, Jy).$$

The following example substantiates the validity of Theorem 9 due to Berinde.

Example 13 Let $Y = [0, \frac{1}{2}]$ be a set. Define a function $d : Y \times Y \to [0, \infty)$ by $d(x, y) = \{\max(x, y)\}^2$. Then (Y, d) is a complete metric space. Let J be a self map defined on Y as follows:

$$J(x) = \frac{x}{3} + x \log(1 + x^{\frac{1}{3}}).$$

Without loss of generality, we take $x > y$. The subsequent figure presents that on utilizing the value of operator $J(x) = \frac{x}{3} + x \log(1 + x^{\frac{1}{3}})$ on the inequality (8) with $\lambda = 0.8$ and $L = 2$, the surface corresponding to the right-hand side is superimposing the surface corresponding to the left-hand side. This validates Berinde's theorem and so J has a unique fixed point $x = 0$ shown by Fig. 10.

In recent investigations, Wardowski [182] considered a new type of contraction, namely, F-contraction, and proved some fixed-point results in a very general and natural setting.

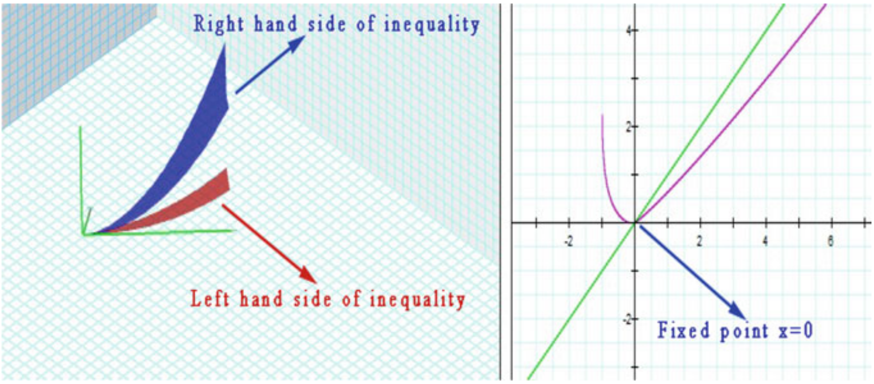

Fig. 10 Pictorial verification of Berinde's result with the fixed point

Definition 5 ([182]) Let (Y, d) be a metric space. A self-mapping $J : Y \to Y$ is said to be an F-contraction, if there exists $\tau > 0$ such that for every $x, y \in Y$

$$d(Jx, Jy) > 0 \Rightarrow \tau + F(d(Jx, Jy)) \leq F(d(x, y)), \tag{9}$$

where the function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following assertions:

$(F1)$ F is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;
$(F2)$ for all sequence $\alpha_n \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(F3)$ there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

By considering various types of mappings F in (9), one can obtain a variety of contractions, some of them are of a type known in the literature. Wardowski [182] described the class of all functions F satisfying $F1$, $F2$ and $F3$ by $\mathscr{F}$.

Theorem 10 *Let (Y, d) be a complete metric space and $J : Y \to Y$ be F-contraction. Then J has precisely one fixed point.*

We verify Wardowski's theorem by the following non-trivial example.

Example 14 Let $Y = [0, 2]$ be a set. Define a function $d : Y \times Y \to [0, \infty)$ by $d(x, y) = |x - y|$. Then (Y, d) is a complete metric space. Let $J : Y \to Y$ be a self map defined on Y as follows

$$Jx = \sin\left(x + \frac{1}{e^{x+1}}\right).$$

Taking $F(a) = \log a$ and $\tau = 0.2$, one can easily see that the left-hand side of the inequality (9) is dominated by its right-hand side as shown in Fig. 11 along with the corresponding fixed point $x^* = 0.841525$.

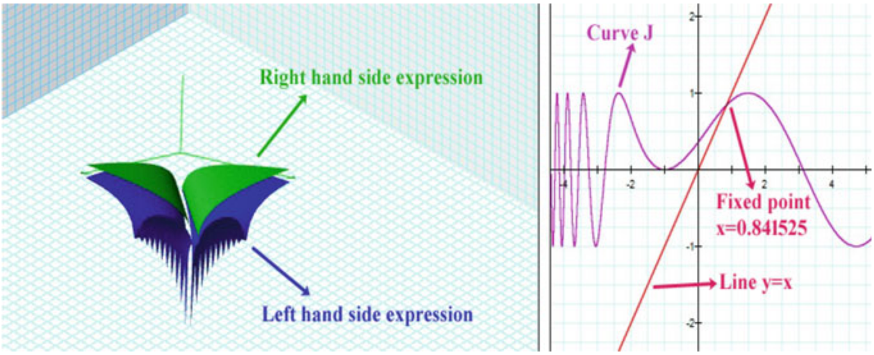

Fig. 11 Verification of inequality $\tau + F(d(Jx, Jy)) \leq F(d(x, y))$ and corresponding fixed point

Later, Secelean et al. [160] and Piri and Kumam [136] extended and refined Definition (5) by establishing some equivalent conditions over the mapping F.

Secelean et al. [160], utilized an equivalent but a more simple condition $(F2')$ instead of condition $(F2)$ as follows:
$(F2')$ $\inf F = -\infty$
or
$(F2')$ there exists a sequence α_n be a sequence of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Piri et al. [136] replaced the condition $(F3)$ by $(F3')$ in the Definition (5) as follows:
$(F3')$ F is continuous on $(0, 1)$.

Detailed description on F-contraction can be also seen in the articles [75, 78, 136, 160, 185, 188, 189] and the related references therein.

In order to prove that a fixed-point theorem is a proper generalization of the Banach contraction principle, the authors usually show the Banach contraction principle to be a direct consequence of their result and construct a map $J : Y \to Y$ to which the Banach contraction principle is not applicable, while the new one is. For more details on these counter-examples, see the proofs of [145, Theorem 1], and [43, Sect. 3].

Moreover, there is an enormous number of works that generalized the contraction on single-valued as well as multi-valued mappings or in terms of space extension. One of these generalizations was carried out recently by considering on partial contractive-type mappings in metric spaces endowed with an arbitrary binary relation, a partial order, and also with a graph.

In 2003, Kirk [105] introduced the notion of cyclic representation which are cyclic relation and cyclic contraction in metric spaces and investigated the existence and uniqueness of the fixed point for cyclical condition. Many papers considered cyclic condition for different contractions and some works introduced new class of cyclic contraction mappings and further in other spaces such as Neammanee [128] extended the concept of cyclic for single-valued to set-valued mappings, Shatanawi

[165] utilized the cyclic mapping for Ω-distance in G-metric spaces, Nashine et al. [124] presented the new formula of cyclic contractive condition for implicit relation and proved the existence and uniqueness of the fixed point for the mappings, while in 2014, Nashine [122] got some fixed-point results for cyclic contraction endowed with implicit relation. In addition, Popa [137] provided cyclic for set-valued mapping, which is more generalized than [122]. Moreover, Kumari and Panthi [107] also considered the cyclic contraction for proving fixed-point theorem in the generating spaces. Moreover, there are many works introduced the new condition of implicit function that we shall see in [25, 26, 123].

Furthermore, the notion of the pair $(\mathscr{F}, h)$ is an upper class which was introduced by Ansari and Shukla [11]. They involved this pair in a contraction condition and proved a fixed-point theorem which generalized many existing results. On the other hand, Samet et al. [156] introduced the notions of α-ψ-contractive and α-admissible mappings and proved fixed-point theorems which also unify several existing fixed point results in the setting of complete metric spaces. Many authors were inspired by the work of Samet et al. [156] and generalized many other results by using the notion of α-admissible mappings. Very recently, Karapinar et al. [93] gave a new type of rational contraction condition for set-valued mappings.

The existence of fixed-point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering has been a relatively new development in metric fixed-point theory and it was initially considered by Ran and Reurings while investigating some applications of matrix equations in [142] (see also [180]). They proved the following result:

Theorem 11 *Let $(Y, \preceq)$ be a partially ordered set such that every pair $x, y \in Y$ has an upper and lower bound and d be a metric on Y such that (Y, d) is a complete metric space. Let $f : Y \to Y$ be a continuous monotone (either order-preserving or order-reversing) mapping. Suppose that the following conditions hold:*

1. *There exists a $k \in (0, 1)$ with*

$$d(fx, fy) \leq kd(x, y) \quad for\ all \ \ x \succeq y.$$

2. *There exists an $x_0 \in Y$ with $x_0 \preceq fx_0$ or $x_0 \succeq fx_0$.*

Then f is a Picard Operator(PO), that is f has a unique fixed point $x^* \in Y$ and for each $x \in Y$, $\lim_{n\to\infty} f^n(x) = x^*$.

Subsequently, Nieto [129, 131] and others [28, 135] modified and improved Ran and Reurings results. There have been so many exciting developments in the field of the existence of fixed points in partially ordered metric spaces. For more details, we can see in the papers by Turinici [179], Nieto and López [130], Agarwal et al. [3], Ćirić et al. [42], Harjani and Sadarangani [70], Jachymski [83] Bhaskar and Lakshmikantham [28], Samet et al. [154, 155], and the references therein.

In 2012, Samet and Turinici [156] extended the concepts of metric spaces and partial ordering to define and construct fixed point theorems in metric spaces with