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Advances in Ring Theory and Applications

WARA22, Messina, Italy, July 18–20, 2022

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Editors

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Organization

WARA22 is organized by the Department of Engineering at University of Messina (Messina, Italy) in cooperation with the Department of Mathematics at Aligarh Muslim University (Aligarh, India), Department of Mathematics at EGE University (Izmir, Turkey), and Department of Mathematics at Chuzhou University (Chuzhou, China).

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Preface

This volume contains the proceedings of the *Workshop on Associative Rings and Algebras* (WARA22) which was held in hybrid mode, that is, both in-person (at University of Messina, Italy) and virtually (on Zoom platform) from July 18 to July 20, 2022. The purpose of the workshop was to present the state of art both in the theory of Lie structures of associative rings and algebras and in the theory of functional identities in rings. The main topics covered referred to rings with involution, Lie and Jordan structures, rings and algebras arising under various constructions, modules, bimodules and ideals in associative algebras, behavior of derivations, automorphisms, and other kinds of additive maps in prime and semiprime rings. The conference was sponsored by University of Messina (Italy), Aligarh Muslim University (India), EGE University (Turkey), and Chuzhou University (China). The main aim of the Workshop was to facilitate the exchange of research ideas and to contribute to mutual communications and collaborations among ring theorists and experts in functional identities theory. A total of 10 invited talks on current topics of algebra and its applications were delivered by distinguished algebraists. The speakers included Profs. Asma Ali (Aligarh Muslim University, India), Mohammad Ashraf (Aligarh Muslim University, India), Luisa Carini (University of Messina, Italy), Çagri Demir (EGE University, Turkey), Münevver Pı́nar Eroğlu (Dokuz Eylul University, Turkey), Alberto Facchini (University of Padova, Italy), Tsiu-Kwen Lee and Jheng-Huei Lin (National Taiwan University, Taipei, Taiwan), Giovanni Scudo (University of Messina, Italy), Faiza Shujat (Taibah University, Saudi Arabia), and Feng Wei (Beijing Institute of Technology, China).

The workshop was particularly devoted to young researchers in order to give them the opportunity to have a correct approach to new research developments and ideas within ring theory, both by attending conferences presented and through the interaction with the invited speakers. More than 100 researchers from all over the world participated in the meetings, thus having the possibility of exchanging new ideas, discussing open problems already known in the literature, proposing new ones, as well as laying the foundations for future research collaborations in the field of algebra in general and with particular regard to the applications of the ring theory in other areas such as, for example, Physics (Lie-admissible algebras), Differential geometry

(Poisson algebras), Mechanics (maps covariant under the action of Lie algebras), Calculus (operator algebras and Banach spaces), and Informatics (cryptography and coding theory). Each of the conference sessions was characterized by a subsequent lively debate on the topics covered. Although it was not the initial intentions of the organizers, it was precisely from these open discussions that the idea of proposing a volume that collected a series of new results was born. This book is then the outcome not only of some invited lectures presented at the conference, but also of research papers by invited algebraists who could not attend the conference. All papers of the volume are peer-reviewed by anonymous experts. To this regard, we would like to thank all those who have contributed to this volume with their papers and those who have kindly and friendly accepted to serve as referees of the submitted papers. We also express our thanks to Springer for giving us the opportunity to publish this volume. Finally, let us thank Dr. Banu Dhayalan (Project Coordinator, Springer), Dr. Francesca Ferrari (Assistant Editor, Springer), and Dr. Francesca Bonadei (Executive Editor, Springer), without whose active assistance and help this project would not have been completed.

Messina, Italy Vincenzo De Filippis
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November 2023

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A Note on Multiplicative (Generalized)-Derivations and Left Sided Ideals in Semiprime Rings

Gurninder S. Sandhu, Basudeb Dhara, and Sourav Ghosh

Abstract Let R be a 2-torsion free semiprime ring with center $Z(R)$ and λ a nonzero left sided ideal of R. Let $F, G : R \to R$ be multiplicative (generalized)-derivations associated with the map $d : R \to R$ (not necessarily additive nor derivation) and $H, T : R \to R$ be any two maps. The main goal of this article is to study identities:

(1) $(d(x)F(y) \pm G(y)d(x)) \pm (H(x)y + yT(x)) = 0$ for all $x, y \in \lambda$;
(2) $(d(x)F(y) \pm G(y)d(x)) \pm (xy \pm yx) \in Z(R)$ for all $x, y \in \lambda$.

Keywords Prime rings · Semiprime rings · Multiplicative (generalized)-derivation

1 Introduction

Let R be an associative ring. Then R is called prime (resp. semiprime) if for any $a, b \in R$, $aRb = (0)$ (resp. $aRa = (0)$) implies $a = 0$ or $b = 0$ (resp. $a = 0$). Recall that an additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In case d is not necessarily additive, then d is called multiplicative derivation of R. If $F : R \to R$ is an additive mapping and d is a derivation of R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, then F is called a generalized derivation of R.

Like multiplicative derivation, if we drop additivity restriction of F, then F is called multiplicative generalized derivation of R. More precisely a mapping F :

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$R \to R$ (not necessarily additive) is called a multiplicative generalized derivation of R, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

It is natural to consider a pair of maps (F, d) which satisfies $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where F (not necessarily additive) and d (not necessarily derivation) are any two maps. In [10], Dhara and Ali introduced the notion of such type of mapping which is called as multiplicative (generalized)-derivation. A mapping $F : R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-derivation, if $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$, where g is any mapping (not necessarily a derivation nor an additive map).

Evidently, these mappings extend the concept of derivations, multiplicative derivations, generalized derivations as well as multiplicative generalized derivations.

After Posner's paper [18], much attention has been devoted to investigate commutative structure (or commutativity) of a ring by imposing polynomial constraints involving derivations and generalized derivations on suitable subsets of it (see [2–5, 13–15, 19, 20] and references therein). In [2], Ali and Huang proved that if R is a 2-torsion free semiprime ring, d a derivation of R and I an ideal of R such that any one of the following holds: (i) $[d(x), d(y)] = \pm[x, y]$ for all $x, y \in I$; (ii) $d(x) \circ d(y) = \pm(x \circ y)$ for all $x, y \in I$, then d is commuting on I, i.e., $[d(x), x] = 0$ for all $x \in I$.

Ashraf et al. [3] studied the commutativity of a prime ring R admitting a generalized derivation F associated with a nonzero derivation d satisfying any one of the following conditions: (i) $[d(x), F(y)] = \pm[x, y]$, (ii) $d(x) \circ F(y) = \pm(x \circ y)$ for all $x, y \in I$, where I is a nonzero ideal of R.

Dhara et al. [9] extended above results by considering R a 2-torsion free semiprime rings and concluded that R contains a nonzero central ideal.

There are ongoing interest to investigate the identities replacing generalized derivation with multiplicative (generalized)-derivation, because multiplicative (generalized)-derivation is a generalization of derivation as well as generalized derivation. Thus multiplicative (generalized)-derivations are the large number of maps which satisfy the above identities. Recently, few papers have investigated identities involving multiplicative (generalized)-derivations in prime and semiprime rings (see [1, 6–8, 10–12, 16, 17, 21, 22]). In [16], Khan studied identities (i) $[d(x), F(y)] \pm [x, y] = 0$ and (ii) $d(x) \circ F(y) \pm (x \circ y) = 0$ for all x, y in some suitable subsets of R, where F is a multiplicative (generalized)-derivation with associated map d.

In [16, Theorem 3.1 and Theorem 3.4], author proved that: *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $d : R \to R$. If $d(x) \circ F(y) \pm (x \circ y) = 0$ for all $x, y \in I$, or $[d(x), F(y)] \pm [x, y] = 0$ for all $x, y \in I$, then* $[[x, d(x)], d(x)] = 0$ for all $x \in I$.

Recently, Dhara et al. [7], improved the second result, that is, $[d(x), F(y)] \pm [x, y] = 0$ for all $x, y \in \lambda$, where λ is a nonzero left ideal of a 2-torsion free semiprime ring R, and obtained that $\lambda[d(\lambda), \lambda] = (0)$. But question is "can we conclude the same conclusion when $d(x) \circ F(y) \pm (x \circ y) = 0$ for all $x, y \in \lambda$?"

In the present article our motivation is to answer this question by considering more generalized situation $(d(x)F(y) \pm G(y)d(x)) \pm (H(x)y + yT(x)) = 0$ for all $x, y \in \lambda$ and obtained the same conclusion $\lambda[d(\lambda), \lambda] = (0)$, where $F, G : R \to R$ are two multiplicative (generalized)-derivations associated with the map $d : R \to R$ and $H, T : R \to R$ are any two maps and λ a nonzero left sided ideal of semiprime ring R.

2 Preliminaries and Auxiliary Lemmas

In the sequel, we shall make extensive use of the basic (anti-)commutator identities and some known facts of the subject, which are stated as follows:

(i) $[xy, w] = x[y, w] + [x, w]y$;
(ii) $[x, yw] = y[x, w] + [x, y]w$;
(iii) $(xy \circ w) = (x \circ w)y + x[y, w] = x(y \circ w) - [x, w]y$;
(iv) $(x \circ yw) = [x, y]w + y(x \circ w) = (x \circ y)w - y[x, w]$.

Lemma 1 *Let R be a semiprime ring. If $F : R \to R$ is a multiplicative (generalized)-derivation of R associated to the map d of R, then d must be multiplicative derivation.*

Proof For any $x, y, z \in R$,

$$F(xyz) = F((xy)z) = F(xy)z + xyd(z) = F(x)yz + xd(y)z + xyd(z) \tag{1}$$

and

$$F(xyz) = F(x(yz)) = F(x)yz + xd(yz). \tag{2}$$

By (1) and (2),

$$x\{d(yz) - d(y)z - yd(z)\} = 0.$$

Since R is semiprime ring, it yields $d(yz) - d(y)z - yd(z) = 0$ implying $d(yz) = d(y)z + yd(z)$ for all $y, z \in R$. Therefore, d is a multiplicative derivation.

Lemma 2 *Let R be a ring with center $Z(R)$ and d be a multiplicative derivation of R. Then $d(Z(R)) \subseteq Z(R)$.*

Proof For any $x \in R$ and $z \in Z(R)$, $xz = zx$ and hence

$$0 = d(xz) - d(zx) = d(x)z + xd(z) - d(z)x - zd(x) = xd(z) - d(z)x = [x, d(z)].$$

This implies $d(z) \in Z(R)$ for any $z \in Z(R)$, as desired.

Lemma 3 *Let R be a 2-torsion free semiprime ring and λ is a nonzero left ideal of R. If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in \lambda$, then $axb = 0 = bxa$ for all $x \in \lambda$.*

Proof Let $x, y \in \lambda$. By using the fact $axb = -bxa$ for all $x \in \lambda$, we have

$$(axb)y(axb) = -(bxa)y(axb) = -\{b(xay)a\}xb = -\{-a(xay)b\}xb$$

$$= a(xaybx)b = -b(xaybx)a = -(bxa)y(bxa) = -(axb)y(axb).$$

Thus $2(axb)y(axb) = 0$. Since R is 2-torsion free semiprime ring, $\lambda(a\lambda b) = (0)$.

Since R is semiprime ring, it must contain a family of prime ideals $\Omega = \{P_\alpha | \alpha \in \Lambda\}$ such that $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$. Let P_α be any member of Ω. Then $\lambda(a\lambda b) = (0)$ implies either $\lambda a \subseteq P_\alpha$ or $\lambda b \subseteq P_\alpha$, so that $b\lambda a \subseteq P_\alpha$ or $a\lambda b \subseteq P_\alpha$. By hypothesis, $b\lambda a \subseteq P_\alpha$ implies that $a\lambda b \subseteq P_\alpha$ and hence $a\lambda b \subseteq P_\alpha$ for any $P_\alpha \in \Omega$. Thus $a\lambda b \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = 0$, that is, $a\lambda b = (0)$.

Lemma 4 *Let R be a prime ring, I a nonzero ideal of R and d a multiplicative derivation of R. If $[d(I), I] = (0)$, then $d : R \to Z(R)$.*

Proof For any $x, y, u \in I$, $r, t \in R$,

$$0 = [d(x), ry] = r[d(x), y] + [d(x), r]y = [d(x), r]y.$$

By primeness of R, $[d(x), r] = 0$ implying $d(I) \subseteq Z(R)$.

Again,

$$0 = [d(xt), r] = [d(x)t + xd(t), r] = d(x)[t, r] + [xd(t), r]. \tag{3}$$

Substituting $x = rx$ and then using (3), we get $d(r)x[t, r] = 0$ for all $x \in I, t, r \in R$. By primeness of R, for each $r \in R$, either $d(r) = 0$ or $r \in Z(R)$. Since $r \in Z(R)$ implies $d(r) \in Z(R)$, therefore in any case, we can conclude that $d(r) \in Z(R)$ for all $r \in R$. Hence conclusion follows.

3 Main Results

Theorem 1 *Let R be a 2-torsion free semiprime ring and λ be a nonzero left ideal of R. If R admits multiplicative (generalized)-derivations $F, G : R \to R$ associated with the same map $d : R \to R$ and any two maps $H, T : R \to R$ such that $(d(x)F(y) \pm G(y)d(x)) \pm (H(x)y + yT(x)) = 0$ for all $x, y \in \lambda$, then $\lambda[d(\lambda), \lambda] = (0)$.*

Proof We assume that

$$(d(x)F(y) \pm G(y)d(x)) \pm (H(x)y + yT(x)) = 0, \quad \forall\, x, y \in \lambda. \tag{4}$$

Substituting yt for y in (4), where $t \in \lambda$, we get

$$d(x)\{F(y)t + yd(t)\} \pm \{G(y)t + yd(t)\}d(x) \pm (H(x)yt + ytT(x)) = 0, \ \ \forall\, x, y, t \in \lambda. \tag{5}$$

By (4), above relation yields

$$d(x)yd(t) \pm G(y)[t, d(x)] + yd(t)d(x) \pm y[t, T(x)] = 0, \ \ \forall\, x, y, t \in \lambda. \tag{6}$$

Writing uy in place of y in (4), where $u \in \lambda$, we obtain

$$d(x)uyd(t) \pm G(uy)[t, d(x)] + uyd(t)d(x) \pm uy[t, T(x)] = 0, \ \ \forall\, x, y, t \in \lambda. \tag{7}$$

Pre-multiplying (6) by u and then subtracting from (7), one can see that

$$[d(x), u]yd(t) \pm (G(uy) - uG(y))[t, d(x)] = 0, \ \ \forall\, x, y, t, u \in \lambda. \tag{8}$$

Replacing t by tw in (8), where $w \in \lambda$ and then using it, we observe that

$$\begin{aligned}[][d(x), u]y(d(t)w + td(w)) \pm (G(uy) - uG(y))[t, d(x)]w \\ \pm (G(uy) - uG(y))t[w, d(x)] = 0,\end{aligned} \tag{9}$$

for all $x, y, t, w, u \in \lambda$. By using (8), (9) gives

$$[d(x), u]ytd(w) \pm (G(uy) - uG(y))t[w, d(x)] = 0, \ \ \forall\, x, y, t, w, u \in \lambda. \tag{10}$$

Put $t = [p, d(v)]t$ in (10) (since $[p, d(v)]t \in \lambda$), we get

$$\pm(G(uy) - uG(y))[p, d(v)]t[w, d(x)] + [d(x), u]y[p, d(v)]td(w) = 0, \tag{11}$$

for all $x, y, t, w, u, p, v \in \lambda$. By using (8) we get

$$-[d(v), u]yd(p)t[d(x), w] + [d(x), u]y[d(v), p]td(w) = 0, \ \ \forall\, x, y, t, w, u, p, v \in \lambda. \tag{12}$$

Since $d(v)u \in \lambda$, we put $u = d(v)u$ in (12) and get

$$[d(x), d(v)]uy[d(v), p]td(w) = 0, \ \ \forall\, x, y, t, w, u, p, v \in \lambda. \tag{13}$$

Put $x = xv$ in (13) and then using (13), we get

$$\Big(d(x)[v, d(v)] + [x, d(v)]d(v)\Big)uy[d(v), p]td(w) = 0, \ \ \forall\, x, y, t, w, u, p, v \in \lambda. \tag{14}$$

Let $q \in \lambda$. Put $x = qx$ in (14) and then using (14), we obtain

$$\Big(d(q)x[v,d(v)]+[q,d(v)]xd(v)\Big)uy[d(v),p]td(w)=0,\ \ \forall\, x,y,t,w,u,p,v,q\in\lambda. \tag{15}$$

For $q=v$, $p=v$, $w=v$, $y=[v,d(v)]y$ in (15)

$$\Big(d(v)x[v,d(v)]+[v,d(v)]xd(v)\Big)u[v,d(v)]y[v,d(v)]td(v)=0,\ \ \forall\, x,y,t,u,v\in\lambda. \tag{16}$$

This gives

$$\Big(d(v)x[v,d(v)]+[v,d(v)]xd(v)\Big)(\lambda[v,d(v)])^3=(0),\ \ \forall\, x,v\in\lambda. \tag{17}$$

that is

$$d(v)x[v,d(v)]b+[v,d(v)]xd(v)b=(0), \tag{18}$$

where $b=(\lambda[v,d(v)])^3$. Put $x=bx$ in (18) and then we have

$$(d(v)b)x([v,d(v)]b)+([v,d(v)]b)x(d(v)b)=(0) \tag{19}$$

By Lemma 3

$$d(v)bx[v,d(v)]b=(0). \tag{20}$$

This gives

$$d(v)\lambda[v,d(v)]^3x[v,d(v)](\lambda[v,d(v)])^3=(0), \tag{21}$$

that is

$$(\lambda[v,d(v)])^8=(0). \tag{22}$$

Since a semiprime ring contains no nonzero nilpotent left ideals, it follows that

$$\lambda[v,d(v)]=(0) \tag{23}$$

Corollary 1 *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. If R admits multiplicative (generalized)-derivations $F, G : R \to R$ associated with the same map $d : R \to R$ and any two maps $H, T : R \to R$ such that $(d(x)F(y) \pm G(y)d(x)) \pm (H(x)y + yT(x)) = 0$ for all $x, y \in I$, then $[d(I), I] = (0)$.*

Proof By Theorem 1, $I[d(I), I] = (0)$. Since R is semiprime, $[d(I), I] \subseteq I \cap ann(I) = 0$.

Corollary 2 *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R, $F : R \to R$ multiplicative (generalized)-derivation associated with the map $d : R \to R$ and $H : R \to R$ any map. If any one of the following holds:*

(i) $[d(x), F(y)] \pm [H(x), y] = 0$ *for all* $x, y \in \lambda$,
(ii) $d(x) \circ F(y) \pm H(x) \circ y = 0$ *for all* $x, y \in \lambda$,
(iii) $[d(x), F(y)] \pm [x, y] = 0$ *for all* $x, y \in \lambda$,

(iv) $d(x) \circ F(y) \pm (x \circ y) = 0$ *for all* $x, y \in \lambda$,

then $\lambda[d(\lambda), \lambda] = (0)$.

Corollary 3 *Let R be a prime ring of char $(R) \neq 2$ and I a nonzero ideal of R. If R admits multiplicative (generalized)-derivations $F, G : R \to R$ associated with the same map $d : R \to R$ and any maps $H, T : R \to R$ such that $(d(x)F(y) \pm G(y)d(x)) \pm (H(x)y + yT(x)) = 0$ for all $x, y \in I$, then d maps R into its center.*

Proof By Corollary 1, $[d(I), I] = (0)$. Then again by Lemmas 1 and 4, conclusion follows.

Corollary 4 *Let R be a prime ring of char $(R) \neq 2$ and I a nonzero ideal of R. If R admits a multiplicative (generalized)-derivation $F : R \to R$ associated with the map $d : R \to R$ such that $[d(x), F(y)] \pm [x, y] = 0$ for all $x, y \in I$, then R must be commutative.*

Proof Assuming $G = F$, $H = Id$ (identity map) and $T = -H$ in Corollary 3, we conclude by Corollary 3 that $d(R) \subseteq Z(R)$ and hence by hypothesis $[I, I] = (0)$. This implies that R is commutative.

Corollary 5 *Let R be a prime ring of char $(R) \neq 2$ and I a nonzero ideal of R. If R admits a multiplicative (generalized)-derivation $F : R \to R$ associated with the map $d : R \to R$ such that $d(x) \circ F(y) \pm x \circ y = 0$ for all $x, y \in I$, then R is commutative and $F(xy) = F(x)y$ for all $x, y \in R$.*

Proof By Corollary 3, $d(R) \subseteq Z(R)$ and hence by hypothesis $2d(x)F(y) \pm x \circ y = 0$ for all $x, y \in I$. Replacing y with yz, we obtain $2d(x)(F(y)z + yd(z)) \pm \{(x \circ y)z - y[x, z]\} = 0$ for all $x, y, z \in I$. By using hypothesis, this relation yields $2d(x)yd(z) \mp y[x, z] = 0$ for all $x, y, z \in I$. Now replacing y with yr, where $r \in R$, in $2d(x)yd(z) \mp y[x, z] = 0$ and using this relation, we get $y[r, [x, z]] = 0$ for all $x, y, z \in I$ and $r \in R$. Primeness of R, implies $[R, [I, I]] = (0)$. This gives that R is commutative. Then from above relation, we have $d(x)yd(z) = 0$ for all $x, y, z \in I$ which gives $d(I) = (0)$. Now $d(I) = (0)$ implies $d(R) = (0)$. Hence $F(xy) = F(x)y$ for all $x, y \in R$.

Theorem 2 *Let R be a 2-torsion free semiprime ring and λ be a nonzero left ideal of R. Let R admits multiplicative (generalized)-derivations $F, G : R \to R$ associated with the same map $d : R \to R$ such that $(d(x)F(y) \pm G(y)d(x)) \pm (xy \pm yx) \in Z(R)$ for all $x, y \in \lambda$. If $d(Z(R)) \neq (0)$, then $\lambda[d(\lambda), \lambda]d(Z(R)) = (0)$ and $\lambda[\lambda, \lambda]d(Z(R)) = (0)$.*

Proof By hypothesis,

$$(d(x)F(y) \pm G(y)d(x)) \pm (xy \pm yx) \in Z(R), \quad \forall\, x, y \in \lambda. \tag{24}$$

Since $Z(R) \neq (0)$, we choose $0 \neq z \in Z(R)$. Since $yz = zy \in \lambda$, taking yz instead of y in our initial hypothesis (24), we observe

$$d(x)(F(y)z + yd(z)) \pm (G(y)z + yd(z))d(x) \pm (xy \pm yx)z \in Z(R), \ \ \forall\, x, y \in \lambda. \tag{25}$$

By using (24) and the fact $d(z) \in Z(R)$, it yields

$$(d(x)y \pm yd(x))d(z) \in Z(R), \ \ \forall\, x, y \in \lambda.$$

It can be re-written as $[(d(x)y \pm yd(x))d(z), r] = 0$ for all $x, y \in \lambda$ and $r \in R$. Since $d(z) \in Z(R)$, it follows that

$$[d(x)y \pm yd(x), t]d(z) = 0, \ \ \forall\, x, y, t \in \lambda. \tag{26}$$

Substituting yt in place of y in (26) and using it, we find

$$[y[d(x), t], t]d(z) = 0, \ \ \forall\, x, y, t \in \lambda. \tag{27}$$

Replacing y by $d(x)y$ in (27), we get $[d(x), t]y[d(x), t]d(z) = 0$ for all $x, y, t \in \lambda$. This implies $(\lambda[d(x), t]d(z))^2 = (0)$ for all $x, t \in \lambda$. It forces

$$\lambda[d(x), t]d(z) = (0), \ \ \forall\, x, t \in \lambda. \tag{28}$$

Writing xu in place of x in (28) and using it, we can easily write after simple calculation that

$$yd(x)[u, t]d(z) + y[x, t]d(u)d(z) = 0, \ \ \forall\, x, y, t, u \in \lambda. \tag{29}$$

Replacing u by uw in (29), to get

$$yd(x)u[w, t]d(z) + y[x, t]ud(w)d(z) = 0, \ \ \forall\, x, t, u, w \in \lambda. \tag{30}$$

Returning to (28), we may look at it as

$$yd(w)ud(z) - yud(w)d(z) = 0, \ \ \forall\, u, w \in \lambda.$$

Writing $y[x, t]$ for y in the last expression, we obtain

$$y[x, t]d(w)ud(z) - y[x, t]ud(w)d(z) = 0, \ \ \forall\, x, t, u, w \in \lambda. \tag{31}$$

Comparing (30) and (31), we have

$$yd(x)u[w, t]d(z) + y[x, t]d(w)ud(z) = 0, \ \ \forall\, x, t, u, w \in \lambda. \tag{32}$$

Substituting uv in place of u in (32) and using it in order to get

$$yd(x)u[[w, t], v]d(z) = 0, \ \ \forall\, x, t, u, w, v \in \lambda. \tag{33}$$

Replacing x by zx in (33), we find

$$yxd(z)u[[w,t],v]d(z) = 0, \quad \forall\, x,t,u,w,v \in \lambda.$$

Taking $[[w,t],v]$ instead of x in the above relation, we find $(\lambda[[w,t],v]d(z))^2 = (0)$ for all $w,t,v \in \lambda$. It forces $x[[w,t],v]d(z) = 0$ for all $x,w,t,v \in \lambda$. Replacing w by wt in the last expression, we find $x[w,t][t,v]d(z) = 0$ for all $w,t,v \in \lambda$. Substituting yv instead of v, we get $x[w,t]y[t,v]d(z) = 0$ for all $x,y,w,t,v \in \lambda$. It yields $(\lambda[w,t]d(z))^2 = (0)$, and it is implying that $\lambda[\lambda,\lambda]d(Z(R)) = (0)$. Thus the proof is completed.

Corollary 6 *Let R be a prime ring of char $(R) \neq 2$ and λ be a nonzero left ideal of R. If R admits multiplicative (generalized)-derivations F and G associated with the same map d such that $(d(x)F(y) \pm G(y)d(x)) \pm (xy \pm yx) \in Z(R)$ for all $x,y \in \lambda$ and $d(Z(R)) \neq (0)$, then $\lambda[d(\lambda),\lambda] = (0)$ and $\lambda[\lambda,\lambda] = (0)$.*

Proof We know that for any prime ring R and $0 \neq z \in Z(R)$, $rz = 0$ implies $r = 0$. Therefore, by Theorem 2, $\lambda[d(\lambda),\lambda] = (0)$ and $\lambda[\lambda,\lambda] = (0)$.

Corollary 7 *Let R be a prime ring with char $(R) \neq 2$ and I be a nonzero ideal of R. If R admits multiplicative (generalized)-derivations F and G associated with the same map d such that $(d(x)F(y) \pm G(y)d(x)) \pm (xy \pm yx) \in Z(R)$ for all $x,y \in I$ and $d(Z(R)) \neq (0)$, then R is commutative.*

Proof In light of Corollary 6, we have $[d(I),I] = (0)$ for all $y,t \in I$. By Lemma 4, $d(R) \subseteq Z(R)$. In view of our initial hypothesis

$$[d(x)\overline{F}(y) \pm (xy \pm yx), t] = 0, \quad \forall\, x,y,t \in I, \tag{34}$$

where $\overline{F} = F \pm G$. Note that $\overline{F}$ is a multiplicative (generalized)-derivation of R associated with the map μd, where $\mu = 2$ or $\mu = 0$. Writing yt in place of y in (34), we have

$$[d(x)\overline{F}(y)t + \mu d(x)yd(t) \pm \{(xy \pm yx)t \mp y[x,t]\}, t] = 0, \quad \forall\, x,y,t \in I, \tag{35}$$

By (34), above relation yields

$$[\mu d(x)yd(t) \mp y[x,t], t] = 0, \quad \forall\, x,y,t \in I.$$

This implies

$$\mu[y,t]d(x)d(t) \mp [y[x,t],t] = 0, \quad \forall\, x,y,t \in I. \tag{36}$$

Replacing y by ry in (36) and then using it, we get

$$\mu[r,t]yd(x)d(t) \mp [r,t]y[x,t] = 0, \quad \forall\, x,y,t \in I,\ r \in R. \tag{37}$$

Substituting yw for y in (37), we obtain

$$\mu[r,t]ywd(x)d(t) \mp [r,t]yw[x,t] = 0, \quad \forall\, x,y,t \in I,\ r \in R. \tag{38}$$

Right multiplying by w in (37) and then subtracting from (38), we obtain $[r,t]y[w,[x,t]] = 0$ that is $[w,[r,t]]y[w,[x,t]] = 0$ for all $x,y,t,w \in I$ and $r \in R$. Primeness of R forces that $[w,[x,t]] = 0$ for all $x,w,t \in I$ which assures commutativity of R.

Corollary 8 *Let R be a prime ring of char $(R) \neq 2$, I a nonzero ideal of R and $F : R \to R$ multiplicative (generalized)-derivation associated with the map $d : R \to R$ such that $d(Z(R)) \neq (0)$. If any one of the following holds:*
(i) $[d(x),F(y)] \pm [x,y] \in Z(R)$ for all $x,y \in I$;
(ii) $d(x) \circ F(y) \pm x \circ y \in Z(R)$ for all $x,y \in I$,
then R is commutative.

Example 3.1 Let Z be the set of all integers and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b,c \in Z \right\}$. Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, R is not prime. Let $I = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| x,y \in Z \right\}$ be an ideal of R.

Define the mappings $F,d : R \longrightarrow R$ by $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c^2 \\ 0 & 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & bc \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$.

We notice that F and d are not additive maps and so F can not be a generalized derivation and d can not be a derivation. It is easy to verify that F satisfies $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. Therefore, F is a multiplicative (generalized)-derivations associated with the map d. We see that $[d(x),F(y)] \pm [x,y] = 0$ and $d(x) \circ F(y) \pm x \circ y = 0$ for all $x,y \in I$. Since R is noncommutative, the primeness hypothesis is not superfluous in Corollaries 4 and 5.

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References

1. Ali, A., Dhara, B., Khan, S., Ali, F.: Multiplicative (generalized)-derivations and left ideals in semiprime rings. Hacettepe J. Math. Stat. **44**(6), 1293–1306 (2015). https://doi.org/10.15672/HJMS.2015449679
2. Ali, S., Huang, S.: On derivations in semiprime rings. Algebr. Represent. Theor. **15**, 1023–1033 (2012). https://doi.org/10.1007/s10468-011-9271-9
3. Ashraf, M., Ali, A., Rani, R.: On generalized derivations of prime rings. Southeast Asian Bull. Math. **29**(4), 669–675 (2005)
4. Ashraf, M., Ali, A., Ali, S.: Some commutativity theorems for rings with generalized derivations. Southeast Asian Bull. Math. **31**, 415–421 (2007)
5. Bell, H.E., Daif, M.N.: On derivations and commutativity of prime rings. Acta Math. Hungar. **66**(4), 337–343 (1995). https://doi.org/10.1007/BF01876049
6. Camci, D.K., Aydin, N.: On multiplicative (generalized)-derivations in semiprime rings. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **66**(1), 153–164 (2017). https://doi.org/10.1501/Commua1_0000000784
7. Dhara, B., Kar, S., Kuila, S.: A note on multiplicative (generalized)-derivations and left ideals in semiprime rings. Rend. Circ. Mat. Palermo Ser. II **70**(2), 631–640 (2021). https://doi.org/10.1007/s12215-020-00515-4
8. Dhara, B., Mozumder, M.R.: Some identities involving multiplicative generalized derivations in prime and semiprime rings. Bol. Soc. Paran. Mat. **36**(1), 25–36 (2018). https://doi.org/10.5269/bspm.v36i1.30822
9. Dhara, B., Ali, S., Pattanayak, A.: Identities with generalized derivations in semiprime rings. Demonstratio Math. **46**(3), 453–460 (2013). https://doi.org/10.1515/dema-2013-0471
10. Dhara, B., Ali, S.: On multiplicative (generalized)-derivation in prime and semiprime rings. Aequat. Math. **86**, 65–79 (2013). https://doi.org/10.1007/s00010-013-0205-y
11. Dhara, B., Pradhan, K.G.: A note on multiplicative (generalized) derivations with annihilator conditions. Georgian Math. J. **23**(2), 191–198 (2016). https://doi.org/10.1515/gmj-2016-0020
12. Gölbaşi, Ö.: Multiplicative generalized derivations on ideals of semiprime rings. Math. Slovaca **66**(6), 1285–1296 (2016). https://doi.org/10.1515/ms-2016-0223
13. Huang, S.: Generalized derivations of prime rings. Internat. J. Math. Math. Sci., Article ID 85612, 6 pages (2007). https://doi.org/10.1155/2007/85612
14. Huang, S.: Derivations with Engel conditions in prime and semiprime rings. Czech. Math. J. **61**(136), 1135–1140 (2011). https://doi.org/10.1007/s10587-011-0053-7
15. Huang, S., Davvaz, B.: Generalized derivations of rings and Banach algebras. Commun. Algebra **41**, 1188–1194 (2013). https://doi.org/10.1080/00927872.2011.642043
16. Khan, S.: On semiprime rings with multiplicative (generalized)-derivations. Beitr. Algebra Geom. **57**, 119–128 (2016). https://doi.org/10.1007/s13366-015-0241-y
17. Koç, E., Gölbaşi, Ö.: Some results on ideals of semiprime rings with multiplicative generalized derivations. Commun. Algebra **46**(11), 4905–4913 (2018). https://doi.org/10.1080/00927872.2018.1459644
18. Posner, E.C.: Derivation in prime rings. Proc. Am. Math. Soc. **8**, 1093–1100 (1957). https://doi.org/10.2307/2032686
19. Quadri, M.A., Khan, M.S., Rehman, N.: Generalized derivations and commutativity of prime rings. Indian J. Pure Appl. Math. **34**(9), 1393–1396 (2003)
20. Sandhu, G.S., Kumar, D.: A note on derivations and Jordan ideals of prime rings. AIMS Math. **2**(4), 580–585 (2017). https://doi.org/10.3934/Math.2017.4.580
21. Sandhu, G.S., Camci, D.K.: Some results on prime rings with multiplicative derivations. Turk. J. Math. **44**, 1401–1411 (2020). https://doi.org/10.3906/mat-2002-24
22. Tiwari, S.K., Sharma, R.K., Dhara, B.: Multiplicative (generalized)-derivations in semiprime rings. Beitr. Algebra Geom. **58**(1), 211–225 (2017). https://doi.org/10.1007/s13366-015-0279-x

On Weakly Generalized Reversible Rings

Nirbhay Kumar and Avanish Kumar Chaturvedi

Abstract We introduce a notion of weakly generalized reversible rings as a proper generalization of generalized reversible rings. In support, we give examples. We show that every central reversible ring is an example of weakly generalized reversible ring. Also, we study some properties and characterizarions of weakly generalized reversible ring.

Keywords Reversible rings · Generalized reversible rings · Central reversible rings

1 Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. We denote the ring of all $n \times n$ matrices over a ring R by $M_n(R)$, the ring of all $n \times n$ upper triangular matrices over a ring R by $M_n(R)$; and a square matrix whose (i, j)th entry is 1_R (the identity of R) and elsewhere 0_R (the zero of R) by E_{ij}. The readers are referred to [5] for all undefined terminologies and notions.

Cohn [2] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. In 2014, Kose et al. [4] introduced the notion of central reversible ring as a generalization of reversible ring. They called a ring R central reversible if for any $a, b \in R$, $ab = 0$ implies $ba \in C(R)$; where $C(R)$ denotes the set of all central elements of R. Recently, Subba et al. [6] introduced the notion of generalized reversible ring as a generalization of reversible ring. They called a ring R generalized reversible if, for any $a, b \in R\backslash\{0\}$, $ab = 0$ implies that there exists $m \in \mathbb{N}$ such that $b^m \neq 0$ and $b^m a = 0$.

By the motivation, we introduced a notion of weakly generalized reversible ring as a generalization of generalized reversible ring.

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2 Definition and Properties

Definition 1 We call a ring R weakly generalized reversible if, for any $a, b \in R\backslash\{0\}$, $ab = 0$ implies that there exist $m, n \in \mathbb{N}$ such that $b^m \neq 0$ and $b^m a^n = 0$.

Example 1 Every generalized reversible ring is weakly generalized reversible. However, a weakly generalized reversible ring need not to be generalized reversible. For example, let F be a field and S be the polynomial ring $F < x, y >$ in two non-commuting indeterminates x and y. If I is the ideal $< x^2, xy, y^2 >$ of a ring S and $R = S/I$, then elements of the ring R are of the form $(a + bx + cy + dyx) + I$, where $a, b, c, d \in F$. Let $\bar{f} = (a + bx + cy + dyx) + I$, $\bar{g} = (a' + b'x + c'y + d'yx) + I \in R\backslash\{I\}$ such that $\bar{f}\bar{g} = I$. Then $aa' + (ba' + ab')x + (ca' + ac')y + (da' + cb' + ad')yx \in I$ which implies that

$$aa' = ba' + ab' = ca' + ac' = da' + cb' + ad' = 0.$$

<u>Claim:- $a = 0$ and $a' = 0$</u>: If possible, suppose that $a \neq 0$. Since F is a field, $aa' = 0$ implies that $a' = 0$. Hence $ba' + ab' = 0 = ca' + ac'$ implies that $b' = 0 = c'$. So, $da' + cb' + ad' = 0$ implies that $d' = 0$. Therefore, $\bar{g} = I$ which is a contradiction. Thus, $a = 0$. Next, since $\bar{f} \neq I$, at least one of b, c and d must be nonzero. If $b \neq 0$ or $c \neq 0$, then $ba' + ab' = ca' + ac' = 0$ gives $a' = 0$. If $b = c = 0$, then d must be nonzero and so, in this case, equation $da' + cb' + ad' = 0$ gives $a' = 0$. Hence $\bar{f} = (bx + cy + dyx) + I$ and $\bar{g} = (b'x + c'y + d'yx) + I$. Therefore, $\bar{f}^2 = bcyx + I$ and so $\bar{g}^1 \bar{f}^2 = I$. Thus, R is weakly generalized reversible. Since $(x + I)(y + I) = I$, $(y + I)(x + I) \neq I$ and $(y + I)^2 = I$, so R is not generalized reversible.

Remark 1 1. Obviously, every subring of a weakly generalized reversible ring is weakly generalized reversible.
2. Quotient of a weakly generalized reversible ring need not be weakly generalized reversible. For example, the ring S in Example 1 is weakly generalized reversible being an integral domain but $R = S/I$ is not weakly generalized reversible.

A central reversible ring need not be generalized reversible. For example, if R is the following subring of matrix ring $M_3(\mathbb{Z})$:

$$\left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} | a, b, c, d \in \mathbb{Z} \right\}$$

then R is central reversible by [4, Example 2.2]. However, R is not generalized reversible as if we take $A = E_{23}$, $B = E_{12} \in R$, then $AB = 0$, $BA = E_{13} \neq 0$ and $B^2 = 0$. However, we find the following result:

Proposition 1 *Every central reversible ring is weakly generalized reversible.*

Proof Let R be a central reversible ring. Let $a, b \in R\backslash\{0\}$ such that $ab = 0$. Then $ba \in C(R)$ and so $ba^2 = (ba)a = a(ba) = (ab)a = 0$. Hence R is weakly generalized reversible.

Proposition 2 *A finite subdirect product of weakly generalized reversible rings is weakly generalized reversible.*

Proof Let R be the subdirect product of two rings P and Q. Then, there exist ideal I and J in R such that $R/I \cong P$, $R/J \cong Q$ and $I \cap J = 0$. Suppose that P and Q are weakly generalized reversible. Then, we need to show that R is weakly generalized reversible. Let x and y be two nonzero elements in R such that $xy = 0$. There are two cases:

Case-I: $x \notin I \cup J$ and $y \notin I \cup J$: Then, x and y are neither in I nor J. So, $x + I \neq I$, $y + I \neq I$ and $x + J \neq J$, $y + J \neq J$. Since R/I is weakly generalized reversible, $x + I \neq I$, $y + I \neq I$ and $(x + I)(y + I) = I$, so there exist $m, n \in \mathbb{N}$ such that $y^m \notin I$ and $y^m x^n \in I$. Similarly there exist $k, l \in \mathbb{N}$ such that $y^k \notin J$ and $y^k x^l \in J$. Let $i = max(m, k)$ and $j = max(n, l)$. Then, clearly $y^i \neq 0$ and $y^i x^j = I \cap J = 0$.

Case-II: $x \in I \cup J$ or $y \in I \cup J$: In this case $yx \in I \cup J$. So, $yx \in I$ or $yx \in J$. If $yx \in I \cap J = 0$, we have nothing to prove. Hence, assume that $yx \in I$ and $yx \notin J$. Now since J is an ideal and $yx \notin J$, clearly $x, y \notin J$. Thus, we have $x + J \neq J$, $y + J \neq J$ and $(x + J)(y + J) = J$. Since R/J is weakly generalized reversible, there exist $m, n \in \mathbb{N}$ such that $y^m \notin J$ and $y^m x^n \in J$. Clearly $y^m \neq 0$ and $y^m x^n \in I \cap J = 0$ as $yx \in I$.

Thus, in both cases, there exist $m, n \in \mathbb{N}$ such that $y^m \neq 0$ and $y^m x^n = 0$. Therefore, R is weakly generalized reversible.

Corollary 1 *A finite product of rings is weakly generalized reversible if and only if each ring of the product is weakly generalized reversible.*

Corollary 2 *For any central idempotent e of a ring R, eR and $(1 - e)R$ are weakly generalized reversible if and only if R is weakly generalized reversible.*

Proposition 3 *For any ideal I of a ring R having no nonzero nilpotent element, R is weakly generalized reversible whenever R/I is so.*

Proof Let x and y be two nonzero elements in R such that $xy = 0$. Then, $(x + I)(y + I) = I$. There are two cases.

Case-I: $x \notin I$ and $y \notin I$. So $x + I \neq I$ and $y + I \neq I$. Since R/I is weakly generalized reversible, there exist $m, n \in \mathbb{N}$ such that $y^m \notin I$ and $y^m x^n \in I$. Now since $y^m x^n \in I$ and $(y^m x^n)^2 = y^m x^{n-1}(xy)y^{m-1}x^n = 0$, $y^m x^n = 0$ as I has no nonzero nilpotent element.

Case-II: $x \in I$ and $y \in I$. So, $yx \in I$ and $(yx)^2 = y(xy)x = 0$. Hence $yx = 0$ as I has no nonzero nilpotent element.

Thus, in both cases, there exist $m, n \in \mathbb{N}$ such that $y^m \neq 0$ and $y^m x^n = 0$. Therefore, R is weakly generalized reversible.

3 Some Extensions

Recall [1], for a ring R,

$$R_n = \left\{ \begin{bmatrix} a & a_{12} & a_{13} & \cdots & a_{1\,n-1} & a_{1\,n} \\ 0 & a & a_{23} & \cdots & a_{1\,n-1} & a_{1\,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & a_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & a \end{bmatrix} | a, a_{ij} \in R \text{ for any } i, j \right\}$$

is a subring of the upper triangular matrix ring $T_n(R)$. We observe that for any ring R and $n \geq 3$, the ring R_n is not generalized reversible as if we take $A = E_{23}$, $B = E_{12} \in R_n$, then $AB = 0$, $BA = E_{13} \neq 0$ and $B^2 = 0$.

Proposition 4 *Let S be a multiplicatively closed subset of a ring R consisting of central regular elements. Then, R is weakly generalized reversible if and only if $S^{-1}R$ is so.*

Proof Suppose that R is weakly generalized reversible and let $0 \neq \gamma = r_1 s_1^{-1}$, $0 \neq \delta = r_2 s_2^{-1} \in S^{-1}R$ such that $\gamma\delta = (r_1 r_2)(s_1 s_2)^{-1} = 0$. Then, $r_1, r_2 \neq 0$ and $r_1 r_2 = 0$. So, there exist $m, n \in \mathbb{N}$ such that $r_2^m \neq 0$ and $r_2^m r_1^n = 0$ as R is weakly generalized reversible. This implies that $\delta^m = (r_2^m)(s_2^m)^{-1} \neq 0$ and $\delta^m \gamma^n = (r_2^m r_1^n)(s_2^m s_1^n)^{-1} = 0$. Thus, $S^{-1}R$ is weakly generalized reversible.

Conversely, suppose that $S^{-1}R$ is weakly generalized reversible and let $0 \neq x, y \in R$ such that $xy = 0$. If we take $\alpha = x1^{-1}$, $\beta = y1^{-1} \in S^{-1}R$, then $\alpha, \beta \neq 0$ and $\alpha\beta = 0$. So, there exist $m, n \in \mathbb{N}$ such that $\beta^m = y^m 1^{-1} \neq 0$ and $\beta^m \alpha^n = (y^m x^n)(1)^{-1} = 0$ as $S^{-1}R$ is weakly generalized reversible. This implies that $y^m \neq 0$ and $y^m x^n = 0$. Thus, R is weakly generalized reversible.

Corollary 3 *For any ring R, $R[x]$ is weakly generalized reversible ring if and only if the Laurent polynomials ring $R[x, x^{-1}]$ is so.*

Proof If we take $S = \{1, x, x^2, x^3, \ldots\}$, then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $S^{-1}(R[x]) = R[x, x^{-1}]$. So, result follows from Proposition 4.

Let R be an algebra over $\mathbb{Z}$. Then the set $R \times \mathbb{Z}$ with operations $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1 r_2 + m_1 r_2 + m_2 r_1, m_1 m_2)$, where $r_i \in R$ and $m_i \in \mathbb{Z}$, form a ring with identity $(0, 1)$. Construction of this ring was due to J. L. Dorroh in [3]. This ring is called as *Dorroh extension* of R by $\mathbb{Z}$ and usually denoted by $D(R, \mathbb{Z})$.

Lemma 1 [6, Lemma 2.11] *For any $(r, s) \in D(R, \mathbb{Z})$ and for any positive integer k,*

$$(r, s)^k = \left(\sum_{i=0}^{k-1} {}^k C_i s^i r^{k-i}, s^k \right).$$

Proposition 5 *A ring R is weakly generalized reversible if and only if the Dorroh extension ring $D(R, \mathbb{Z})$ is so.*

Proof Suppose that R is weakly generalized reversible and let $(0, 0) \neq (a, b)$, $(c, d) \in D(R, \mathbb{Z})$ such that $(a, b)(c, d) = (ac + bc + da, bd) = (0, 0)$. Then, $ac + bc + da = 0$ and $bd = 0$. Since $c, d \in \mathbb{Z}$ and $bd = 0$, $b = 0$ or $d = 0$. Thus, there are two cases:

Case-I: $b = 0$ In this case, $a(c + d1_R) = 0$ and $a \neq 0$ as $ac + bc + da = 0$ and $(a, b) \neq (0, 0)$. If $c + d1_R \neq 0$, then by the assumption that R is weakly generalized reversible, there exist $m, n \in \mathbb{N}$ such that $(c + d1_R)^m \neq 0$ and $(c + d1_R)^m a^n = 0$. Since Lemma 1 shows that sum of components of $(c, d)^m$ is $(c + d1_R)^m$, so $(c, d)^m \neq (0, 0)$. Also,

$$
\begin{aligned}
(c, d)^m (a, b)^n &= \left(\sum_{i=0}^{m-1} {}^m C_i d^i c^{m-i}, d^m\right)(a, 0)^n \\
&= \left(\sum_{i=0}^{m-1} {}^m C_i d^i c^{m-i}, d^m\right)(a^n, 0) \\
&= \left(\left(\sum_{i=0}^{m-1} {}^m C_i d^i c^{m-i}\right) a^n + d^m a^n, 0\right) \\
&= \left(\left(\sum_{i=0}^{m-1} {}^m C_i d^i c^{m-i} + d^m 1_R\right) a^n, 0\right) \\
&= \big((c + d1_R)^m a^n, 0\big) = (0, 0).
\end{aligned}
$$

If $c + d1_R = 0$, then $(c, d)(a, b) = (c, d)(a, 0) = (ca + da, 0) = ((c + d1_R) a, 0) = (0, 0)$.

Case-II: $d = 0$ In this case, $(a + b1_R)c = 0$ and $c \neq 0$ as $ac + bc + da = 0$ and $(c, d) \neq (0, 0)$. If $a + b1_R \neq 0$, then by the assumption that R is weakly generalized reversible, there exist $m, n \in \mathbb{N}$ such that $c^m \neq 0$ and $c^m(a + b1_R)^n = 0$. Clearly $(c, d)^m = (c, 0)^m = (c^m, 0) \neq (0, 0)$ and

$$
\begin{aligned}
(c, d)^m (a, b)^n &= (c, 0)^m \left(\sum_{i=0}^{n-1} {}^n C_i b^i a^{n-i}, b^n\right) \\
&= (c^m, 0)\left(\sum_{i=0}^{n-1} {}^n C_i b^i a^{n-i}, b^n\right)
\end{aligned}
$$

$$= \left(c^m \left(\sum_{i=0}^{n-1} {}^nC_i b^i a^{n-i}\right) + c^m b^n, 0\right)$$

$$= (\left(c^m \left(\sum_{i=0}^{n-1} {}^nC_i b^i a^{n-i} + b^n 1_R\right), 0\right)$$

$$= \left(c^m (a + b1_R)^n, 0\right) = (0, 0).$$

If $a + b1_R = 0$, then $(c, d)(a, b) = (c, 0)(a, b) = (ac + bc, 0)$=$((a+b1_R)c, 0) = (0, 0)$.

Thus, in all cases, there exist $m, n \in \mathbb{N}$ such that $(c, d)^m \neq (0, 0)$ and $(c, d)^m (a, b)^n = (0, 0)$. Therefore, $D(R, \mathbb{Z})$ is weakly generalized reversible.

The converse follows from the fact that R is a subring of $D(R, \mathbb{Z})$.

Recall that if B is a subring of a ring A having same identity, then the set, denoted and defined by $R[A, B] = \{(a_1, \ldots, a_n, b, b, \ldots) | n \in \mathbb{N}, a_i \in A, b \in B\}$ form a ring with respect to the component wise addition and multiplication.

Proposition 6 *Let B be a subring of a ring A having same identity. Then, A is weakly generalized reversible if and only if $R[A, B]$ is so.*

Proof Suppose that A is weakly generalized reversible and let $0 \neq X = (a_1, \ldots, a_n, a_{n+1}, a_{n+1}, \ldots)$, $0 \neq Y = (b_1, \ldots, b_n, b_{n+1}, b_{n+1}, \ldots) \in R[A, B]$ such that $XY = (a_1b_1, \ldots, a_nb_n, a_{n+1}b_{n+1}, a_{n+1}b_{n+1}, \ldots) = 0$. Let $i_1, i_2, \ldots, i_r$ be all indices for which a_j and b_j both are nonzero. Since A is weakly generalized reversible, there exist $m_j, n_j \in \mathbb{N}$ such that $b_j^{m_j} \neq 0$ and $b_j^{m_j} a_j^{n_j} = 0$ for all $j \in \{i_1, i_2, \ldots, i_r\}$. Clearly, $b_i a_i = 0$ for all $i \in \{1, 2, \ldots, n, n+1\} \backslash \{i_1, i_2, \ldots, i_r\}$. Let $m = max\{m_{i_1}, \ldots, m_{i_r}, 1\}$ and $n = max\{n_{i_1}, \ldots, n_{i_r}, 1\}$. Then, clearly, $Y^m \neq 0$ and $Y^m X^n = 0$. Thus, $R[A, B]$ is weakly generalized reversible.

The converse follows from the fact that A is a subring of $R[A, B]$.

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References

1. Chaturvedi, A.K., Kumar, N., Shum, K.P.: On Z-symmetric rings. Math. Slovaca **71**(6), 1361–1374 (2021)
2. Cohn, P.M.: Reversible rings. Bull. London Math. Soc. **31**(6), 641–648 (1999)
3. Dorroh, J.L.: Concerning adjunctions to algebras. Trans. Am. Math. Soc. **26**(6), 85–88 (1932)
4. Kose, H., Ungor, B., Halicioglu, S., Harmanci, A.: A generalization of reversible rings. Iranian J. Sci. Tech. **38**(1), 43–48 (2014)

5. Lam, T.Y.: A First Course in Noncommutative rings, 2nd edn., Graduate Texts in Mathematics, vol 131. Springer, New York (2001)
6. Roy, D., Subedi, T.: On a generalization of reversible rings. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **65**(2), 331–340 (2019)