Grigorios Dimitriadis

Unsteady Aerodynamics

Potential and Vortex Methods

Aerospace Series
Editors Peter Belobaba, Jonathan Cooper
and Allan Seabridge

WILEY

Unsteady Aerodynamics

Aerospace Series

David Allerton. *Principles of Flight Simulation*
Dora Musielak. *Scramjet Propulsion: A Practical Introduction*
Allan Seabridge, Mohammad Radaei. *Aircraft Systems Classifications: A Handbook of Characteristics and Design Guidelines*
Douglas M. Marshall. *UAS Integration into Civil Airspace: Policy, Regulations and Strategy*
Paul G. Fahlstrom, Thomas J. Gleason, Mohammad H. Sadraey. *Introduction to UAV Systems, 5th Edition*
James W. Gregory, Tianshu Liu. *Introduction to Flight Testing*
Ashish Tewari. *Foundations of Space Dynamics*
Egbert Torenbeek. *Essentials of Supersonic Commercial Aircraft Conceptual Design*
Mohammad H. Sadraey. *Design of Unmanned Aerial Systems*
Saeed Farokhi. *Future Propulsion Systems and Energy Sources in Sustainable Aviation*
Rama K. Yedavalli. *Flight Dynamics and Control of Aero and Space Vehicles*
Allan Seabridge, Ian Moir. *Design and Development of Aircraft Systems, 3rd Edition*
Gareth D. Padfield. *Helicopter Flight Dynamics: Including a Treatment of Tiltrotor Aircraft, 3rd Edition*
Craig A. Kluever. *Space Flight Dynamics, 2nd Edition*
Trevor M. Young. *Performance of the Jet Transport Airplane: Analysis Methods, Flight Operations, and Regulations*
Andrew J. Keane, Andros Sobester, James P. Scanlan. *Small Unmanned Fixed-wing Aircraft Design: A Practical Approach*
Pascual Marques, Andrea Da Ronch. *Advanced UAV Aerodynamics, Flight Stability and Control: Novel Concepts, Theory and Applications*
Farhan A. Faruqi. *Differential Game Theory with Applications to Missiles and Autonomous Systems Guidance*
Grigorios Dimitriadis. *Introduction to Nonlinear Aeroelasticity*
Nancy J. Cooke, Leah J. Rowe, Winston Bennett Jr., DeForest Q. Joralmon. *Remotely Piloted Aircraft Systems: A Human Systems Integration Perspective*
Stephen Corda. *Introduction to Aerospace Engineering with a Flight Test Perspective*
Wayne Durham, Kenneth A. Bordignon, Roger Beck. *Aircraft Control Allocation*
Ashish Tewari. *Adaptive Aeroservoelastic Control*
Ajoy Kumar Kundu, Mark A. Price, David Riordan. *Theory and Practice of Aircraft Performance*
Peter Belobaba, Amedeo Odoni, Cynthia Barnhart, Christos Kassapoglou. *The Global Airline Industry, 2nd Edition*
Jan R. Wright, Jonathan Edward Cooper. *Introduction to Aircraft Aeroelasticity and Loads, 2nd Edition*
Tapan K. Sengupta. *Theoretical and Computational Aerodynamics*
Andros Sobester, Alexander I.J. Forrester. *Aircraft Aerodynamic Design: Geometry and Optimization*
Roy Langton. *Stability and Control of Aircraft Systems: Introduction to Classical Feedback Control*
T. W. Lee. *Aerospace Propulsion*
Ian Moir, Allan Seabridge, Malcolm Jukes. *Civil Avionics Systems, 2nd Edition*
Wayne Durham. *Aircraft Flight Dynamics and Control*
Konstantinos Zografos, Giovanni Andreatta, Amedeo Odoni. *Modelling and Managing Airport Performance*
Egbert Torenbeek. *Advanced Aircraft Design: Conceptual Design, Analysis and Optimization of Subsonic Civil Airplanes*
Christos Kassapoglou. *Design and Analysis of Composite Structures: With Applications to Aerospace Structures, 2nd Edition*
Keith A. Rigby. *Aircraft Systems Integration of Air-Launched Weapons*
Doug McLean. *Understanding Aerodynamics: Arguing from the Real Physics*
Mohammad H. Sadraey. *Aircraft Design: A Systems Engineering Approach*
G.D. McBain. *Theory of Lift: Introductory Computational Aerodynamics in MATLAB/Octave*
Plamen Angelov. *Sense and Avoid in UAS: Research and Applications*
John Valasek. *Morphing Aerospace Vehicles and Structures*
Reg Austin. *Unmanned Aircraft Systems: UAVS Design, Development and Deployment*
Visit www.wiley.com to view more titles in the Aerospace Series.

Unsteady Aerodynamics

Potential and Vortex Methods

Grigorios Dimitriadis
University of Liège
Belgium

WILEY

This edition first published 2024.
© 2024 John Wiley & Sons Ltd.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, except as permitted by law. Advice on how to obtain permission to reuse material from this title is available at http://www.wiley.com/go/permissions.

The right of Grigorios Dimitriadis to be identified as the author of this work has been asserted in accordance with law.

Registered Offices
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

For details of our global editorial offices, customer services, and more information about Wiley products visit us at www.wiley.com.

Wiley also publishes its books in a variety of electronic formats and by print-on-demand. Some content that appears in standard print versions of this book may not be available in other formats.

Trademarks: Wiley and the Wiley logo are trademarks or registered trademarks of John Wiley & Sons, Inc. and/or its affiliates in the United States and other countries and may not be used without written permission. All other trademarks are the property of their respective owners. John Wiley & Sons, Inc. is not associated with any product or vendor mentioned in this book.

Limit of Liability/Disclaimer of Warranty
While the publisher and authors have used their best efforts in preparing this work, they make no representations or warranties with respect to the accuracy or completeness of the contents of this work and specifically disclaim all warranties, including without limitation any implied warranties of merchantability or fitness for a particular purpose. No warranty may be created or extended by sales representatives, written sales materials or promotional statements for this work. This work is sold with the understanding that the publisher is not engaged in rendering professional services. The advice and strategies contained herein may not be suitable for your situation. You should consult with a specialist where appropriate. The fact that an organization, website, or product is referred to in this work as a citation and/or potential source of further information does not mean that the publisher and authors endorse the information or services the organization, website, or product may provide or recommendations it may make. Further, readers should be aware that websites listed in this work may have changed or disappeared between when this work was written and when it is read. Neither the publisher nor authors shall be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages.

Library of Congress Cataloging-in-Publication Data

Names: Dimitriadis, Grigorios, 1972- author.
Title: Unsteady aerodynamics : potential and vortex methods / Grigorios
Dimitriadis.
Description: Hoboken, NJ : Wiley, 2024. | Series: Aerospace series |
Includes index.
Identifiers: LCCN 2023030217 (print) | LCCN 2023030218 (ebook) | ISBN
9781119762478 (cloth) | ISBN 9781119762539 (adobe pdf) | ISBN
9781119762553 (epub)
Subjects: LCSH: Unsteady flow (Aerodynamics)
Classification: LCC TL574.U5 D56 2024 (print) | LCC TL574.U5 (ebook) |
DDC 629.132/3–dc23/eng/20230719
LC record available at https://lccn.loc.gov/2023030217
LC ebook record available at https://lccn.loc.gov/2023030218

Cover Design: Wiley
Cover Image: Image provided by G. Dimitriadis

Set in 9.5/12.5pt STIXTwoText by Straive, Chennai, India

Contents

Preface

The term ‘unsteady aerodynamics’ is used to denote fluid flow problems whereby either a body moves in a fluid in a time-varying fashion or the flow is time-varying in itself. The reader should also take note of the subtitle: potential and vortex methods. Although physical phenomena related to unsteady fluid motion are discussed in detail, the focus of this book is on modelling methods. Furthermore, the subtitle makes it clear that there will be no discussion in this book of what is commonly referred to as Computational Fluid Dynamics. Even though many numerical approaches will be presented, none of them will rely on discretising the entire flowfield around the body. All numerical solutions will be obtained by discretising the surface of the body and shedding vorticity in its wake, propagating the latter using a Lagrangian approach.

The book is called ‘Unsteady Aerodynamics’ and not Unsteady Fluid Dynamics because the main focus is on flow over wings. Nevertheless, many of the methodologies presented here could be applied to hydrodynamic problems, and the validation data used for some of the theories were obtained in water tunnels. Even though unsteadiness is the main subject area, several steady aerodynamic problems are presented and discussed in detail. However, the book is not an introduction to all aerodynamics or fluid dynamics in general. This means that the reader should have good background knowledge of basic aerodynamics.

The emphasis of this book is on application so that all theories are accompanied by practical examples solved by means of Matlab and C codes. These codes are available to the reader on the Wiley website; they have been tested on Matlab version 2020a but could also be compatible with other versions. The C codes are written as Matlab mex functions and should be compiled appropriately. It is the reader’s responsibility to do so, neither Wiley nor the author will provide technical support. The Mathworks website includes a list of compatible compilers for different architectures here: https://www.mathworks.com/support/requirements/supported-compilers.html. The reader should note that the purpose of the codes is to illustrate the examples and the underlying theories. They solve the particular problems for which they were written, but they should not be seen as general unsteady aerodynamic analysis codes that can be directly applied to different problems.

Chapter 1 is a brief introduction to steady and unsteady aerodynamics and provides a more detailed outline of the book. Chapter 2 presents the fundamentals of unsteady flow, focusing on the concepts and equations that will be of use throughout the book. Classical 2D unsteady potential theory is presented in Chapter 3, with a focus on analytical solutions. Chapter 4 discusses numerical 2D potential flow solutions and concentrates on interesting physical phenomena such as thrust production and propulsive efficiency. Chapter 5 introduces the aerodynamic analysis of finite wings in incompressible flows, initially by means of analytical methods and then using numerical techniques. Compressible unsteady flows are treated in Chapter 6, which addresses subsonic, supersonic and transonic aerodynamic problems. Finally, Chapter 7 introduces viscous unsteady flows featuring significant flow separation.

I would like to take this opportunity to thank my colleagues Thomas Andrianne, Adrien Crovato, Thierry Magin and Ludovic Noels who took over my teaching during my sabbatical year. A particularly warm thank you goes to Kyros Iakinthos and Pericles Panagiotou, who welcomed me into their research group at the Aristotle University of Thessaloniki during that year. I would also like to thank Adrien Crovato for spotting a negative steady drag issue in the source and doublet panel code I developed for this book, Thomas Lambert for our exchanges on the Vatistas model, Mariano Sánchez Martínez for carrying out Euler simulations on the LANN wing, Johan Boutet for our collaboration on unsteady lifting line methods and Vincent Terrapon and Boyan Mihaylov for a late-night Zoom session discussing compound rotations (among other things) during the lockdown.

August 2023

Grigorios Dimitriadis
Liège and Thessaloniki

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/dimitriadis/unsteady_aerodynamics

The website includes sample programs with MATLAB codes and solutions.

1

Introduction

Unsteady aerodynamics refers to flow of air over bodies whose velocity field is changing in time. The causes of unsteadiness can be

- Translational and rotational acceleration of the body relative to the fluid. The vast majority of this book is devoted to this type of unsteadiness.
- Upstream or free stream unsteadiness. In the atmosphere, this phenomenon is referred to as atmospheric turbulence and is caused by a variety of meteorological and geographical phenomena. In the laboratory, we encounter wind tunnel turbulence. There will be no discussion of this type of unsteadiness in this book.
- Turbulence in boundary layers, which is a ubiquitous source of unsteadiness in most practical flows. A short discussion of laminar and turbulent boundary layers is included in Chapter 7.
- Flow that separates from the surface of the body, either instantaneously or permanently. This type of flow is inherently unsteady, even when the motion of the body is steady. We will discuss this type of unsteadiness in Chapter 7.

Other sources of unsteadiness, such as acoustics, jet impingement, wake interaction, and thermal effects, lie beyond the scope of this book.

The vast majority of practical airflows will feature some degree of turbulence upstream in the boundary layer and in the wake of the body. As a consequence, nearly all aerodynamics is unsteady. However, aircraft, rotors, wind turbines and other engineering structures are generally designed to operate under attached flow conditions, so that the turbulence is confined to a thin layer of fluid in contact with the surface. Under such conditions, the effect of turbulence is averaged and therefore the flow can be treated as steady. Then, the major source of unsteadiness becomes the motion of the body itself. Conversely, civil engineering structures are mostly aerodynamically bluff bodies; even though they seldom move, they are subjected to significant unsteadiness due to separated flow and upstream turbulence.

This book deals mostly with wings and therefore the source of unsteadiness it will address most of the time is body motion. Our ancestral prototype of flight is bird flight, which involves flapping wings. Yet the first man-made flying objects were kites which in their simplest form do not flap or deform in any way. The first gliders and aircraft also had fixed wings, and flapping blades were introduced in helicopter rotors much later. From a practical point of view, it is clearly easier to work with steady aerodynamics. This is also the case

Unsteady Aerodynamics: Potential and Vortex Methods, First Edition. Grigorios Dimitriadis.

Companion website: www.wiley.com/go/dimitriadis/unsteady_aerodynamics

from a mathematical point of view; the flow equations are simpler and easier to solve. From the experimental point of view too, setting up and measuring a steady flow is more straightforward.

Even fixed-wing aircraft undergo unsteady motion, both rigid and flexible. Rigid aircraft motion is the field of study of flight dynamics; aircraft have both oscillatory and non-oscillatory rigid body eigenmodes that cannot be predicted adequately using purely steady aerodynamic analysis. We will give an example of the calculation of aerodynamic stability derivatives in Chapter 5. Furthermore, aircraft structures are flexible and are becoming increasingly so. The study of vibrating structures in an airflow is the subject area of aeroelasticity. Again, a steady or quasi-steady aerodynamic analysis is insufficient to predict aeroelastic phenomena. Chapter 3 includes one example of a direct application of unsteady aerodynamics to flutter prediction. Nevertheless, all of the methods presented in this book can be used for flight dynamic, aeroelastic or combined aeroservoelastic analysis.

1.1 Why Potential and Vortex Methods?

The equations of fluid flow are notorious for being unsolvable. The Millennium Prize (Clay Mathematics Institute, 2000) for proving the existence and smoothness of solutions to the 3D Navier–Stokes equations was still unclaimed at the time of writing of this book and the original US$1 million prize money had already depreciated to US$575,000 due to inflation. Numerical solutions of these equations are possible, but turbulence renders them impractical. In order to capture all the spatial scales of turbulence at a Reynolds number encountered in aeronautical practice, the computational requirements of a direct numerical simulation of the Navier–Stokes equations exceed the capabilities of even the fastest and biggest modern computers. Therefore, in order to model practical problems, we resort to solving easier equations. These can be averaged or filtered versions of the original Navier–Stokes relations or simpler equations that are developed after making assumptions about the physics of the flow.

The fastest solutions are obtained for potential flow equations, whereby the flow is assumed to be inviscid, irrotational and isentropic, if not incompressible. Even though a significant amount of the physics of fluid flow is discarded in order to obtain such solutions, their range of validity can include many aeronautical applications under nominal operating conditions. For example, potential flow methods are the industrial standard for aircraft aeroelastic calculations. As long as the flow remains attached to the surface, its Reynolds number is high and there are no strong shock waves, potential methods can provide fast and reliable solutions to practical engineering problems. Their main advantage is that they do not require the calculation of the solution in the entire flowfield; calculations on the surface of the body and in its wake are sufficient, and the computational cost of such solutions is very low. Even separated flows can be approximated in this manner, by shedding vortices from the separated flow region of the body into the wake.

Potential flow approaches for steady aerodynamics are presented in detail in many textbooks, notably Katz and Plotkin (2001). Gülçat (2016) discusses many potential flow methods for unsteady aerodynamics for various flow conditions, from incompressible to hypersonic. Potential flow techniques for unsteady transonic flows are also presented in Landahl (1961) and Nixon (1989). The present book focuses on application; each method is presented in detail and applied to practical, usually experimental test cases. Furthermore, the book is accompanied by computer codes in the Matlab programming environment that can be used to solve these test cases. The text and computer codes should be studied in parallel. It is hoped that this application-based approach will help the reader to develop a deeper understanding of the various methodologies.

The focus on potential and vortex methods means that the reader will not find any information in this book on what is commonly referred to as Computational Fluid Dynamics (CFD). The latter requires the numerical solution of the flow in a very wide region around the body, usually by means of finite volume, finite element or finite difference discretization. Even though most of the methods discussed in this book are numerical, they only require the discretization of the surface of the body; the wake is treated in a Lagrangian manner so that its vorticity propagates at the local flow velocity. Readers interested in unsteady CFD can consult alternative texts, such as Tucker (2014).

1.2 Outline of This Book

Chapter 2 constitutes an introduction to the mathematics of unsteady flow. The full flow equations are presented, and their simplifications using specific flow assumptions are derived. Both compressible and incompressible flow equations are presented, and their boundary conditions are discussed. Solutions to these equations are developed, and their implementation by means of Green's theorem is described in detail. The chapter finishes with a discussion of vorticity and the viscous flow equations.

Chapter 3 introduces the classical unsteady aerodynamic theories for 2D incompressible inviscid flow. The modelling of a flat plate airfoil oscillating in a flow is presented, and analytical equations for the resulting aerodynamic loads are derived. The example of impulsive airfoil motion is used in order to introduce the Wagner function while oscillating motion is used for the definition of Theodorsen's function. It is shown how general small amplitude motion can be represented using these theories and the generation of thrust or drag due to unsteady phenomena is explained. Finally, finite state theory is derived in detail.

Chapter 4 introduces numerical methods that can be used to model 2D inviscid unsteady flow with higher fidelity, for example by modelling more accurately the wake behind an oscillating airfoil or by representing the geometry of airfoils with non-negligible thickness. Three such methods are presented, all with their advantages and disadvantages. They are used in order to demonstrate the physics of the wake behind oscillating airfoils and the mechanisms of thrust generation. Comparison of the predicted aerodynamic loads to experimental results demonstrates how the higher fidelity of numerical methods can represent more of the physics of the real phenomenon. Furthermore, it is shown that all numerical panel methods can be linearized and transformed to the frequency domain in order to obtain faster aerodynamic load predictions for harmonically oscillating wings.

Chapter 5 presents unsteady aerodynamic theories for 3D finite wings. It starts with a description of finite wing geometry. Then, analytical solutions are developed, starting with an impulsively started elliptical wing. Unsteady lifting line theories are discussed before detailing two numerical panel methods, the Vortex Lattice Method and the Source and

Doublet Panel Method. These approaches are applied to several practical problems, such as the calculation of aerodynamic stability derivatives and prediction of the unsteady pressure distribution on a flexible wing.

Chapter 6 treats 3D compressible unsteady flows. Subsonic flow is treated first, with the presentation of the Doublet Lattice Method and the subsonic Source and Doublet Panel Method. Supersonic flow is modelled by means of the Mach box and Mach panel techniques. Then, unsteady transonic flows are discussed, and their modelling by means of field panel techniques is outlined. Finally, steady and unsteady corrections that allow subsonic approaches to model transonic flows are presented.

Chapter 7, which is the last chapter of this book, addresses viscous flows. It starts with a brief presentation of the boundary layer and its separation and then proceeds to discuss leading edge separation and dynamic stall. Finally, the Discrete Vortex Method is used to model highly separated flows around bluff bodies.

References

Clay Mathematics Institute (2000). Millennium problems. https://www.claymath.org/millennium-problems (accessed 14 March 2023).

Gülçat, U. (2016). *Fundamentals of Modern Unsteady Aerodynamics*, 2e. Springer.

Katz, J. and Plotkin, A. (2001). *Low Speed Aerodynamics*. Cambridge University Press.

Landahl, M.T. (1961). *Unsteady Transonic Flow*. Dover Publications, Inc.

Nixon, D. (ed.) (1989). *Unsteady Transonic Aerodynamics*, *Progress in Astronautics and Aeronautics*, vol. 120. AIAA.

Tucker, P.G. (2014). *Unsteady Computational Fluid Dynamics in Aeronautics*, *Fluid Mechanics and Its Applications*, vol. 104. Springer.

2

Unsteady Flow Fundamentals

2.1 Introduction

This chapter introduces the concepts and equations that will be used throughout the rest of the book. It is not intended as an introduction to all fluid dynamics and many results will be taken for granted. We will not introduce the continuum assumption or constitutive fluid models and we will not derive the flow equations; there are several good textbooks that do. The main focus lies in deriving the compressible and incompressible potential flow equations, discussing their boundary conditions and developing fundamental solutions for these equations.

2.2 From Navier–Stokes to Unsteady Incompressible Potential Flow

For a Newtonian fluid, the flow equations (see for example Anderson Jr. (1985) or Kuethe and Chow (1986)) are given by the continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{2.1}$$

and the momentum equations, also known as the Navier–Stokes equations,

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{2.2}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2.3}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{2.4}$$

where ρ is the fluid density, u, v, w the flow velocities in the x, y and z directions, p the pressure and μ the dynamic viscosity. The Navier–Stokes equations reduce to the Euler equations for inviscid flow by setting $\mu = 0$, such that

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x} \tag{2.5}$$

Unsteady Aerodynamics: Potential and Vortex Methods, First Edition. Grigorios Dimitriadis.

Companion website: www.wiley.com/go/dimitriadis/unsteady_aerodynamics

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho v w)}{\partial z} = -\frac{\partial p}{\partial y} \tag{2.6}$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho u w)}{\partial x} + \frac{\partial(\rho v w)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} \tag{2.7}$$

We can write the Euler equations in a different form that does not contain derivatives of the density. We first multiply the continuity equation by u to obtain

$$u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z} = 0 \tag{2.8}$$

and then we write Eq. (2.5) as

$$u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + \rho v\frac{\partial u}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

Subtracting Eq. (2.8) from this latest expression yields

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

Carrying out similar operations to Eqs. (2.6) and (2.7), we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.9}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.10}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{2.11}$$

Despite the lack of density derivatives, Eqs. (2.9)–(2.11) can still describe compressible flow if ρ is not constant; the only simplification we have imposed is to ignore viscosity.

2.2.1 Irrotational Flow

We can apply a further simplification by assuming that the flow is irrotational, that is

$$\nabla \times \mathbf{u} = \mathbf{0} \tag{2.12}$$

where $\mathbf{u} = (u, v, w)$ and

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

from which we obtain the three irrotationality relationships:

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0,\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0,\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{2.13}$$

Equation (2.9) can be rewritten as

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2 + w^2\right) - v\frac{\partial v}{\partial x} - w\frac{\partial w}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and re-arranged in the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2 + w^2\right) - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Substituting from the irrotationality relationships of Eq. (2.13), the x momentum equation simplifies to

$$\rho\frac{\partial u}{\partial t}+\frac{1}{2}\rho\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)=-\frac{\partial p}{\partial x}$$

Carrying out similar operations to Eqs. (2.10) and (2.11), we obtain the irrotational Euler equations

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.14}$$

$$\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.15}$$

$$\frac{\partial w}{\partial t}+\frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{2.16}$$

Furthermore, we define the velocity potential function $\Phi(x,y,z,t)$ such that

$$u=\frac{\partial\Phi}{\partial x},\ v=\frac{\partial\Phi}{\partial y},\ w=\frac{\partial\Phi}{\partial z} \tag{2.17}$$

Substituting these definitions into Eqs. (2.14)–(2.16) leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial x}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.18}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.19}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial z}\right)+\frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{2.20}$$

2.2.2 Laplace's and Bernoulli's Equations

We now apply the final simplification by assuming that the flow is incompressible so that the density is constant everywhere in the flowfield and at all times. The continuity Eq. (2.1) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{w}{\partial z}=0 \tag{2.21}$$

Substituting from the definition of the potential in expressions (2.17) results in

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{2.22}$$

which is known as Laplace's equation and can also be expressed as

$$\nabla^2\Phi=0 \tag{2.23}$$

where $\nabla^2=\nabla\cdot\nabla$.

The inviscid, irrotational and incompressible assumptions have simplified the continuity equation from the form (2.1) to the form (2.22), which is a linear partial differential equation with a single unknown, the potential $\Phi(x,y,z)$. The irrotational Euler equations can also be simplified using the incompressible assumption; we can multiply Eqs. (2.18)–(2.20) by their respective spatial differentials to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial x}\right)\mathrm{d}x+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)\mathrm{d}x=-\frac{1}{\rho}\frac{\partial p}{\partial x}\mathrm{d}x \tag{2.24}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial y}\right)\mathrm{d}y+\frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)\mathrm{d}y=-\frac{1}{\rho}\frac{\partial p}{\partial y}\mathrm{d}y \tag{2.25}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z}\right)\mathrm{d}z+\frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)\mathrm{d}z=-\frac{1}{\rho}\frac{\partial p}{\partial z}\mathrm{d}z \tag{2.26}$$

Since ρ is constant, exchanging the order of the time and space derivatives in the first terms of each of the equations and integrating in space results in

$$\frac{\partial \Phi}{\partial t}+\frac{1}{2}\left(u^2+v^2+w^2\right)=-\frac{p}{\rho}+\text{constant}$$

which is known as the unsteady Bernoulli equation. Note that all three momentum equations lead to the same Bernoulli expression. The constant of integration can be written as p_{ref}/ρ without loss of generality, where p_{ref} is a reference pressure. Then the unsteady Bernoulli equation becomes

$$\rho\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+\frac{\partial \Phi}{\partial t}\right)+p=p_{\text{ref}} \tag{2.27}$$

Using the definition of the potential, it can also be written as

$$\rho\left(\frac{1}{2}(\nabla\Phi)^2+\frac{\partial \Phi}{\partial t}\right)+p=p_{\text{ref}} \tag{2.28}$$

where $(\nabla\Phi)^2=\nabla\Phi\cdot\nabla\Phi$. Bernoulli's equation is valid everywhere in an incompressible, inviscid and irrotational flowfield but the reference pressure needs to be specified. As it stands, Eq. (2.27) relates conditions at any point x, y, z to a point in space where the pressure is equal to p_{ref}, the total flow speed is zero and the time derivative of the potential is also equal to zero. Bernoulli's equation can be written more intuitively by relating conditions at point x, y, z to a faraway point where the pressure is p_∞, the total flow speed is Q_∞ and the potential is constant or zero, that is

$$\rho\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+\frac{\partial \Phi}{\partial t}\right)+p=\frac{1}{2}\rho Q_\infty^2+p_\infty \tag{2.29}$$

The quantity $1/2\rho\left(u^2+v^2+w^2\right)$ is known as the dynamic pressure and $1/2\rho Q_\infty^2$ is the far-field dynamic pressure. The pressure coefficient is defined as

$$c_p=\frac{p-p_\infty}{\frac{1}{2}\rho Q_\infty^2} \tag{2.30}$$

and, using Eq. (2.29)

$$c_p(x,y,z,t)=1-\frac{\left(u^2+v^2+w^2\right)}{Q_\infty^2}-\frac{2}{Q_\infty^2}\frac{\partial \Phi}{\partial t} \tag{2.31}$$

Equations (2.23) and (2.31) will be used to solve all incompressible flow problems presented in this book, for which viscous phenomena are not important. Laplace's equation is a second-order linear partial differential equation with a single unknown, the potential $\Phi(x, y, z, t)$. Once it is solved, the potential can be substituted into Bernoulli's equation in order to calculate the pressure anywhere in the flow. Since Laplace's equation is of second order, it requires two boundary conditions.

2.2.3 Motion in an Incompressible, Inviscid, Irrotational Fluid

Consider a body in unsteady motion in still air. Points on the body's surface are denoted by vector $\mathbf{x}_s(t) = (x_s(t), y_s(t), z_s(t))$ with respect to a static origin; they are defined as the solutions of the equation $S(\mathbf{x}_s, t) = 0$. The body is rotating, translating and deforming so that the velocity of the points on the surface is denoted by $\mathbf{V}_S(\mathbf{x}_s, t)$. Consequently, if the equation of the surface is $S(\mathbf{x}_s(t), t) = 0$ at time t, at time $t + \Delta t$, it becomes

$$S(\mathbf{x}_s + \Delta\mathbf{x}, t + \Delta t) = 0$$

Expanding $S(\mathbf{x}_s + \Delta\mathbf{x}, t + \Delta t)$ as a Taylor series around $S(\mathbf{x}_s, t) = 0$ gives

$$S(\mathbf{x}_s + \Delta\mathbf{x}, t + \Delta t) = S(\mathbf{x}_s, t) + \left.\frac{\partial S}{\partial \mathbf{x}}\right|_{\mathbf{x}_s, t} \cdot \Delta\mathbf{x} + \left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} \Delta t + \cdots = 0$$

where $\partial S/\partial \mathbf{x}$ is the vector $(\partial S/\partial x, \partial S/\partial y, \partial S/\partial z) = \nabla S$. Therefore, recalling that $S(\mathbf{x}_s, t) = 0$,

$$\nabla S(\mathbf{x}_s, t) \cdot \Delta\mathbf{x} + \left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} \Delta t + \cdots = 0 \tag{2.32}$$

where the notation $\nabla S(\mathbf{x}_s, t)$ signifies 'evaluate ∇S at $\mathbf{x}_s$ and t'. Now, as the surface of the body is moving with velocity $\mathbf{V}_S(\mathbf{x}_s, t)$, we can approximate $\Delta\mathbf{x}$ as

$$\Delta\mathbf{x} = \mathbf{V}_S(\mathbf{x}_s, t)\Delta t$$

Substituting back into Eq. (2.32) yields

$$\nabla S(\mathbf{x}_s, t) \cdot \mathbf{V}_S(\mathbf{x}_s, t)\Delta t + \left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} \Delta t + O(\Delta t^2) = 0$$

or dividing throughout by Δt

$$\nabla S(\mathbf{x}_s, t) \cdot \mathbf{V}_S(\mathbf{x}_s, t) + \left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} + O(\Delta t) = 0$$

Finally, taking the limit of this latest expression as $\Delta t \to 0$,

$$\nabla S(\mathbf{x}_s, t) \cdot \mathbf{V}_S(\mathbf{x}_s, t) + \left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} = 0 \tag{2.33}$$

Equation (2.33) describes how the surface of the body deforms with respect to its own velocity $\mathbf{V}_S(\mathbf{x}_s, t)$. However, we are also interested in how the fluid deforms due to the motion of the body's surface.

We denote the coordinates of any point in the fluid by $\mathbf{x} = (x, y, z)$. Assuming that the air is still, far from the body its velocity will be zero. This is known as the far-field boundary condition, which states that the flow disturbance caused by the body decays to zero as $r \rightarrow \infty$, where

$$r = ||\mathbf{x} - \mathbf{x}_s|| = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}$$

As the flow velocity is given by $\nabla\Phi = (u, v, w)$, the far-field condition can be formulated mathematically as

$$\lim_{r\rightarrow\infty} \nabla\Phi(\mathbf{x}, t) = 0 \tag{2.34}$$

Far from the body, the potential must be constant so that its derivatives in all directions are equal to zero. This constant value of the potential may be chosen to be equal to zero.

Close to the body, the flow will be disturbed by the body's motion and therefore the flow velocity and potential will be non-zero. The objective of unsteady potential flow modelling is to calculate the flow velocities $u(\mathbf{x}_s, t)$, $v(\mathbf{x}_s, t)$, $w(\mathbf{x}_s, t)$ on the surface of the body. From these, the pressure around the body, $p(\mathbf{x}_s, t)$, can be evaluated from Eq. (2.29); the total aerodynamic force acting on the body is then given by

$$\mathbf{F} = \int_S p(\mathbf{x}_s, t)\mathbf{n}(\mathbf{x}_s, t)\mathrm{d}S \tag{2.35}$$

where $\mathbf{n}(\mathbf{x}_s, t)$ is a unit vector normal to the surface at point $\mathbf{x}_s$ and time t, while $\int_S$ denotes an integral over the entire surface of the body and $\mathrm{d}S$ is an infinitesimal element of this surface. The fundamental flow equation to be solved is Laplace's equation (2.22), which requires two boundary conditions. One of them is the far-field condition but we still need to define the second. Assuming that the surface of the body is impermeable, the layer of fluid in contact with the surface will also obey $S(\mathbf{x}_s, t) = 0$ and, hence, Eq. (2.32). The velocity of the fluid on the surface is given by $\nabla\Phi(\mathbf{x}_s, t)$ so that for the fluid,

$$\Delta\mathbf{x} = \nabla\Phi(\mathbf{x}_s, t)\Delta t$$

Substituting in Eq. (2.32), dividing by Δt and taking the limit as $\Delta t \rightarrow 0$ leads to

$$\nabla S(\mathbf{x}_s, t) \cdot \nabla\Phi(\mathbf{x}_s, t) + \left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} = 0$$

Furthermore, dividing throughout by $||\nabla S(\mathbf{x}_s, t)||$ gives

$$\frac{\nabla S(\mathbf{x}_s, t)}{||\nabla S(\mathbf{x}_s, t)||} \cdot \nabla\Phi(\mathbf{x}_s, t) + \frac{1}{||\nabla S(\mathbf{x}_s, t)||}\left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} = 0$$

The quantity $\nabla S(\mathbf{x}_s, t)/||\nabla S(\mathbf{x}_s, t)||$ is in fact the unit vector normal to the surface $\mathbf{n}(\mathbf{x}_s, t)$, so that

$$\mathbf{n}(\mathbf{x}_s, t) \cdot \nabla\Phi(\mathbf{x}_s, t) + \frac{1}{||\nabla S(\mathbf{x}_s, t)||}\left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}_s, t} = 0 \tag{2.36}$$

Finally, we solve Eq. (2.33) for $\partial S/\partial t|_{\mathbf{x}_s, t}$ and substitute the result into Eq. (2.36) to obtain

$$\left(\nabla\Phi(\mathbf{x}_s, t) - \mathbf{V}_S(\mathbf{x}_s, t)\right) \cdot \mathbf{n}(\mathbf{x}_s, t) = 0 \tag{2.37}$$

Equation (2.37) is known as the impermeability boundary condition, or zero normal flow condition. It states that the relative velocity between the fluid and the surface in a direction normal to the surface must be equal to zero so that no flow can cross the solid boundary.