Modeling and Simulation in Science,
Engineering and Technology

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Mathematical Modeling and Control in Life and Environmental Sciences

Regional Control Problems

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Preface

To the planet Earth: our Environment.

Control theory has been intensively used in Engineering, but recently has moved to other sciences, such as Medicine, e.g., for optimizing therapeutic regimens in oncology (see, e.g., Jarrett, A. J., et al., Optimal control theory for personalized therapeutic regimens in oncology: Background, history, challenges, and opportunities. *J. Clin. Med.* 2020, 9, 1314; doi:10.3390/jcm9051314; see also [140, 186]). The current COVID-19 pandemic has revealed the inadequacy of usual methods of control not supported by the required body of expertise integrating mathematics, biology, physics, engineering, logistics, etc. This has been the main motivation for producing a monograph addressing the mathematical problem of controlling a dynamical system modelled by ordinary and/or partial differential equations; a problem which is common to a variety of real-life problems. It is neither a tract nor a recipe book as such; rather, it aims to providing an account of fundamental concepts as they are required in relevant modern applications and in existing literature. A set of significant case studies provide a substantial motivation to the knowledge of the proposed mathematical and computational methods. The originality of the approach presented in this monograph is in its “leit motiv": regional control. In many real structured populations, it is practically impossible, due to anyhow limited resources, to control the relevant system by global interventions in the whole habitat. In geographical epidemiology (for human populations, or in agriculture), this might also be due to unavoidable logistic restrictions. This has motivated, about 20 years ago, two of the authors [SA and VC] to propose regional control methods, along the principle of “Think Globally, Act Locally." They have shown that, under suitable assumptions, it is possible to eradicate an epidemic in the whole habitat by interventions only in a bounded subset of it.

For age-dependent populations, a similar problem, arisen in connection with social distancing and vaccination policies, concerns the optimal selection of the age segments on which to act for a fast and effective eradication of an epidemic. This problem has been of crucial importance in connection with COVID-19 vaccination strategies.

In geographical economics, a recent key issue concerns the interplay of economic growth and the induced environmental pollution.

We make no pretense of its being complete. Indeed, we have omitted many results that we feel are not directly related to the main theme or that are available in easily accessible sources.

Consistent with the concept of "learning by doing," we intend to guide the reader on how to treat control problems, starting with modelling, and continuing with mathematical concepts and methods from control theory, including their computational aspects.

Special attention has been paid on case studies concerning the control of relevant examples of epidemic phenomena, and the impact of human production activities on environmental pollution.

We may compare our book with a couple of other excellent books in the literature devoted to the control in mathematical biology: Lenhart S. , Workman J.T., Optimal Control Applied to Biological Models, Chapman and Hall/CRC, 2007; Schättler H., Ledzewicz U., Optimal Control for Mathematical Models of Cancer Therapies, Springer, 2015. In addition, there are several very good monographs devoted to the general theory of optimal control or to controllability and stabilizability problems (see, e.g., [40] and references therein).

We have not restricted ourselves to usual optimal control, controllability, and stabilizability problems, but we have considered some nonstandard control problems as well.

Indeed, our plan is to guide the reader on the modelling of some realistic and affordable control problems, including the way one may derive information about the structure of the control. All supported by analytical (whenever possible) and computational experiments. A special feature of the book is its focusing on regional control problems for structured populations, in particular age and space structured populations, that is models described by partial differential equations.

The plan is to present optimal control, as well as stabilization and controllability problems. Preventing the spread of a pest population or of an epidemic via a control is another type of problem which has been discussed.

As already mentioned above, due to practical limitations/constraints, in real-world applications, controls usually act only on subsets of a habitat or on subsets of the relevant age interval; this is the main reason why we have paid particular attention to regional control problems, which means not only to address the magnitude of the relevant control, but also the location and geometry of the region where the controls acts.

We aim to addressing three main groups of readers: first, mathematicians working in a different field; second, other scientists and professionals from a private or academic background; third, graduate or advanced undergraduate students of a quantitative subject related to the analysis and applications of dynamical systems and their control. This topic has become increasingly popular in terms of current research output and applications; many pure as well as applied mathematicians have become interested in learning about the fundamentals of control theory and modern applications. We have tried to write this volume in a language that all addressed groups can understand and in its content and structure will allow them to learn the essentials profoundly and in a time-efficient manner. Other scientistpractitioners and academics from fields like biology, medicine, and economics might be very familiar with a less mathematical approach to their specific fields and thus be interested in the mathematical methods required for modelling their applications. Furthermore, this book would be suitable as a textbook accompanying a graduate or advanced undergraduate course or as secondary reading for students of mathematical or computational sciences.

The book is divided into four main parts. In Part I, comprising Chapters 1 and 2, we consider spatially structured epidemics, with a particular attention to man–environment epidemics and vector-borne epidemics. In particular, in Chapter 1, we introduce the main ideas related to our paradigm of “THINK GLOBALLY, ACT LOCALLY” which is the *leit motiv* of the whole book. General concepts about eradicability and control of a paradigmatic epidemic model are presented, with a view to the existence of endemic states, their stability, and the existence of travelling waves. Within this excursus, a challenging open problem has been raised, concerning the possibility to control the advance of a travelling front, for which some interesting computational experiments have been presented. Moreover it has been reported about the use of nonlocal diffusion operators as an alternative to the Laplace operator which models random diffusion; anyhow a diffusion approximation of nonlocal operators is presented too.

In Chapter 2, an extension of the spatially structured epidemic model discussed in the previous chapter is presented concerning malaria as an example of vector-borne epidemics. In this chapter, the regional control problem has been analyzed, with respect to both the parameters of the model, and with respect to the region of intervention. All has been supported by a set of numerical simulations, based on the gradient method.

In Part II, comprising Chapters 3 and 4, we investigate the regional control for some optimal harvesting problems related to population dynamics. We consider the problem of maximizing the profit for structured populations with respect to both the harvesting effort and the selected subregion where the control acts.

In Chapter 3, we treat the regional optimal harvesting problem for space-structured population dynamics with a logistic term (nonlocal as well as local). Chapter 4 concerns two regional control problems related to continuous models for age-dependent populations. The first problem is devoted to an optimal control problem when the harvesting effort acts only on an age subinterval. The second one is related to a nonlinear model in a time-periodic environment. The attention is focused on the determination of the position and length of the time-subinterval when the harvesting is allowed/prohibited.

In Part III, comprising Chapters 5 and 6, recent results concerning the possible control of a devastating epidemic in olive orchards have been presented. This is an additional case of vector-borne disease in which the role of the human population is taken by olive trees, and the vector is a pest insect. In Chapter 5, the main three steps are presented, i.e., (i) the mathematical modelling of the population dynamics of the ecosystem in presence of the infection, (ii) possible strategies for the eventual eradication of the disease, and (iii) computational experiments supporting the theoretical outcomes. In Chapter 6, the role of a predator is analyzed as a possible biocontrol agent of epidemics in agriculture. The limits of this strategy are discussed with respect to specialist and generalist predators.

In Part IV, comprising only Chapter 7, as an additional application of the methods presented in this volume, we introduce an environmental issue concerning the control of the pollution produced by human activities. Based again on a spatially structured reaction diffusion model, the optimal control problem concerning the role of taxation for reducing the amount of produced pollution by human activities has been analyzed. A set of challenging open problems are submitted to interested readers.

All chapters lead to challenging open problems for the interested reader.

The Appendix includes two chapters.

In Appendix A, we have collected and deduced some mathematical definitions and results concerning Rayleigh's formula and Krein–Rutman theorem and their applications related to the principal eigenvalue and eigenvectors of elliptic and integro-elliptic problems, as well as some comparison results for integro-parabolic equations.

In Appendix B, we introduce a non-expert reader to the field of numerical approximation of partial differential equations with low-order finite element methods and to their implementation. We do not enter into the theoretical details, since our focus is to provide a simple tutorial for the implementation of the algorithms adopted throughout the book. We assume that the reader has some basic notions of numerical linear algebra.

The chosen computational platform has been MATLAB (www.mathworks.com), that has successfully emerged as one of the most used programming and numerical computing platforms for scientific computing. The MATLAB codes of most of the numerical simulations are available at the web site:

https://github.com/sn-code-inside/Mathematical-Modeling-and-Control-in-Life-and-Environmental-Sciences

Chapter by chapter, it is also provided a direct link to the code used for the corresponding numerical simulations. In the computational section of each chapter, we have also included a comprehensive description of the numerical methods implemented in the codes used for the numerical simulations.

It is worth reporting here a statement (abridged) taken from a very recent paper [156] concerning the possible role of our investigations:

> a model is only an approximate interpretation of reality and it is always wrong in some small or relevant elements. The destiny of the model presented here is to be rapidly improved thanks to novel knowledge coming from new observations and better assumptions. The Authors hope that many and more brilliant minds will read the present pages, will identify and highlight putative mistakes, will get inspiration for their research, and will produce better, more complete, and useful models........ If the speculations presented here on implications for surveillance, control, and therapy will contribute, even only minimally, to save ..... [human life - environment]........., then the Authors have accomplished their small mission.

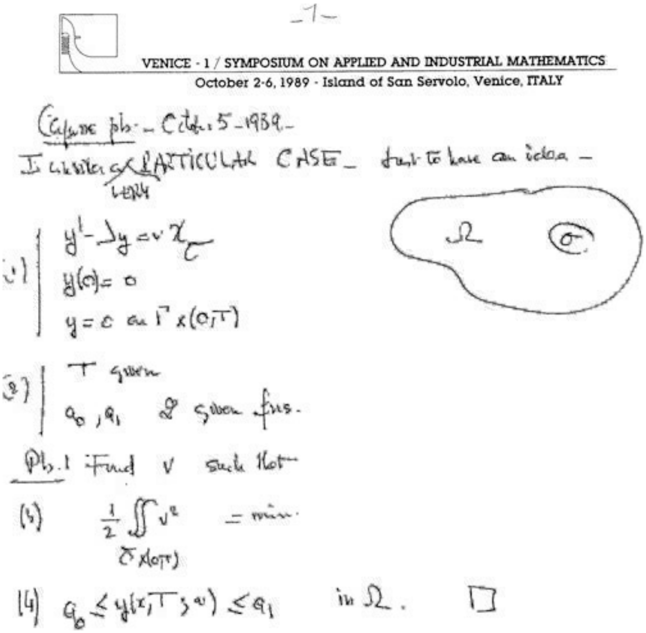

Fig. 1 Notes by Jacques L. Lions concerning a first discussion on regional control

We are indebted to many colleagues for discussions which have led to significant improvements of our results in the years.

But it is a particular duty to acknowledge the role played by the late Jacques L. Lions (see e.g. Figure 1) and by Viorel Barbu for having recognized the importance of regional control for reaction-diffusion systems, and for their consequent encouragement.

It is our duty and pleasure to acknowledge the editorial assistance of Dr. Rossana Caselli, to whom we are grateful in particular for the thorough revision of the bibliography.

Iaşi, Romania — Sebastian Aniţa
Milan, Italy — Vincenzo Capasso
Milan, Italy — Simone Scacchi
September, 2023

Contents

Nomenclature

"increasing" is used with the same meaning as "nondecreasing"; "decreasing" is used with the same meaning as "nonincreasing." In strict cases, "strictly increasing/strictly decreasing" is used.

$:=$	Equal by definition
$\equiv$	Coincide
$\emptyset$	Empty set
∇	Gradient
∂A	Boundary of a set A
$\square$	End of a proof
$\|a\|$	Absolute value of a number a, or modulus of a complex number a, or Euclidean norm on $\mathbb{R}^n$
$\|\|x\|\|$	Norm of a vector x
$\langle f, g\rangle$	Scalar product of two elements f and g in a Hilbert space
$\langle g, f\rangle_{V,V^*}, \langle f, g\rangle_{V^*,V}$	The duality between V and V^*
$\{x \in E : \mathcal{P}(x)\}$	set of elements of E, satisfying property $\mathcal{P}$
$\bar{A}$	Closure of a set A, depending upon context
A'	Transpose of matrix A, depending upon context
$A \setminus B$	Set of elements of A that do not belong to B
$C(A)$	Set of continuous functions from A to $\mathbb{R}$
$C(A; B)$	Set of continuous functions from A to B
$C^k(A)$	Set of functions from A to $\mathbb{R}$ with continuous derivatives up to order k

$L^p(\Omega)$	Set of equivalence classes of a.e. equal p-integrable functions with respect to the Lebesgue integral
$\|\cdot\|_p$ or $\|\cdot\|_{L^p(\Omega)}$	the usual norm in $L^p(\Omega)$
$L^\infty(\Omega)$	Set of equivalence classes of a.e. equal (with respect to the Lebesgue measure) essentially bounded functions
$\|\cdot\|_\infty$ or $\|\cdot\|_{L^\infty(\Omega)}$	the usual norm in $L^\infty(\Omega)$
$\mathbb{I}_A$	Characteristic function associated with a set A, i.e., $\mathbb{I}_A(x) = 1$, if $x \in A$, otherwise $\mathbb{I}_A(x) = 0$
a.e.	Almost everywhere
$\exp\{x\}$ or $\exp(x)$	Exponential function e^x
$f * g$	Convolution of functions f and g
f^-, f^+	Negative (positive) part of f, i.e., $f^- = \max\{-f, 0\}$ ($f^+ = \max\{f, 0\}$)
$\text{sign}\{x\}$ or $\text{sign}(x)$	Sign function: 1, if $x > 0$; 0, if $x = 0$; -1, if $x < 0$
$\text{sgn}\{x\}$ or $\text{sgn}(x)$	The multivalued sign function: 1, if $x > 0$; $[-1, 1]$, if $x = 0$; -1, if $x < 0$
$\mathbb{C}$	Complex plane
$\mathbb{N}$	Set of natural nonnegative integers
$\mathbb{N}^*$	Set of natural (strictly) positive integers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}_+$	Set of positive (nonnegative) real numbers
$\mathbb{R}_+^*$	Set of (strictly) positive real numbers
$\mathcal{A}$	Infinitesimal generator of a semigroup
$\mathcal{D}_A$	Domain of definition of an operator A
Δ	Laplace operator
div_x	divergence with respect to the variable x
δ_x	Dirac delta function localized at x

Part I
Regional Control of Spatially Structured Epidemics

Chapter 1
Regional Control for a Class of Spatially Structured Epidemics: *Think Globally, Act Locally*

1.1 Man–Environment Epidemics: A Nonlocal Reaction–Diffusion System

A widely accepted model for the spatial spread of epidemics in a habitat Ω, via environmental pollution produced by an infective population, e.g., via the excretion of pathogens in the environment, is the following one, as proposed in [66, 67] (see also [69] and the references therein). The model below is a more realistic generalization of a previous model proposed in [73, 75, 76] to describe fecal-orally transmitted diseases (cholera, typhoid fever, infectious hepatitis, etc.) which are typical for the European Mediterranean regions; anyhow it can be applied to other infections and other regions, which are propagated by similar mechanisms (see, e.g., [90]); schistosomiasis in Africa is a typical additional example [160].

$$
(\mathbf{E}) \quad \begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) = d_1 \Delta u_1(x,t) - a_{11}u_1(x,t) + \displaystyle\int_\Omega k(x,x')u_2(x',t)dx', \\ \dfrac{\partial u_2}{\partial t}(x,t) = -a_{22}u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{1.1}
$$

in $\Omega \subset \mathbb{R}^N$ $(N \geq \mathbb{N}^*)$, a nonempty bounded domain with a sufficiently smooth boundary $\partial\Omega$; for $t \in (0, +\infty)$, where $a_{11} \geq 0$, $a_{22} \geq 0$, and $d_1 > 0$ are constants.

In this volume by "domain" we have meant a nonempty open and connected set; by the wording "sufficiently smooth" we have meant that the required regularity is satisfied, depending on the context.

- $u_1(x,t)$ denotes the concentration of the pollutant (pathogen material) at a spatial location $x \in \overline{\Omega}$ and a time $t \geq 0$.
- $u_2(x,t)$ denotes the spatial distribution of the infective population.
- The terms $-a_{11}u_1(x,t)$ and $-a_{22}u_2(x,t)$ model natural decays.

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- The total susceptible population is assumed to be sufficiently large with respect to the infective population, so that it can be taken as constant.

As an example, for fecal-orally transmitted infectious diseases, the infectious agent is multiplied by the infective human population and then sent to the environment (sea, lakes, ponds, etc.) mainly through sewage; because of the peculiar habits of the population of these regions, the agent may return via some diffusion-transport mechanism to any point of the habitat Ω, where the infection process is further activated; thus the nonlocal operator

$$\int_{\Omega} k(x, x')u_2(x', t)dx'$$

expresses the fact that the pollution produced at any point $x' \in \Omega$ of the habitat is made available at any other point $x \in \Omega$; when dealing with human pollution, this may be due to either malfunctioning of the sewage system or improper dispersal of sewage in the habitat. The linearity of the above integral operator is just a simplifying option.

The Laplace operator takes into account the pure random dispersal of the infectious agent in the habitat Ω, due to uncontrolled additional causes of dispersion (we take a constant diffusion coefficient to avoid purely technical complications).

As a further simplification, we assume that the infective population does not diffuse; possible extensions of the above model might include a more general nonlocal dispersion mechanism (see Section 1.6).

Under a suitable definition of its modelling ingredients, Model (E) in System (1.1) can be adopted as a good model for the spatial propagation of an infection in agriculture and forests, as well (see the case of *Xylella* in Chapter 5).

Concerning the local "incidence rate" of the human population at point $x \in \Omega$ and time $t \geq 0$, it is given by

$$(i.r.)(x, t) = g(u_1(x, t)),$$

depending upon the local concentration of the pollutant.

1.1.1 The "Force" of Infection

Based on the discussion originated in [77] concerning the functional response g on the *force of infection* due to the existing infective population, in [67] the following generalized form has been proposed for the force of infection due to the environmental pollution (see also [69, 127]):

$$g(z) = \frac{k_0 z^p}{\alpha_0 + \beta_0 z^q}, \qquad k_0, \alpha_0 > 0,\ \beta_0 \geq 0,\ p, q > 0. \tag{1.2}$$

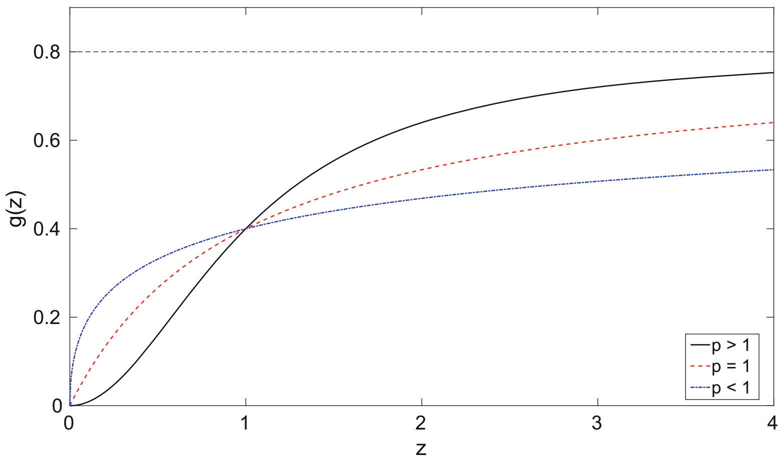

Fig. 1.1 Nonlinear forces of infection [77], according to Equation (1.2), with $k_0 = 0.8$, $\alpha_0 = \beta_0 = 1$, $p = q$, and $p = 2$ (continuous black line), $p = 1$ (dashed red line), and $p = 0.5$ (dashed-dotted blue line)

Particular cases are

$$g(z) = k_0 z^p, \qquad p > 0.$$

For the case $p = q$ we have the behaviors described in Figure 1.1.

Additional shapes of $g(z)$, as proposed in [77], which may decrease for large values of z ($p < q$ in (1.2)) may be interpreted as "awareness" effects in the contact rates (see Figure 1.2). Significant contributions to this concept and related epidemiological issues in recent literature can be found in [100].

System (1.1) has been the subject of a large literature (see, e.g., [196] and the references therein). From now on we shall call it Model (E).

1.1.2 *Large Time Behavior*

In this section we shall report the outcomes of [67] concerning the large time behavior of System (1.1) in a nonempty bounded domain Ω of $\mathbb{R}^2$, with a sufficiently smooth boundary $\partial\Omega$.

The first equation in (1.1) is subject to a boundary condition of the form

$$\beta(x)\frac{\partial u_1}{\partial n}(x,t) + \alpha(x)\,u_1(x,t) = 0, \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{1.3}$$

where $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, and α and β are sufficiently smooth nonnegative functions such that $\alpha(x) + \beta(x) > 0$, for $x \in \partial\Omega$.

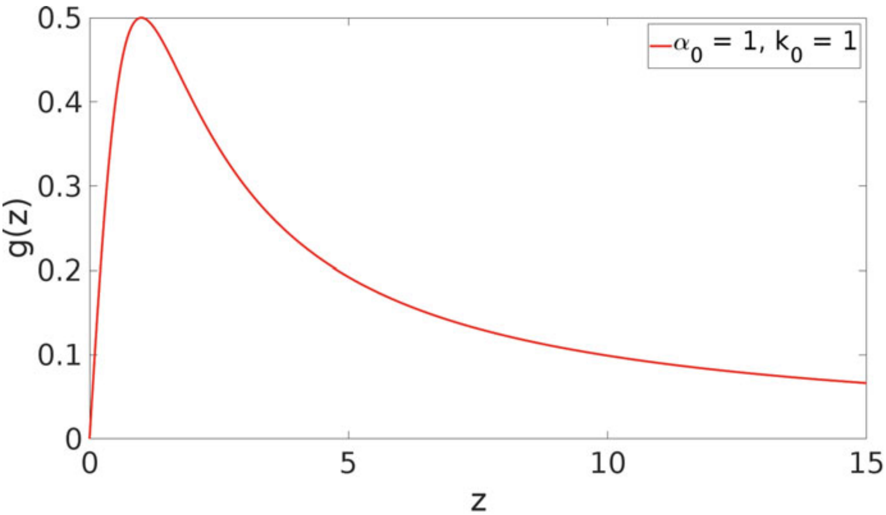

Fig. 1.2 Force of infection with behavioral response [77], according to Equation (1.2), with $k_0 = \alpha_0 = \beta_0 = 1$, $p = 1$, and $q = 2$

If $\alpha(x) = 0$ (a pure Neumann condition), (1.3) means the complete isolation of System (1.1) at $\partial\Omega$.

We shall assume that a_{11} and a_{22} are positive real constants and that $g : [0, +\infty) \longrightarrow (0, +\infty)$ satisfies the following assumptions which include most of the cases described by (1.2):

(i) If $z_1, z_2 \in [0, +\infty)$, $0 < z_1 < z_2$, then $0 < g(z_1) < g(z_2)$.
(ii) $g(0) = 0$.
(iii) g is twice continuously differentiable, and $g''(z) < 0$, for $z \in (0, +\infty)$.

Furthermore we assume that the kernel

$$k : \overline{\Omega} \times \overline{\Omega} \longrightarrow [0, +\infty)$$

is a given continuous function such that if we denote

$$a_{12} = \max_{x \in \overline{\Omega}} \int_{\Omega} k(x, x')dx',$$

then

$$\limsup_{z \to +\infty} \frac{g(z)}{z} < \frac{a_{11}\, a_{22}}{a_{12}} \; .$$

Let $X := C(\overline{\Omega}; \mathbb{R}^2)$ and consider the initial–boundary value problem for (1.1), supplemented by the boundary condition

$$\frac{\partial u_1}{\partial n}(x,t) = 0, \quad \text{for } x \in \partial\Omega,\ t > 0 \tag{1.4}$$

and the initial condition

$$u(\cdot, 0) = u^0 = (u_1^0, u_2^0) \in X_+, \tag{1.5}$$

where $u = (u_1, u_2)$ and

$$X_+ := \{v = (v_1, v_2) \in X :\ v_1(x), v_2(x) \geq 0,\ \text{for all } x \in \overline{\Omega}\}.$$

Definition 1.1 We say that $u = (u_1, u_2)$ is a (classical) solution to the above mentioned initial–boundary value problem (1.1), (1.4), (1.5) if

$$u \in \left(C^{2,1}(\Omega \times (0, +\infty)) \cap C^{1,0}(\overline{\Omega} \times (0, +\infty))\right)$$
$$\times \left(C^{0,1}(\Omega \times (0, +\infty)) \cap C(\overline{\Omega} \times (0, +\infty))\right)$$

satisfies (1.1), (1.4), and

$$\lim_{t \to 0+} u(x,t) = u^0(x), \quad \text{for any } x \in \Omega.$$

It can be shown the following existence result (see [67] and [112] for the case of even more general boundary conditions).

Theorem 1.2 *There exists a unique (classical) solution $u = (u_1, u_2)$ of the above problem. Both components of the solution are nonnegative.*

For a matter of simplicity, in (1.3) we have assumed that $\alpha(x) = 0$ and $\beta(x) = 1$, for $x \in \partial\Omega$. In this case the eigenvalue problem below

$$\begin{cases} -\Delta\phi(x) = \lambda\phi(x), & x \in \Omega, \\ \dfrac{\partial\phi}{\partial n}(x) = 0, & x \in \partial\Omega \end{cases}$$

admits $\lambda_1 = 0$ as a principal eigenvalue; as a corresponding eigenfunction, we may choose

$$\phi_1 \equiv 1, \quad x \in \overline{\Omega}.$$

Let us denote

$$H(u_2)(x) = \int_\Omega k(x, x')\, u_2(x')dx', \qquad u_2 \in C(\overline{\Omega}),\ x \in \overline{\Omega}.$$

For $\phi_1 \equiv 1$,

$$H(\phi_1)(x) = \int_\Omega k(x, x')\phi_1(x')dx' = \int_\Omega k(x, x')dx'.$$

We shall denote

$$\overline{H} = \max_{x \in \overline{\Omega}} \int_\Omega k(x, x')dx', \quad \underline{H} = \min_{x \in \overline{\Omega}} \int_\Omega k(x, x')dx'.$$

We may notice that if $k(x, x')$ is an isotropic symmetric probability density function, multiplied by a_{12},

$$\overline{H} = \underline{H} = a_{12}.$$

Under the above assumptions Theorem 3.1 in [67] becomes

Theorem 1.3 *If*

$$\theta_1 := \frac{g'(0)a_{12}}{a_{11}\, a_{22}} < 1,$$

then the trivial solution is globally asymptotically stable in X_+ for System (1.1).
If $\theta_1 > 1$, then the trivial solution is unstable for System (1.1).

For the case of more general boundary conditions, we refer to [67]. See later for the possible existence of travelling waves in an unbounded domain.

1.2 Regional Control

Let us consider System (1.1) in $\Omega \subset \mathbb{R}^2$, a nonempty bounded domain with a sufficiently smooth boundary $\partial\Omega$; for $t \in (0, T)$, where $a_{11} \geq 0$, $a_{22} > 0$, $d_1 > 0$, and $T > 0$ are constants.

The public health concern consists of reducing the disease in the relevant habitat, as fast as possible, at an optimal cost. The underlying issue of our presentation is that very often the entire domain of interest for the epidemic is either unknown or difficult to manage, for an affordable implementation of suitable environmental sanitation programs. Think of malaria, schistosomiasis, and alike, in Africa, Asia, etc.

This has led the authors [VC and SA] of this monograph to face the problem of an effective diminishment of an epidemic in the whole habitat Ω by implementing possible control strategies only in a suitably chosen subregion $\omega \subset \Omega$ (Figure 1.3). A satisfactory mathematical treatment of this issue has been obtained in [10] (see

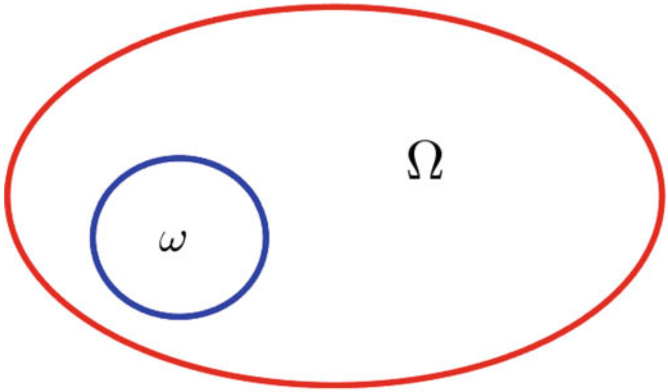

Fig. 1.3 Think globally, act locally

also [11, 14]). This practice may have an enormous importance in real cases with respect to both financial and practical affordability. Furthermore, since we propose to act on the elimination of the pollution only, this practice means an additional nontrivial social benefit on the human population, since it would not be limited in his personal and social habits.

Let $\omega \subset \Omega$ be a nonempty open subset; we denote by $\mathbb{I}_\omega$ the characteristic function of ω and use the convention

$$\mathbb{I}_\omega(x)w(x) = 0, \qquad x \in \mathbb{R}^2 \setminus \omega,$$

even if function w is not defined on the whole set $\mathbb{R}^2 \setminus \omega$.

Our goal is to study System (1.1) subject to suitable controls, as follows:

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) - d_1 \Delta u_1(x,t) = -a_{11}u_1(x,t) + \displaystyle\int_\Omega k(x,x')u_2(x',t)dx' \\ \qquad -\gamma_1(x,t)\mathbb{I}_\omega(x)u_1(x,t), & (x,t) \in Q, \\ \dfrac{\partial u_2}{\partial t}(x,t) = -(a_{22} + \gamma_2(x,t)\mathbb{I}_\omega(x))u_2(x,t) \\ \qquad +(1 - \gamma_3(x,t)\mathbb{I}_\omega(x))g(u_1(x,t)), & (x,t) \in Q, \end{cases}$$

subject to suitable boundary and initial conditions. Here $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$.

The idea underlying the controls is the following:

$$\gamma_1(x,t)u_1(x,t)\mathbb{I}_\omega(x)$$

represents the sanification of the environment, say by the combined action of different chemical and/or mechanical practices, in the subregion ω; $\gamma_1(x,t)u_1(x,t)$

represents the corresponding reduction of the polluting agent at location $x \in \Omega$ and time $t > 0$.

$$\gamma_2(x,t)u_2(x,t)\mathbb{I}_\omega(x)$$

describes the medical treatment of the infected population in the subregion ω; $\gamma_2(x,t)u_2(x,t)$ is then the corresponding healed population at location $x \in \Omega$ and time $t > 0$.

$$\gamma_3(x,t)\mathbb{I}_\omega(x)$$

is the reduction of the contact rate of humans–environment by means of behavioral and/or distancing practices, implemented in the subregion ω.

Each of the above-mentioned control practices implies a cost to be taken into account in the total cost functional for optimal control problems. Below we have, respectively, denoted by $\zeta_1(\gamma_1)$ the cost for obtaining the actual control γ_1, by $\zeta_2(\gamma_2 u_2)$ the cost of the treatment per individual, and by $\zeta_3(\gamma_3)$ the cost for obtaining the actual control γ_3.

When the costs of the controls to reduce the epidemic of the damages produced by the disease and of the intervention in the subset ω in the time interval $[0,T]$ are considered, then we get the total cost function:

$$\begin{aligned} J(\gamma,\omega) = &\int_0^T\int_\Omega C(x)\zeta_1(\gamma_1(x,t))\mathbb{I}_\omega(x)dx\,dt \\ &+\int_0^T\int_\Omega \zeta_2(\gamma_2(x,t)u_2(x,t))\gamma_2(x,t)u_2(x,t)\mathbb{I}_\omega(x)dx\,dt \\ &+\int_0^T\int_\Omega C(x)\zeta_3(\gamma_3(x,t))\mathbb{I}_\omega(x)dx\,dt \\ &+\int_0^T\int_\Omega l(u_2(x,t))dx\,dt + \alpha_1\ \text{area}(\omega) + \beta_1\ \text{perimeter}(\omega). \end{aligned} \tag{1.6}$$

Here ω is a subset of Ω and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ belongs to a suitable set $\mathcal{G}_T$ of feasible values of the $\gamma' s$, and $C(x)$ is the spatial density of the total population. As far as the last three terms in (1.6) are concerned, the term $\int_0^T\int_\Omega l(u_2(x,t))dx\,dt$ is meant to take into account the costs, over the whole domain and the total observation/control period, deriving from loss of work hours, hospitalization, drugs, etc., while the terms α_1 area$(\omega) + \beta_1$ perimeter(ω) take into account costs of transport of intervention devices (nets, chemicals, personnel, etc.) which may depend on the geometry of the subregion of intervention ω. For our scopes, the geometry of a planar set is indeed characterized by its area and perimeter, the coefficients α_1 and β_1 take into account the specific logistic structure of the habitat, and we may assume $\alpha_1, \beta_1 \geq 0$. This aspect has been taken care in some detail for the malaria model in Chapter 2.

1.3 Eradicability

In this section we face the problem of possible eradicability of an epidemic modelled by System (1.1) by just reducing the concentration of the pollutant in a nonempty and sufficiently large subset of the spatial domain. Later, in applications, relevant optimal control problems will be presented, as mentioned in the previous section. Here we will show that, under suitable assumptions, it is indeed possible to diminish exponentially the epidemic process.

Consider again the man–environment epidemic system in $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, a nonempty bounded domain with a sufficiently smooth boundary $\partial\Omega$. Let $\omega \subset \Omega$ be a nonempty bounded domain with a C^2-boundary, such that $\overline{\omega} \subset \Omega$ and $\Omega \setminus \overline{\omega}$ a domain. Denote by $\mathbb{I}_\omega$ the characteristic function of ω. We use the convention

$$\mathbb{I}_\omega(x)h(x) = 0, \qquad x \in \mathbb{R}^N \setminus \omega,$$

even if function h is not defined on the whole set $\mathbb{R}^N \setminus \omega$.

Our goal is to study if there exists a control $v \in L^\infty_{loc}(\overline{\omega} \times [0, +\infty))$ (which implies that $supp(v(t)) \subset \overline{\omega}$, for $t \geq 0$) such that the (weak) solution (u_1, u_2) of the following system

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) = d_1 \Delta u_1(x,t) - a_{11}u_1(x,t) + \displaystyle\int_\Omega k(x,x')u_2(x',t)dx' \\ \qquad + \mathbb{I}_\omega(x)\, v(x,t), & (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u_2}{\partial t}(x,t) = -a_{22}u_2(x,t) + g(u_1(x,t)), & (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u_1}{\partial n}(x,t) + \alpha u_1(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u_1(x,0) = u_1^0(x), \quad u_2(x,0) = u_2^0(x), & x \in \Omega \end{cases} \tag{1.7}$$

satisfies

$$u_1(x,t) \geq 0, \quad u_2(x,t) \geq 0, \qquad \text{a.e. } x \in \Omega, \text{ for any } t \geq 0, \tag{1.8}$$

and

$$\lim_{t \to +\infty} \|u_1(t)\|_{L^\infty(\Omega)} = \lim_{t \to +\infty} \|u_2(t)\|_{L^\infty(\Omega)} = 0. \tag{1.9}$$

Here $d_1, a_{22} \in (0, +\infty)$ and $a_{11}, \alpha \in [0, +\infty)$ are constants.

We work under the following assumptions:

(H1) $g : \mathbb{R} \to [0, +\infty)$ is a function satisfying

$g(x) = 0$, for $x \in (-\infty, 0]$,
g is Lipschitz continuous and increasing,
$g(r) \leq a_{21}r$, for any $r \in [0, +\infty)$, where $a_{21} > 0$ is a constant.

(H2) $k \in L^\infty(\Omega \times \Omega),\ k(x, x') \geq 0$, a.e. in $\Omega \times \Omega$,

$$\int_\Omega k(x, x')dx > 0, \qquad \text{a.e. } x' \in \Omega.$$

(H3) $u_1^0,\ u_2^0 \in L^\infty(\Omega), \quad u_1^0(x),\ u_2^0(x) \geq 0$, a.e. in Ω.

Definition 1.4 We say that (u_1, u_2) is a (weak) solution to (1.7) if for any $T \in (0, +\infty)$,

$u_1 \in C([0, T]; L^2(\Omega)) \cap AC((0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^2_{loc}((0, T]; H^2(\Omega))$,

$u_2 \in L^2(\Omega \times (0, T))$ such that for almost any $x \in \Omega$, $u_2(x, \cdot) \in AC([0, T])$,

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x, t) = d_1 \Delta u_1(x, t) - a_{11} u_1(x, t) + \displaystyle\int_\Omega k(x, x') u_2(x', t) dx' \\ \qquad + \mathbb{I}_\omega(x)\, v(x, t), & \text{a.e. } (x, t) \in \Omega \times (0, +\infty), \\ \dfrac{\partial u_2}{\partial t}(x, t) = -a_{22} u_2(x, t) + g(u_1(x, t)), & \text{a.e. } (x, t) \in \Omega \times (0, +\infty), \\ \dfrac{\partial u_1}{\partial n}(x, t) + \alpha u_1(x, t) = 0, & \text{a.e. } (x, t) \in \partial\Omega \times (0, +\infty), \\ u_1(x, 0) = u_1^0(x), \quad u_2(x, 0) = u_2^0(x), & \text{a.e. } x \in \Omega. \end{cases}$$

Let us prove that Problem (1.7) admits a unique (weak) solution (u_1, u_2). For an arbitrary but fixed $T \in (0, +\infty)$, consider the space

$$V = C([0, T]; L^2(\Omega)) \times C([0, T]; L^2(\Omega)),$$

with the distance

$$d((\zeta_1, \zeta_2), (\zeta_3, \zeta_4)) = \sup_{t \in [0,T]} \left(e^{-\xi t} (\|\zeta_1(t) - \zeta_3(t)\|^2_{L^2(\Omega)} + \|\zeta_2(t) - \zeta_4(t)\|^2_{L^2(\Omega)}) \right)^{\frac{1}{2}}.$$

Here ξ is a positive constant to be made precise later. (V, d) is a complete metric space (actually it is even a Banach space).

For any $(y_1, y_2) \in V$, let $\mathcal{T}(y_1, y_2)$ be the weak solution to

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x, t) = d_1 \Delta u_1(x, t) - a_{11} u_1(x, t) + \displaystyle\int_\Omega k(x, x') y_2(x', t) dx' \\ \qquad + \mathbb{I}_\omega(x) v(x, t), & (x, t) \in \Omega \times (0, T), \\ \dfrac{\partial u_1}{\partial n}(x, t) + \alpha u_1(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \dfrac{\partial u_2}{\partial t}(x, t) = -a_{22} u_2(x, t) + g(y_1(x, t)), & (x, t) \in \Omega \times (0, T), \\ u_1(x, 0) = u_1^0(x), \quad u_2(x, 0) = u_2^0(x), & x \in \Omega. \end{cases} \tag{1.10}$$

Problem (1.10) contains a linear parabolic equation and a collection of ordinary differential equations. Notice that $\int_\Omega k(x, x') y_2(x', t) dx' \in L^2(\Omega \times (0, T))$ and

$g(y_1(x,t)) \in L^2(\Omega \times (0,T))$. It follows that there exists a unique weak solution to (1.10).

For any $(y_1, y_2), (\tilde{y}_1, \tilde{y}_2) \in V$, let $(u_1, u_2) = \mathcal{T}(y_1, y_2)$ be the weak solution to (1.10) and $(\tilde{u}_1, \tilde{u}_2) = \mathcal{T}(\tilde{y}_1, \tilde{y}_2)$ be the weak solution to (1.10) corresponding to $y_i := \tilde{y}_i$. We get that $(w_1, w_2) = (u_1 - \tilde{u}_1, u_2 - \tilde{u}_2)$ is the weak solution to

$$\begin{cases} \dfrac{\partial w_1}{\partial t}(x,t) = d_1 \Delta w_1(x,t) - a_{11} w_1(x,t) + \displaystyle\int_\Omega k(x,x') & (y_2 - \tilde{y}_2)(x',t)dx', \\ & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial w_1}{\partial n}(x,t) + \alpha w_1(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \dfrac{\partial w_2}{\partial t}(x,t) = -a_{22} w_2(x,t) + g(y_1(x,t)) - g(\tilde{y}_1(x,t)), & (x,t) \in \Omega \times (0,T), \\ u_1(x,0) = 0, \quad u_2(x,0) = 0, & x \in \Omega. \end{cases}$$

Let us multiply the equation in w_1 by w_1 and the equation in w_2 by w_2, add them, and integrate over $\Omega \times (0,t)$ $(t \in [0,T])$. We get after an easy calculation that

$$\frac{1}{2}\|w_1(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|w_2(t)\|^2_{L^2(\Omega)}$$

$$\leq \int_0^t \int_\Omega w_1(x,s) \int_\Omega k(x,x')(y_2 - \tilde{y}_2)(x',s)dx'\, dx\, ds$$

$$+ \int_0^t \int_\Omega w_2(x,s)(g(y_1(x,s)) - g(\tilde{y}_1(x,s)))dx\, ds$$

$$\leq \|k\|_{L^2(\Omega\times\Omega)} \int_0^t \|w_1(s)\|_{L^2(\Omega)} \|y_2(s) - \tilde{y}_2(s)\|_{L^2(\Omega)} ds$$

$$+ L \int_0^t \|w_2(s)\|_{L^2(\Omega)} \|y_1(s) - \tilde{y}_1(s)\|_{L^2(\Omega)} ds$$

(here L is a Lipschitz constant for g). So, there exists a positive constant C such that for any $t \in [0,T]$,

$$\|w_1(t)\|^2_{L^2(\Omega)} + \|w_2(t)\|^2_{L^2(\Omega)}$$

$$\leq C \int_0^t (\|y_1(s) - \tilde{y}_1(s)\|^2_{L^2(\Omega)} + \|y_2(s) - \tilde{y}_2(s)\|^2_{L^2(\Omega)}) ds$$

$$\leq C \int_0^t e^{\xi t} e^{-\xi s} (\|y_1(s) - \tilde{y}_1(s)\|^2_{L^2(\Omega)} + \|y_2(s) - \tilde{y}_2(s)\|^2_{L^2(\Omega)}) ds$$

$$\leq \frac{C}{\xi} e^{\xi t} d((y_1, y_2), (\tilde{y}_1 - \tilde{y}_2))^2.$$

It follows that

$$d(\mathcal{T}(y_1, y_2), \mathcal{T}(\tilde{y}_1, \tilde{y}_2))^2 \leq \frac{C}{\xi} d((y_1, y_2), (\tilde{y}_1 - \tilde{y}_2))^2.$$

If we choose $\xi > C$, then $\mathcal{T}$ is a contraction and so by Banach's fixed point theorem it has a unique fixed point.

We may conclude that (1.7) has a unique weak solution (u_1, u_2).

Our hypothesis on v does not assure the nonnegativity of the components u_1 and u_2. However, the nonnegativity of u_1 and u_2 is a natural requirement (due to the biological significance of u_1 and u_2). The nonnegativity of the components of the weak solution follows for different choices of v via the comparison results for parabolic equations and applying Banach's fixed point result in (V_+, d), where

$$V_+ = \{(\zeta_1, \zeta_2) \in V : \ 0 \leq \zeta_1(x,t), \ \zeta_2(x,t), \text{ a.e. } x \in \Omega, \text{ for any } t \in [0, T]\}.$$

Definition 1.5 We say that the disease modelled by (1.7) is *eradicable* if, for any u_0^1 and u_0^2 satisfying (H3), there exists $v \in L^\infty_{loc}(\overline{\omega} \times [0, +\infty))$ such that (1.7) admits a unique weak solution (u_1, u_2) with nonnegative components (i.e., satisfying (1.8)) and satisfying (1.9).

Preliminary eradicability results for System (1.7) have been obtained in [10]. For eradicability (stabilizability) results related to some reaction–diffusion systems in population dynamics, we refer to [1, 4, 10–15, 17, 27], while for general distributed systems see [149].

A strong eradicability result (which provides a stabilizing control with a simple structure) has been established in [4], using a different approach from the one we will show in this section. Here we will report the results obtained in [11], with simplified proofs, providing a stabilizing feedback control with a very simple structure. This is an easily implementable feedback control.

1.3.1 An Eigenvalue Problem

For an arbitrary but fixed constant $\gamma > 0$, consider the following eigenvalue problem associated with Problem (1.7):

$$\begin{cases} -d_1 \Delta\varphi(x) + a_{11}\varphi(x) - \displaystyle\int_\Omega k(x, x')\psi(x')dx' \\ \qquad\qquad + \mathbb{I}_\omega(x)\gamma\varphi(x) = \lambda\varphi(x), & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial n}(x) + \alpha\varphi(x) = 0, & x \in \partial\Omega, \\ -a_{21}\varphi(x) + a_{22}\psi(x) = 0, & x \in \Omega. \end{cases} \tag{1.11}$$