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Armen M. Jerbashian
Joel E. Restrepo

Functions of Omega-Bounded Type

Basic Theory

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Devoted to the memory
of Mkhitar Djrbashian

Preface

In view of the contemporary development of the theory of A^p_α spaces and its applications, it is natural and interesting to come out of the frames of the weights $(1-r)^\alpha$ and consider A^p_ω spaces with functional parameters ω, which are associated with the M.M. Djrbashian integral operator L_ω. In some particular cases, this operator becomes the classical integral operators of Riemann–Liouville, Hadamard, Erdélyi–Kober, and many other ones.

The book gives the basic results of the theory of the spaces A^p_ω of functions holomorphic in the unit disc, halfplane, and in the finite complex plane, which depend on functional weights ω permitting any rate of growth of functions near the boundary of the domain. This continues and essentially improves M.M. Djrbashian's theory of spaces A^p_α (1945) of functions holomorphic in the unit disc, the English translation of the detailed and complemented version of which (1948) is given in Addendum to the book. Besides, the book gives the ω-extensions of M.M. Djrbashian's two factorization theories of functions meromorphic in the unit disc of 1945–1948 and 1966–1975 to classes of functions delta-subharmonic in the unit disc and in the halfplane.

The book can be useful for a wide range of readers. It can be a good handbook for Master and PhD students and Postdoctoral Researchers for enlarging their knowledge and analytical methods, and also it can be very useful for scientists to extend their investigation fields.

Yerevan, Armenia — Armen M. Jerbashian
Ghent, Belgium — Joel E. Restrepo
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Introduction

The origins of investigations related to the spaces of holomorphic functions, the squares of modules of which are summable over the area of the unit disc $\mathbb{D} \equiv \{z : |z| < 1\}$, can be found in the paper of L. Biberbach [5] and in other references of the classical book of J.L. Walsh [85]. While L. Biberbach studied approximations by rational functions in the space of holomorphic functions, the derivatives of which satisfy the mentioned summability condition, W. Wirtinger [87] studied approximations in the space H_2' of holomorphic in the unit disc $\mathbb{D}$ functions f which themselves satisfy the summability condition. This space is being denoted by A_0^2 in the modern literature:

$$A_0^2 \;\; (\equiv H_2') : \quad \|f\|^2 = \iint_{\mathbb{D}} |f(z)|^2 dS < +\infty,$$

where dS is for the Lebesgue area measure. In the same work, W. Wirtinger, in particular, proved the representation formula for the functions $f \in A_0^2$ and found the orthogonal projection from the same type Lebesgue space L_0^2 to A_0^2. Due to numerous misunderstandings on the issue in the contemporary literature, below we present the mentioned results of W. Wirtinger as they are given in J.L. Walsh's book [85] (pp. 150–151), where C' means the unit disc.

> …**Theorem 20.** *Let $F(z)$ be of class L^2 in C'. The essentially unique function $f(z)$ of class H_2' such that*
>
> $$\iint_{C'} \left|F(z) - f(z)\right|^2 dS$$
>
> *is least is given by*
>
> $$f(z) \equiv \frac{1}{\pi} \iint_{C'} F(\xi) \frac{dS}{(1 - \bar{\xi} z)^2}, \quad |z| < 1. \tag{58}$$

The formal development of $F(z)$ on C' in terms of the functions z^k is

$$\sum_{k=0}^{\infty} a_k z^k, \quad a_k = \frac{k+1}{\pi} \iint_{C'} F(\xi)\overline{\xi}^k dS; \tag{59}$$

this series converges to $f(z)$ of class H_2' in the mean on C', hence (§5.8, Theorem 17) converges to $f(z)$ uniformly on any closed set interior to C'. Interior to C', the function represented by (59) is

$$f(z) \equiv \frac{1}{\pi} \iint_{C'} F(\xi)\big[1 + 2\overline{\xi}z + 3\overline{\xi}z^2 + \cdots\big]dS, \quad |z| < 1,$$

for the series in square brackets converges uniformly for $|\xi| \le 1$ when z is fixed. This equation for $f(z)$ can be rewritten in form (58). Of course if $F(z)$ is an arbitrary function of class H_2', then (58) is valid with $f(z) \equiv F(z)$.

Theorem 20 is due to Wirtinger [1932], by a quite different method...

... Theorems 20 and 21 and the remark just made extend to more general regions by the use of conformal mapping; compare §11.4.

The study of extremal problems and their solution by methods of approximation is to be resumed in §11.3 and A 3.

Of course one may study approximation in a *multiply connected* region (compare §1.6 and 1.7) in the sense of least squares, by orthogonalizing a suitable set of rational functions; see Ghika [1936] and Bergman [2] ...

Note that the real novelty on the A_0^2 space in the mentioned work of A. Ghika (1936) (no publication data in [85]), also in the monograph of S. Bergman [4], where his results were summarized, was the consideration of the *unweighted* space A_0^2 in some multiply connected domains. The complicated nature of the considered domains permitted S. Bergman only to prove the existence of the corresponding reproducing kernels and establish an analog of W. Wirtinger's Theorem 20.

Later, W. Wirtinger's projection Theorem 20 was extended by V.P. Zakaryuta and V.I. Yudovich [90] to the *unweighted* spaces A_0^p $(1 < p < +\infty)$ in $\mathbb{D}$, the form of bounded linear functionals was revealed, and it was proved that the dual space of A_0^p is A_0^q $(1/p + 1/q = 1)$ in the sense of isomorphism. In W. Rudin's books [72] and [73], the same was done in the polydisc and the unit ball of $\mathbb{C}^n$, where the extension has some explicit forms of kernels, evident in view of W. Wirtinger's Theorem 20 and the result of V.P. Zakaryuta—V.I. Yudovich. In W. Rudin's books, the extension of W. Wirtinger's Theorem 20 was called "Bergman projection," and after that many investigators are attributing the terms "Bergman projection," "Bergman space," "Bergma kernel," and even "Bergman-Nevanlinna classes" to any result on regular functions summable over the area of a complex domain. In fact, this cuts off the original information sources for numerous specialists and causes the above-mentioned misunderstandings on the origins in the contemporary literature.

Coming to weighted spaces, the earliest paper of M.M. Djrbashian [8] is to be referred, the English translation of the detailed and complemented version [9] of which is presented in Addendum of this book. The mentioned papers were aimed mainly at improving R. Nevanlinna's result of 1936 (see [66], page 216) on the density of zeros and poles of functions f meromorphic in $\mathbb{D}$, for which the Riemann–Liouville fractional integral of the growth characteristic $T(r, f)$ is bounded, i.e.,

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-r)^\alpha T(r, f)dr < +\infty$$

for a given $\alpha > -1$. This improvement results in a complete factorization formula for meromorphic in $\mathbb{D}$ functions satisfying the above condition. The factorization formula contains some special Blaschke type product and a surface integral with the degree $2+\alpha$ of the Cauchy kernel in the exponent and becomes the well-known Nevanlinna factorization of functions of bounded type in $\mathbb{D}$ as $\alpha \to -1+0$. The same works [8, 9] contain a large investigation of the similar Hardy type spaces $H^p(\alpha)$ of holomorphic in $|z| < 1$ functions for which the notation A^p_α is used nowadays. M.M. Djrbashian [8] introduced these spaces by the boundedness of the Riemann-Liouville fractional primitive of the integral means $M_p(r, f)$, i.e., by the condition

$$H^p(\alpha) \equiv A^p_\alpha : \quad \int_0^1 (1-r)^\alpha M_p(r, f) r dr$$
$$\equiv \frac{1}{2\pi}\iint_{|\zeta|<1} (1-|\zeta|)^\alpha |f(\zeta)|^p d\sigma(\zeta) < +\infty,$$

where $\alpha \in (-1, +\infty)$ and $p \in [1, +\infty)$ are any fixed numbers and $d\sigma$ is for Lebesgue's area measure. In particular, in [8,9] W. Wirtinger's Theorem 20 was extended to the spaces A^2_α. Also, there are to be mentioned the paper of M.V. Keldysch [59] and the monograph by A.E. Djrbashian and F.A. Shamoyan [19], where they extended the W. Wirtinger–M.M. Djrbashian orthogonal projection theorem for A^2_α to the spaces A^p_α $(-1 < \alpha < +\infty)$ and continued the M.M. Djrbashian's theory of A^p_α spaces by numerous new results. More about the prehistory of the spaces A^p_α in $\mathbb{D}$ and associated results can be found in the survey [56].

The results on weighted spaces of functions in a sense regular and area-integrable over the unit disc remain in considerable interest, since they find development and application in numerous contemporary investigations some of which are described and referred in the survey [56].

Part I of the present book is devoted to the construction of a theory continuing and essentially improving the theory of M.M. Djrbashian of 1945–1948. The spaces A^p_ω of functions holomorphic in the unit disc $\mathbb{D}$, upper halfplane $G^+ \equiv \{z : \operatorname{Im} z > 0\}$, and in the whole finite complex plane $\mathbb{C}$ are investigated. These spaces depend on a functional parameter ω which compensate any growth of several integral means of functions near the boundaries of the considered domains. Thanks to this, the spaces A^p_ω in the unit disc and A^2_ω in the whole complex plane cover the whole sets of functions holomorphic in these domains. Besides, the factorization result of [8] is extended to some ω-weighted classes of functions delta-subharmonic in the unit disc and similar classes in the upper halfplane, and the classes in the unit disc cover all functions delta-subharmonic in that domain.

In the period of 1966–1975, an application of the Riemann—Liouville fractional integrodifferentiation and a more general operator depending on a functional parameter ω directly to the considered functions led M.M. Djrbashian (see [11], Ch. IX and [10, 12, 13, 16, 20–22]) to the factorization theory of his Nevanlinna type $N\{\omega\}$ classes, the union of which coincides with the *whole* set of functions meromorphic in the unit disc. Because of this comprehensiveness of the theory, M.M. Djrbashian first designated the last letter ω of the Greek alphabet for the functional parameter.

The new theory in a sense was more elegant, since it contains the Blaschke product and the classical formulas of Nevanlinna factorization, Jensen-Nevanlinna, Poisson–Jensen, and the Jensen inversion formula in the particular case $\omega \equiv 1$. The classes $N\{\omega\}$ were introduced in [13] by application of a general Riemann–Liouville type operator L_ω [12]; see also in the monograph of S.G. Samko et al. [74] and many contemporary investigations. In a particular case, this operator takes the simple form

$$L_\omega \log|f(z)| = -\int_0^1 \log|f(tz)|d\omega(t), \quad |z| < 1,$$

when applied to the logarithm of the modulus of a meromorphic in $\mathbb{D}$ function. In the later extension of this theory [39] to the set of all functions *delta-subharmonic* in the unit disc $\mathbb{D}$, a growth condition was posed on $L_\omega u$ with an arbitrary delta-subharmonic in $\mathbb{D}$ function u which replaced $\log|f|$. This led to a very explicit understanding of the Nevanlinna type T_ω characteristic and N_ω classes.

Part II of the present book gives an extension of the M.M. Djrbashian factorization theory to a Riesz type representation theory of functions delta-subharmonic in the unit disc $\mathbb{D}$ and the construction of a similar theory in the upper halfplane. Also, it contains some results on Banach spaces of functions delta-subharmonic in the unit disc and in the upper halfplane.

August, 2023

Armen M. Jerbashian
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Contents

Part I

Omega-Weighted Classes of Area Integrable Regular Functions

1 Preliminary Results

1.1 M.M. Djrbashian Operators L_ω and His Omega-Kernels

We start by reducing M.M. Djrbashian's general integrodifferential operator L_ω used in his factorization theory of functions meromorphic in the unit disc [12, 13, 16, 20–22] to some simple forms, which are used in this book.

In the mentioned theory, a function ω is said to be of the class Ω, if $\omega > 0$ in $[0, 1)$, $\omega(0) = 1$ and $\omega \in L^1[0, 1]$. Further, for any $\omega \in \Omega$ it is set

$$p(0) = 1 \quad \text{and} \quad p(t) \equiv t \int_t^1 \frac{\omega(x)}{x^2} dx, \quad 0 < t < 1, \tag{1.1}$$

and for a measurable in $|z| < R \le +\infty$ function u it is introduced the operator

$$L_\omega u(re^{i\varphi}) \equiv -\frac{d}{dr}\left\{ r \int_0^1 u(tre^{i\varphi}) dp(t) \right\}. \tag{1.2}$$

This formula for L_ω was used by M.M. Djrbashian, since it writes his fractional integration and differentiation operator in a united form for the cases $\omega(1-0) = 0$ and $\omega(1-0) = +\infty$ and even for $\omega \equiv 1$ when L_ω becomes the identical operator.

Now, we introduce some classes $\Omega(\mathbb{D})$ and $\widetilde{\Omega}(\mathbb{D})$ of parameter-functions ω, for which the operator L_ω of (1.2) can be written in some simplified forms.

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Definition 1.1

1°. $\Omega(\mathbb{D})$ is the class of all positive, strictly decreasing, continuously differentiable in $[0, 1)$ functions ω, such that $\omega(0) = 1$ and for some $\alpha > 0$

$$\omega(t) \leq O(1-t)^{\alpha} \quad \text{as} \quad t \to 1-0. \tag{1.3}$$

2°. $\widetilde{\Omega}(\mathbb{D})$ is the set of all functions $\omega(t)$ in $[0, 1)$, such that:

(i) $\omega > 0$ and is continuous and non-decreasing in $[0, 1)$,

(ii) $\omega(0) = 1$ and $(1-t)\omega(t) \to 0$ as $t \to 1-0$,

(iii) ω satisfies the Lipschitz condition with $\lambda_t \in (0, 1]$ at all points $t \in [0, 1)$.

Lemma 1.1

1°. *Let $\omega \in \Omega(\mathbb{D})$, and let u be a subharmonic function in $|z| < R \leq +\infty$. Then, the function $L_\omega u$ of the form (1.2) coincides with*

$$L_\omega u(z) \equiv -\int_0^1 u(tz)d\omega(t) \tag{1.4}$$

almost everywhere in $|z| < R$.

2°. *Let $\omega \in \widetilde{\Omega}(\mathbb{D})$, and let $u(z)$ be a harmonic function in $|z| < R \leq +\infty$. Then, the function $L_\omega u$ of the form (1.2) in $|z| < R$ coincides with*

$$L_\omega u(z) \equiv u(0) + L_{\omega_1} U(z), \tag{1.5}$$

where L_{ω_1} is of the form (1.4) and is applied to the harmonic function

$$U(z) = |z|\frac{\partial}{\partial|z|}u(z), \quad \text{and} \quad \omega_1(t) = \int_t^1 \frac{\omega(x)}{x}dx. \tag{1.6}$$

Proof

1°. By (1.1), $\omega(t) = p(t) - tp'(t)$. Thus, for any $z = re^{i\varphi}$ with $|z| = r < R$

$$\int_0^r L_\omega u(xe^{i\varphi})dx = -\int_0^1 \left(\int_0^r u(txe^{i\varphi})dx\right)d\big[p(t) - tp'(t)\big], \tag{1.7}$$

where the integrals are absolutely convergent. Denoting

$$J(te^{i\varphi}) = tp'(t)\int_0^r u(txe^{i\varphi})dx, \quad 0 < t \le 1,$$

where the last integral obviously is a continuous function for $0 \le t \le 1$, with at most a logarithmic singularity at $t = 0$, observe that

$$tp'(t) = t\int_t^1 \frac{\omega(x)}{x^2}dx - \omega(t) = t\int_t^1 \frac{\omega'(x)}{x}dx.$$

Therefore by (1.1) and (1.3) we easily get $J(te^{i\varphi}) \to 0$ as $t \to 1-0$ and $t \to +0$. Consequently, by (1.7)

$$\int_0^r L_\omega u(xe^{i\varphi})dx = -\int_0^1\left[\int_0^r u(txe^{i\varphi})dx\right]dp(t) - \int_0^1 tp'(t)d\left[\int_0^r u(txe^{i\varphi})dx\right]$$
$$= -\int_0^1 p'(t)d\left[t\int_0^r u(txe^{i\varphi})dx\right] = -r\int_0^1 u(tre^{i\varphi})dp(t),$$

and our statement holds by differentiation.

2°. Integrating by parts, for any $z = re^{i\vartheta}$ with $|z| = r < R$ we get

$$-r\int_0^1 u(tre^{i\vartheta})dp(t) = ru(0) + r\int_0^r\left[\frac{\partial}{\partial t}u(te^{i\vartheta})\right]p\left(\frac{t}{r}\right)dt.$$

Hence

$$L_\omega u(re^{i\vartheta}) = u(0) + \frac{\partial}{\partial r}\left\{r\int_0^r\left[\frac{\partial}{\partial t}u(te^{i\vartheta})\right]p\left(\frac{t}{r}\right)dt\right\}$$
$$= u(0) + \int_0^r\left[\frac{\partial}{\partial t}u(te^{i\vartheta})\right]\left\{p\left(\frac{t}{r}\right) - \frac{t}{r}p'\left(\frac{t}{r}\right)\right\}dt,$$

and the equality $p(x) - xp'(x) = \omega(x)$ leads to formulas (1.5) and (1.6). □

Remark 1.1 The general integral operator L_ω (1.4) in particular cases becomes the classical integral operators of Riemann–Liouville, Hadamard [32], Erdélyi [24]–Kober [60] and many other operators.

It is easy to verify that in both cases $\omega \in \Omega(\mathbb{D})$ and $\omega \in \widetilde{\Omega}(\mathbb{D})$ the application of the operator L_ω to a holomorphic in a disc $|z| < R \leq +\infty$ function means multiplication of its Taylor series coefficients by the moments $\Delta_0 = 1$, $\Delta_k = k\int_0^1 x^{k-1}\omega(x)dx$, $k \geq 1$, i.e. if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \text{then} \quad L_\omega f(z) = \sum_{k=0}^{\infty} a_k \Delta_k z^k,$$

where the function $L_\omega f$ is holomorphic in the same disc, since $\lim_{k\to\infty} \sqrt[k]{\Delta_k} = 1$. Indeed,

$$\sqrt[k]{\Delta_k} \leq \sqrt[k]{k}\sqrt[k]{\int_0^1 \omega(x)dx} \to 1 \quad \text{as} \quad k \to \infty. \tag{1.8}$$

On the other hand, for any $\delta \in (0, 1)$

$$\sqrt[k]{\Delta_k} \geq \sqrt[k]{k\int_\delta^1 x^{k-1}\omega(x)dx} \geq \delta^{1-\frac{1}{k}}\sqrt[k]{\int_\delta^1 \omega(x)dx} \to \delta \quad \text{as} \quad k \to \infty, \tag{1.9}$$

and the passage $\delta \to 1-0$ gives $\liminf_{k\to\infty} \sqrt[k]{\Delta_k} \geq 1$.

This means that L_ω transforms the holomorphic in a neighborhood of the origin functions to functions of the same kind, and this mapping is one-to-one, since the converse transform means just a division of the Taylor coefficients by Δ_k, which again does not change the convergence radius.

Further, for $\omega(x) = (1-x)^\alpha/\Gamma(1+\alpha)$ $(-1 < \alpha < +\infty)$ the operator L_ω becomes the classical Riemann–Liouville fractional integrodifferentiation with integration over the complex interval $[0, z]$. Namely, for any $z = re^{i\vartheta}$ formulas (1.4) and (1.5) respectively take the forms

$$L_{\frac{(1-x)^\alpha}{\Gamma(1+\alpha)}} u(re^{i\vartheta}) = \frac{r^{-\alpha}}{\Gamma(\alpha)}\int_0^r (r-t)^{\alpha-1}u(te^{i\vartheta})dt, \quad 0 < \alpha < +\infty,$$

$$L_{\frac{(1-x)^\alpha}{\Gamma(1+\alpha)}} u(re^{i\vartheta}) = u(0) + \frac{r^{-\alpha}}{\Gamma(1+\alpha)}\int_0^r (r-t)^\alpha \frac{\partial}{\partial t}u(te^{i\vartheta})dt, \quad -1 < \alpha < 0.$$

The next theorem proves that the operator L_ω is a one-to-one mapping in some wider sets of functions in starshaped, regular domains.[1] Namely, we shall apply L_ω to delta-subharmonic functions, i.e., differences of two subharmonic functions. Note that the equality $u = v$ of two delta-subharmonic functions $u = u_1 - u_2$ and $v = v_1 - v_2$ means

[1] A domain $G \subset \mathbb{C}$ is said to be starshaped, if it contains the closed straight line interval $[0, z]$ along with any point $z \in G$. For the definition of regular domains, see [33], Section 2.6.2, also formula (0.5). For a simpler case, see [31], Chapter 1, formula (1.6), also Theorem 1.1.

the equality $u_1 + v_2 = v_1 + u_2$. Besides, the subharmonic functions are a generalization of $\log |f|$ of holomorphic functions f, while the delta-subharmonic functions are a generalization of $\log |f|$ of meromorphic functions f.

Theorem 1.1 *Let $D \subseteq \mathbb{C}$ be a star-shaped, regular domain. Then the following statements are true:*

1°. *Let $\omega \in \Omega(\mathbb{D})$ and let L_ω be defined by (1.4), then:*

- (i) *If u is a harmonic function in D, then also the function $L_\omega u$ is harmonic in G. Besides, $L_\omega u \equiv 0$ in D if and only if $u \equiv 0$ in D.*
- (ii) *If u is a subharmonic function in D, with an associated Riesz measure ν such that $\min_{\zeta \in \overline{supp\ \nu}} |\zeta| = d_0 > 0$, then the function $L_\omega u$ is continuous and subharmonic in D.*
- (iii) *If u and v are delta-subharmonic in D and the supports of their Riesz measures are distanced from the origin by some $d_0 > 0$, then $L_\omega u$ and $L_\omega v$ are delta-subharmonic in D and $L_\omega u \equiv L_\omega v$ in D if and only if $u \equiv v$ in D.*

2°. *Let $\omega \in \widetilde{\Omega}(\mathbb{D})$ and let L_ω be that defined by (1.5) and (1.6).*

- (i′) *If u is a harmonic function in D, then also the function $L_\omega u$ is harmonic in D. Besides, $L_\omega u \equiv 0$ in D if and only if $u \equiv 0$ in D.*
- (ii′) *If u is a subharmonic function in D, with an associated Riesz measure ν such that $\min_{\zeta \in \overline{supp\ \nu}} |\zeta| = d_0 > 0$, then the function $L_\omega u$ is continuous and superharmonic in $D \setminus supp\ \nu$.*

Proof

1°. (i) If u is a real, harmonic function in D, then the function $f = u + iv$, where v is the harmonic conjugate of u, is holomorphic in D, and it is easy to verify the validity of the Cauchy–Riemann polar equations for $L_\omega f$ at any point $z \in D$. Thus, $L_\omega f$ is holomorphic and $L_\omega u$ is harmonic in D. Further, evidently f is holomorphic in a neighborhood $|z| < \rho$ of the origin, where also $L_\omega f$ is holomorphic, and the identity $L_\omega u \equiv \mathrm{Re}\, L_\omega f \equiv 0$ in $|z| < \varepsilon$ implies $f \equiv iC$, and $u \equiv 0$ in D.

(ii) Assuming that u is subharmonic in D, ν is its associated Borel measure and $\min\{|\zeta| : \zeta \in \overline{\mathrm{supp}\nu}\} = d_0 > 0$, observe that for any $\delta \in (0, 1)$ and $R \in (0, +\infty)$ the Riesz representation is true in the domain $\delta D_R = \{\delta z : z \in D, \ \ |z| < R\}$:

$$u(z) = -\iint_{\delta D_R} G(z, \zeta) d\nu(\zeta) + \frac{1}{2\pi} \int_{\partial \delta D_R} u(s) \frac{\partial G(s, z)}{\partial n} ds, \quad z \in \delta D_R,$$

where G is the Green function of the domain δD_R, $\partial/\partial n$ is differentiation along the outer normal and ds is the curve length element (see, e.g. [31], Ch. I, Sec. 2). Since ν is bounded in $\overline{\delta D_R}$, we can write the latter formula also in the form

$$u(z) = \iint_{\delta D_R} \log\left|1 - \frac{z}{\zeta}\right| d\nu(\zeta) + U(z) \equiv P^*(z) + U(z), \quad z \in \delta D_R, \tag{1.10}$$

where P^* is subharmonic and U is harmonic in δD_R, and hence also $L_\omega U$ is harmonic in δD_R. Besides, P^* is harmonic in $\delta D_R \setminus \mathrm{supp}\ \nu$, since its integral is absolutely and uniformly convergent inside any compact $\mathcal{K} \subset \delta D_R \setminus \mathrm{supp}\ \nu$ due to the obvious inequality $\big|\log|1 - z/\zeta|\big| \le M_1 < +\infty$ in $\mathcal{K}$, where M_1 is a constant depending on the distance from $\mathcal{K}$ to $\delta D_R \setminus \mathrm{supp}\ \nu$.

Assuming now that $\mathcal{K} \subset \delta D_R$ is any compact, we shall prove that the function $L_\omega P^*$ is continuous in $\mathcal{K}$ and

$$L_\omega P^*(z) = \iint_{\zeta \in \delta D_R} L_\omega \log\left|1 - \frac{z}{\zeta}\right| d\nu(\zeta), \quad z \in \delta D_R. \tag{1.11}$$

To this end, observe that for any $\zeta \in \delta D_R$, $|\zeta| \ge d_0 > 0$, the function

$$J(z) \equiv L_\omega \log\left|1 - \frac{z}{\zeta}\right| = -\int_0^1 \log\left|1 - \frac{tz}{\zeta}\right| d\omega(t) \tag{1.12}$$

is continuous in δD_R. Indeed, if the compact $\mathcal{K}$ does not intersect with the infinite interval $\ell_\zeta \equiv \{z : \arg z = \arg \zeta, \ \ |\zeta| \le |z| < +\infty\}$, then $\log\big|1 - tz/\zeta\big| \le M_2 < +\infty$ $(0 \le t \le 1)$, where M_2 is a constant depending solely on the distance from $\mathcal{K}$ to the mentioned interval. Hence, J is harmonic in $\mathcal{K}$ and, consequently, in $\delta D_R \setminus \ell_\zeta$. For proving the continuous extension of J to ℓ_ζ, by integration by parts we get

$$L_\omega \log\left|1 - \frac{z}{\zeta}\right| = -\mathrm{Re} \int_0^1 \frac{\omega(t)}{\zeta/z - t} dt = -\mathrm{Re} \int_0^1 \frac{\omega(t)}{\lambda - t} dt, \quad \lambda = \frac{\zeta}{z}, \tag{1.13}$$

where the last Cauchy -type integral is understood in the sense of its principal value for $\lambda \in [0, 1]$. When a complex λ tends to a point of $(0, 1]$, then $z = \zeta/\lambda$ tends to a point of ℓ_ζ, and the continuity of the Cauchy- type integral when λ

crosses l_ζ holds by the properties of $\omega \in \Omega(\mathbb{D})$ and the well-known properties of the Cauchy-type integrals (see, eg., [29], Sec. 4.2, 4.4, 8.1). So, the function $L_\omega \log|\zeta - z|$ is continuous in δD_R and harmonic in $\delta D_R \setminus \ell_\zeta$. Besides, $L_\omega \log|\zeta - z|$ is subharmonic in δD_R, which is easy to verify by applying L_ω to both sides of the inequality for $\log|\zeta - z|$, its integral mean and changing the integration order.

To prove formula (1.11), observe that this formula holds at least in $|z| < d_0$ by applying L_ω to P^*, due to the absolute convergence of its integral inside $|z| < d_0$. On the other hand, P^* is harmonic outside of the support of ν, and hence, $L_\omega P^*$ is harmonic in the star-shaped domain $D_R \setminus \bigcup_{\zeta \in \overline{\text{supp}\,\nu}} \ell_\zeta$, and the right-hand side integral is harmonic in the same domain and continuous in δD_R, due to its absolute and uniform convergence of its integral inside the mentioned domain and the already proved properties of $L_\omega \log|\zeta - z|$. Hence, formula (1.11) is true in $D_R \setminus \bigcup_{\zeta \in \overline{\text{supp}\,\nu}} \ell_\zeta$ by the uniqueness of harmonic function, it is true also in the whole δD_R, where P^* has a continuous extension. Thus, the proof of the statement (ii) is complete by the arbitrariness of δ and R.

(iii) If u and v are subharmonic in D and $L_\omega u \equiv L_\omega v$ in D, then for u and v the Poisson-Jensen formula is true in the finite, regular, starshaped domain δD. Besides, in δD, there are decompositions $u = U + P_1$ and $v = V + P_2$, where U and V are harmonic functions and $P_{1,2}$ are Green potentials. Consequently, $L_\omega\big(U - V\big) \equiv L_\omega\big(P_2 - P_1\big)$ in δD, where $L_\omega\big(U - V\big)$ is a harmonic function, while $L_\omega\big(P_2 - P_1\big)$ is not harmonic in δD for δ close enough to 1 and the associated measures of u and v are different. So, $P_1 \equiv P_2$. Hence, $L_\omega\big(U - V\big) \equiv L_\omega\big(P_2 - P_1\big) \equiv 0$, which implies $U \equiv V$ in δD by the already proved statement (ii), and consequently $u \equiv v$ in δD. Exhausting D by the domains δD, where we let $\delta \to 1 - 0$, we get $u \equiv v$ in the whole D. Further, for two delta-subharmonic functions $u = u_1 - u_2$ and $v = v_1 - v_2$, the identity $u \equiv v$ is understood in the sense that $u_1 + v_2 \equiv v_1 + u_2$. Hence, the identity $L_\omega u \equiv L_\omega v$ means $L_\omega\big(u_1 + v_2\big) \equiv L_\omega\big(v_1 + u_2\big)$, where $u_1 + v_2$ and $v_1 + u_2$ are subharmonic in D. Consequently $u = u_1 - u_2 \equiv v_1 - v_2 = v$, i.e. $u \equiv v$.

2°. (i′) The proof is the same as that of the statement 1°(i).

(ii′) The proof is almost the same as that of 1°(ii). Therefore, we show only the differences. Applying the operator of the form (1.5)–(1.6) to $\log|1 - z/\zeta|$, we get the same formula (1.13) with the only differences that $\omega(t) \in \widetilde{\Omega}(\mathbb{D})$ and the last integral is with the sign "+." This excludes continuity at the point $z = \zeta$ $(\lambda = 1)$, since $\omega(t)$ does not vanish as $t \to 1 - 0$. The rest of the proof differs from that of 1°(ii) only by a change of the domain δD_R by $\delta D_{R,\varepsilon} = \delta D_R \setminus \bigcup_{\zeta \in \text{supp}\,\nu}\{\zeta : |z - \zeta| < \varepsilon\}$, where the superharmonicity of $L_\omega P^*$ in D is true due to the mentioned integral sign difference, and then letting $\delta \to 1 - 0$ and $\varepsilon \to +0$. □

The next lemma gives a decomposition of the operator L_ω.

Lemma 1.2 *Let $\omega_{1,2} \in \Omega(\mathbb{D})$ and additionally, let $\omega_2(x) \equiv 1$ for $0 \le x \le \varepsilon < 1$ with a fixed ε. Then*

$$\omega_3(x) \equiv -\int_x^1 \omega_2\Big(\frac{x}{t}\Big)d\omega_1(t) = -\int_x^1 \omega_1\Big(\frac{x}{t}\Big)d\omega_2(t) \in \Omega(\mathbb{D}) \tag{1.14}$$

and

$$\omega_3'(x) = -\int_x^1 \omega_2'\Big(\frac{x}{t}\Big)\omega_1'(t)\frac{dt}{t} = -\int_x^1 \omega_1'\Big(\frac{x}{t}\Big)\omega_2'(t)\frac{dt}{t}. \tag{1.15}$$

Besides, if a function u is subharmonic in a starshaped domain D, and the associated Borel measure of u is supported in a ring $0 < d_0 \le |\zeta| < 1$, then

$$L_{\omega_3}u(z) = L_{\omega_1}L_{\omega_2}u(z) = L_{\omega_2}L_{\omega_1}u(z), \quad z \in D. \tag{1.16}$$

Proof The second equality in (1.14) holds by an integration by parts and a simple change of variable. Further, the integrals in (1.14) are uniformly convergent with respect to $x \in [0, 1]$, and hence $\omega_3(0) = 1$. If x sufficiently close to 1, then by (1.3)

$$\omega_3(x) = \int_x^1 \omega_2(t)d\omega_1\Big(\frac{x}{t}\Big) \le \omega_1(x) \le O(1-x)^{\alpha_1},$$

where $\alpha_1 > 0$ is that of (1.3) for ω_1. Further, changing the variable as $\lambda = x/t$, by a well-known differentiation formula we get equalities (1.15):

$$\omega_3'(x) = -\omega_2\Big(\frac{x}{\lambda}\Big)\omega_1'(\lambda)\Big|_{\lambda=x}^{1} - \int_x^1 \omega_2'\Big(\frac{x}{\lambda}\Big)\omega_1'(\lambda)\frac{d\lambda}{\lambda} = -\int_x^1 \omega_2'\Big(\frac{x}{\lambda}\Big)\omega_1'(\lambda)\frac{d\lambda}{\lambda}.$$

Hence, $\omega_3' < 0$ in $[0, 1)$ and is continuous in $[0, 1]$, and the equalities (1.14) hold. At last, the equalities (1.16) are easy to verify by (1.15). □

As we have seen, to get a holomorphic in $\mathbb{D}$ function, which becomes f after the application of L_ω, it is just necessary to divide the Taylor coefficients of f by Δ_k. This is the way in which the M.M. Djrbashian Cauchy and Schwarz type ω-kernels are introduced in $\mathbb{D}$:

$$C_\omega(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k}, \quad \text{and} \quad S_\omega(z) = 2C_\omega(z) - 1, \quad z \in \mathbb{D}, \tag{1.17}$$

where

$$\Delta_0 \equiv 1. \quad \Delta_k \equiv k\int_0^1 x^{k-1}\omega(x)dx, \quad k \geq 1, \tag{1.18}$$

or

$$\Delta_k = -\int_0^1 x^k d\omega(x) \ \ (k \geq 0), \quad \text{if } \ \omega(0) = 1 \ \text{ and } \ \omega(1) = 0. \tag{1.19}$$

Note that the functions C_ω and S_ω are holomorphic in $\mathbb{D}$ because of the relations (1.8) and (1.9), besides for any $\alpha \in (-1, +\infty)$

$$C_{\frac{(1-x)^\alpha}{\Gamma(1+\alpha)}}(z) \equiv C_\alpha(z) = \frac{1}{(1-z)^{1+\alpha}}, \quad S_{\frac{(1-x)^\alpha}{\Gamma(1+\alpha)}}(z) \equiv S_\alpha(z) = 1 - \frac{2}{(1-z)^{1+\alpha}} \tag{1.20}$$

and

$$L_\omega C_\omega(z) = C_0(z) = \frac{1}{1-z}, \quad L_\omega S_\omega(z) = S_0(z) = \frac{1+z}{1-z}.$$

In this book, also an analog of the operator L_ω with infinite integration contour is used for constructing in a sense similar theory in the upper halfplane $G^+ = \{z = x + iy : 0 < y < +\infty\}$, where the Taylor series apparatus is replaced by that of the Laplace transform. It is natural to use the notation L_ω also for the new operator and again call it M.M. Djrbashian operator.

For any acceptable functions u in G^+, ω in $(0, +\infty)$ and $\omega_1(t) = \int_0^t \omega(\lambda)d\lambda$ we set

$$L_\omega u(z) \equiv \int_0^{+\infty} u(z + it)d\omega(t) \quad \text{or} \quad L_\omega u(z) \equiv L_{\omega_1}\Big(-\frac{\partial}{\partial y}u(z)\Big). \tag{1.21}$$

The way in which this operator acts to Laplace transforms is very similar to that in which the considered before L_ω was acting to Taylor series. Namely, if for some acceptable functions ω and μ in $(0, +\infty)$

$$f(z) = \int_0^{+\infty} e^{izt}d\mu(t), \ \ \text{then} \ \ L_\omega f(z) = \int_0^{+\infty} e^{izt}\Big(t\int_0^{+\infty} e^{-t\lambda}\omega(\lambda)d\lambda\Big)d\mu(t)$$

for any $z \in G^+$. The properties of this operator are more complicated than those of L_ω for the unit disc theory. Their study is given in those chapters of the book where they are used. We just notice that the introduced operator is a generalization of the Liouville fractional integrodifferentiation, which holds as a particular case. Namely, for an acceptable function $u(z) \equiv u(x+iy)$ given in G^+

$$L_{\frac{t^\alpha}{\Gamma(1+\alpha)}} u(z) = \frac{1}{\Gamma(\alpha)} \int_y^{+\infty} (t-y)^{\alpha-1} u(x+it)dt, \quad 0 < \alpha < +\infty,$$

$$L_{\frac{t^\alpha}{\Gamma(1+\alpha)}} u(z) = L_{\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}} \Big(\frac{\partial}{\partial y} u(z) \Big), \quad -1 < \alpha < 0.$$

Also, we introduce a Cauchy-type kernel C_ω in G^+, which again is natural to call M.M. Djrbashian kernel. To introduce this kernel, we give some definitions.

Definition 1.2 $\Omega_\alpha(G^+)$ $(-1 \le \alpha < +\infty)$ is the class of functions ω given in $[0, +\infty)$ and such that:

(i) $\omega \nearrow$ (is non-decreasing) in $(0, +\infty)$, $\omega(0) = \omega(+0)$ and there exists a sequence $\delta_k \downarrow 0$ such that $\omega(\delta_k) \downarrow$ (is strictly decreasing);
(ii) $\omega(t) \asymp t^{1+\alpha}$ for $\Delta_0 \le t < +\infty$ and some $\Delta_0 \ge 0$.

Note that $f \asymp g$ means that $m_1 f \le g \le m_2 f$ for some constants $m_{1,2} > 0$. Evidently, if $\omega \in \Omega_\alpha(G^+)$ $(\alpha \ge -1)$, then (ii) is true for any $\Delta \in (0, \Delta_0]$.

Definition 1.3 $\Omega_\alpha^{\mathfrak{N}}(G^+)$ is the set consisting of $\omega \equiv 1$ and all decreasing, continuous functions $\omega > 0$ in $(0, +\infty)$, such that

$$\omega(x) \asymp x^\alpha \quad \text{for some} \quad -1 < \alpha < 0 \quad \text{and any} \quad x \ge \Delta_0 > 0,$$

where Δ_0 is a fixed number. Besides, we set

$$\omega_1(x) = \int_0^x \omega(t)dt < +\infty, \quad 0 < x < +\infty.$$

Assuming that $\omega \in \Omega_\alpha(G^+)$ $(\alpha \ge -1)$ or $\omega \in \Omega_\alpha^{\mathfrak{N}}(G^+)$ $(-1 < \alpha < 0)$, we define

$$C_\omega(z) = \int_0^{+\infty} e^{itz} \frac{dt}{I_\omega(t)}, \quad I_\omega(t) = t \int_0^{+\infty} e^{-tx} \omega(x)dx, \quad z \in G^+, \tag{1.22}$$

where we may write

$$I_\omega(t) = \int_0^{+\infty} e^{-tx} d\omega(x), \quad \text{if} \quad \omega(0) = 0. \tag{1.23}$$

Note that, being an obvious generalization of the ordinary Cauchy kernel in the one-dimensional case of $\mathbb{D}$, the ω-kernel C_ω was first used A.H. Karapetyan in [57], where it was constructed in the multidimensional case of tube domains.

Further, note that for $\omega \in \Omega_\alpha(G^+)$ the function C_ω and for $\Omega_\alpha^{\mathfrak{N}}(G^+)$ the functions C_ω and C_{ω_1} are holomorphic in G^+. Indeed, for any $k \ge 1$ the integral of C_ω uniformly converges in $G^+_{\delta_k} = \{z : \operatorname{Im} z > \delta_k\}$ because of the estimate

$$\left|I_\omega(t)\right| \ge \left|\int_{\delta_k}^{\delta_{k-1}} e^{-tx} d\omega(x)\right| \ge e^{-t\delta_k}\left|\omega(\delta_{k-1}) - \omega(\delta_k)\right| > 0, \quad k \ge 1,$$

and the same estimate with ω_1. Besides, one can see that for any $\alpha > -1$

$$C_{\frac{t^\alpha}{1+\alpha}}(z) \equiv C_0(z) = \frac{1}{(-iz)^{1+\alpha}} \quad \text{and} \quad L_\omega C_\omega(z) = C_0(z) = \frac{1}{-iz}, \quad z \in G^+. \tag{1.24}$$

1.2 Evaluation of M.M. Djrbashian C_ω-Kernels

This section gives some useful asymptotic estimates of C_ω-kernels in the unit disc and in the halfplane.

As M.M. Djrbashian often stated, one of the most significant problems related to his factorization theory is the evaluation of the C_ω-kernels. The main assumption was that under some additional conditions on the behavior of the parameter-functions ω in $(0, 1)$ or in $(0, +\infty)$, the following estimates have to be true:

$$|C_\omega(z)| \le \frac{M}{|(1-z)^2\omega'(|z|)|} \quad (z \in \mathbb{D}), \quad |C_\omega(z)| \le \frac{M}{|z^2\omega'(\operatorname{Im} z)|} \quad (z \in G^+) \tag{1.25}$$

which are natural to expect because of the equalities (1.20) and (1.24).

Here, in the first subsection, we use a united evaluation method to prove (1.25) for both kernels, which we use under some conditions in which the derivative of ω in $(0, 1)$ (or $(0, +\infty)$) decreases as $x \to 1 - 0$ (or $x \to +0$) not more rapidly than the function $(1 - x)^\alpha$ (or x^α). The found estimates are exact on the positive radius $z = r \in (0, 1)$ and on the imaginary half-axis $z = iy$, $y \in (0, +\infty)$.

In the second subsection, the C_ω-kernels are evaluated for certain scales of ω, which are exponentially decreasing as $x \to 1 - 0$ (or $x \to +0$). These estimates *differ* from (1.25), though they also are exact on the positive radius of $\mathbb{D}$ and on the imaginary half-axis of G^+.

1.2.1 ω Decreases Not More Rapidly than a Power Function

We start by the "model" argument on evaluation of the C_ω-kernel for the halfplane. Beforehand some necessary technical apparatus is to be prepared.

We shall use the below easily verifiable inequalities, with $\nearrow$ and $\searrow$ meaning the non-decreasing and non-increasing of a function. Namely, for any monotone in $(0, +\infty)$ function $\varphi > 0$

$$\int_{1/y}^{+\infty} e^{-ty}\varphi(t)dt \Big/ \int_0^{1/y} e^{-ty}\varphi(t)dt \begin{cases} \geq M_1, & \text{if} \quad \varphi \nearrow \\ \leq M_2, & \text{if} \quad \varphi \searrow \end{cases}, \quad y > 0, \tag{1.26}$$

where $M_{1,2}$ are some positive constants. Besides, for small enough values $v > 0$

$$\int_{1/v}^{+\infty} e^{-tv}\varphi(t)d[t] \Big/ \int_{+0}^{1/v} e^{-tv}\varphi(t)d[t] \begin{cases} \geq M_3, & \text{if} \quad \varphi(t) \nearrow \\ \leq M_4, & \text{if} \quad \varphi(t) \searrow \end{cases}, \tag{1.27}$$

where $[t]$ means the integral part of t and $M_{3,4} > 0$ are some constants. We shall often use also the inequality

$$\frac{t}{1+t} < 1 - e^{-t} < \frac{4}{3}\frac{t}{1+t}, \quad 0 < t < +\infty.$$

The main tool of this section is the following, perhaps known statement.

Lemma 1.3 *Let $\varphi > 0$ be a function defined in $(0, +\infty)$.*

1°. *If $\varphi(t) \nearrow$ but $t^{-\alpha}\varphi(t) \searrow$ in $(0, +\infty)$ for some $\alpha > 0$, then*

$$\int_0^{+\infty} e^{-tx}\varphi(t)\,dt \asymp \frac{\varphi(1/x)}{x}, \quad 0 < x < +\infty. \tag{1.28}$$

2°. *If $\varphi(t) \nearrow$ but $(1 - e^{-t})^{-\alpha}\varphi(t) \searrow$ for some $\alpha > 0$, then for small enough $v > 0$*

$$\int_{+0}^{+\infty} e^{-tv}\varphi(t)\,d[t] \asymp \frac{\varphi(1/v)}{v}. \tag{1.29}$$

3°. *If $\varphi(t) \searrow$ in $(0, +\infty)$ but $t^\delta\varphi(t) \nearrow$ or $(1 - e^{-t})^\delta\varphi(t) \nearrow$ for a $\delta \in (0, 1)$, then (1.28) and (1.29) are true, respectively.*

Proof

1°. Evidently

$$\int_0^{+\infty} e^{-tx}\varphi(t)dt \geq x^\alpha \varphi(1/x) \int_0^{1/x} e^{-tx} t^\alpha dt = \frac{\varphi(1/x)}{x} \int_0^1 e^{-\lambda}\lambda^\alpha d\lambda.$$

On the other hand, by (1.26)

$$\int_0^{+\infty} e^{-tx}\varphi(t)dt \leq \left(1+\frac{1}{M_1}\right) \int_{1/x}^{+\infty} e^{-tx}\varphi(t)dt \leq \left(1+\frac{1}{M_1}\right)\frac{\varphi(1/x)}{x} \int_1^{+\infty} e^{-\lambda}\lambda^\alpha d\lambda.$$

2°. If $v > 0$ is small enough, then

$$\begin{aligned}
\int_{+0}^{+\infty} e^{-tv}\varphi(t)d[t] &\geq \int_{+0}^{1/v} e^{-tv}\varphi(t)d[t] \geq M'\varphi(1/v) \int_{+0}^{1/v} e^{-tv} \left(\frac{t}{1+t}\right)^\alpha d[t] \\
&> M''\varphi(1/v) \int_{1/2}^{1/v} e^{-tv} d[t] \\
&= M'' e^{-v} \varphi(1/v) \left[1 + e^v + \cdots + e^{-v([1/v]-1)}\right] \\
&= M'' e^{-v} \varphi(1/v) \frac{1 - e^{-v[1/v]}}{1 - e^{-v}} > M''' \frac{\varphi(1/v)}{v}.
\end{aligned}$$

On the other hand, if $v > 0$ is small enough, then by (1.27)

$$\begin{aligned}
\int_{+0}^{+\infty} e^{-tv}\varphi(t)d[t] &\leq \left(1 + \frac{1}{M_3}\right) \int_{1/v}^{+\infty} e^{-tv}\varphi(t)d[t] \\
&\leq \left(1 + \frac{1}{M_3}\right) \left(1 - e^{-1/v}\right)^{-\alpha} \varphi(1/v) \int_{1/v}^{+\infty} e^{-tv}(1 - e^{-t})^\alpha d[t] \\
&\leq M^{IV} \varphi(1/v) \int_{1/v}^{+\infty} e^{-tv} d[t] \leq M^V \frac{\varphi(1/v)}{v}.
\end{aligned}$$

3°. The proof is similar and even more simple. □

We start the evaluations by the case of the halfplane kernel C_ω of (1.22).

Theorem 1.2 *Let $\omega > 0$ be a non-decreasing, continuously differentiable function in $(0, +\infty)$, such that*

1°. $\omega(+0) = 0$ *and* $\lim_{x\to+\infty} e^{-\varepsilon x}\omega(x) = 0$ *for any* $\varepsilon > 0$,

2°. (i) $\omega'(x) \nearrow$ *but* $x^{-\alpha}\omega'(x) \searrow$ *for some* $\alpha > 0$ *or, alternatively,*

(ii) $\omega'(x) \searrow$ *but* $x^{\delta}\omega'(x) \nearrow$ *for some* $\delta \in (0, 1)$,

then

$$C_\omega(iy) \asymp \frac{1}{y^2\omega'(y)}, \quad 0 < y < +\infty. \tag{1.30}$$

If along with 1° *and* 2°(i) *we have*

3°. $\omega'(+\infty) = +\infty$ *and* $x^{-1}\omega'(x) \nearrow$ *or* $x^{-1}\omega'(x) \searrow$ *but* $x^{-\delta}\omega'(x) \nearrow$ *for some* $\delta \in (0, 1)$, *then there is a constant* M $(\equiv M_\omega) > 0$ *for which*

$$|C_\omega(z)| \le \frac{M}{|z|^2\omega'(y)}, \quad z = x + iy \in G^+. \tag{1.31}$$

Proof If 2°(i) is true, then by (1.22)–(1.23) and (1.28)

$$C_\omega(iy) \asymp \int_0^{+\infty} e^{-yt}\frac{t}{\omega'(1/t)}dt, \quad 0 < y < +\infty. \tag{1.32}$$

But $t[\omega'(1/t)]^{-1} \nearrow$ and $x^{-\alpha}\omega'(x) \searrow$. Hence

$$\frac{1}{(1/t)^{-\alpha}\omega'(1/t)} = \frac{t^{-\alpha}}{\omega'(1/t)} = t^{-(1+\alpha)}\frac{t}{\omega'(1/t)} \searrow .$$

Thus, the function $t[\omega'(1/t)]^{-1}$ satisfies the condition 1° of Lemma 1.3, and (1.30) follows from (1.32) by (1.28). Under the assumption 2°(i), (1.30) is proved similarly.

For proving (1.31), observe that the function $\varphi(t) = t^n\omega'(t)$ satisfies the condition 1° of Lemma 1.3 for any $n \ge 0$. Hence, an integration by parts gives

$$C_\omega(z) = \frac{1}{(iz)^2}\left\{\int_0^{+\infty} e^{izt}\frac{\int_0^{+\infty} e^{-\sigma t}\sigma^2\omega'(\sigma)d\sigma}{\left[\int_0^{+\infty} e^{-\sigma t}\omega'(\sigma)d\sigma\right]^2}dt\right.$$
$$\left.-\,2\int_0^{+\infty} e^{izt}\frac{\left[\int_0^{+\infty} e^{-\sigma t}\sigma\omega'(\sigma)d\sigma\right]^2}{\left[\int_0^{+\infty} e^{-\sigma t}\omega'(\sigma)d\sigma\right]^3}dt\right\} \equiv \frac{1}{(iz)^2}\{I_1 - I_2\},$$