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Preface

The theory of partial differential equations (PDE) is a vast and exciting area of mathematics. PDE started with applications in mind. In the late eighteenth to early nineteenth centuries, through seminal works by giants such as d'Alembert, Euler, Lagrange, Laplace, Poisson, and Fourier, the three major linear PDE, namely, the wave equation, the Laplace (and Poisson) equation, and the heat equation, have been in place to study vibration phenomena, potential theory, and heat flow, respectively. These equations formed the prototype for hyperbolic, elliptic, and parabolic PDE, respectively. The contributions of PDE to mathematics, however, extended well beyond their ubiquitous applicability, about which plenty has already been said. The theory of PDE in fact developed hand in hand with many important areas of mathematics, such as complex analysis, harmonic analysis, functional analysis, and calculus of variations, and contributed significantly to other areas of mathematics such as differential geometry as well as theoretical physics.

The rigorous study of PDE probably started in the late nineteenth century. In the last decade of the nineteenth century, Poincaré completed a series of rigorous study of elliptic PDE, which played a fundamental role in the modern development of potential theory, spectral theory, and nonlinear analysis.

In the celebrated problems that Hilbert posed at the start of the twentieth century, 2 out of 23 were directly devoted to PDE, specifically nonlinear elliptic PDE, namely, Problem 19 which concerns the regularity of solutions and Problem 20 which concerns the existence of solutions and their variational properties.

For a fascinating account of the development of PDE, the reader is referred to the article "PDE in the 20th Century" by H. Brezis and F. Browder (Adv. Math. 135(1998), 76–144).

In this book, we focus on boundary value problems of second-order nonlinear elliptic partial differential equations and systems. In particular, we mainly concern ourselves with the existence of solutions, more specifically non-constant positive solutions, as well as the uniqueness, stability, and asymptotic behavior of such solutions. We concentrate on two major approaches to the study of existence of solutions, namely, the upper and lower solutions method and the topological degree method.

Theoretically, the upper and lower solutions method appears straightforward. It relies on comparison principles and the monotone iterative method. However, its successful application hinges on the construction of suitable upper and lower solutions, which often requires some degree of ingenuity. We illustrate this method in detail through many concrete examples. The topological degree method, on the other hand, involves the computation of fixed point indices, which is often technically highly demanding. A significant value of this book is the development of an effective framework for the computation of fixed point indices for the homogeneous Neumann boundary value problem for elliptic systems (Chap. 6). This framework bypasses the complicated estimates of eigenvalues and facilitates the computation of fixed point indices by simply determining the sign of the roots of some simple polynomials. This is based on the authors' work (Strategy and stationary pattern in a three-species predator-prey model, J. Differential Equations 200(2004), 245–273).

A key feature of this book is the thorough treatment of these two methods. For the upper and lower solutions method, we deal with single equations (Chap. 3) and systems (Chap. 4). We delve into different boundary conditions, different equation types, and different domains. More importantly, we strive to illustrate the method in many concrete examples. On the other hand, for the topological degree method, we focus on the theory of topological degree in cones and its applications to Dirichlet boundary value problems (Chap. 5), while Neumann boundary value problems are treated separately (Chap. 6). For the topological degree method, the approach of illustration using concrete examples persists.

To provide a foundation for the two methods, we included a chapter on eigenvalue problems of the second-order linear elliptic operators (Chap. 2). We then extended the treatment to p-Laplace equations and systems in Chap. 7.

We are very much aware that our focus on eigenvalue problem, upper and lower solutions method, and topological degree method clearly does not do full justice to the theory of nonlinear elliptic PDE, as many other methods such as variational methods are not given due coverage. We must admit that we have not been motivated by comprehensiveness, and simply wished to illustrate some methods that we have found useful.

As mentioned repeatedly above, the illustration of the theories and methods using concrete examples is core to the design of this book. Because of the authors' own preference and previous work, examples from mathematical biology are often chosen. However, we note emphatically that this is not a book on mathematical biology. Indeed, we have not included the biological background of the examples (to which we refer the reader to the original papers contained in the bibliography). These examples are chosen ultimately for their merit in illustrating the theories and methods.

This book is primarily intended as a textbook for intermediate to advanced graduate students who have already had an introductory course on PDE and some familiarity with functional analysis and nonlinear functional analysis. We have summarized briefly some basic results of Sobolev spaces and nonlinear functional analysis, and basic theory of elliptic equations, in the two appendices. These two appendices provide a good gauge of the prerequisites for this book.

While we have not intended to make this book entirely self-contained, we have attempted to include some basic results. Some of these materials are based on and rewritten from *Nonlinear Elliptic Partial Differential Equations* (Science Press, Beijing, 2010) by M.X. Wang [187] (published in Chinese).

With the focused theoretical content, the ample illustrative examples and inclusion of exercises, we hope that this book will be of use to graduate students.

While working on this book, the first author was partially supported by NSFC Grants 11371113, 11771110, and 12171120. We would also like to take this opportunity to thank many colleagues and students, especially those at the Harbin Institute of Technology, who have provided feedback to drafts of this work.

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Contents

Chapter 1
Preliminaries

In this chapter we first lay down some notations, conventions and basic assumptions. Then, for later applications, we briefly review some basic theories and results of the calculus in Banach spaces and unconditional local extrema, which are elementary knowledge of variational methods. Finally, we give two applications which will be used in Chap. 7.

1.1 Notations, Conventions and Basic Assumptions

Given two sets $A, B \subset \mathbb{R}^n$, we use $\overline{A}$ to represent the closure of A, and use $d(A, B)$ or $\operatorname{dist}(A, B)$ to represent the distance between A and B. The notation $A \Subset B$ means that A is bounded and $\overline{A} \subset B$.

Given a set $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^m$, we use $f(A)$ to represent the image of A under f.

Given a set $A \subset \mathbb{R}^n$, we use $|A|$ to represent the measure of A.

For a given function u, we define the positive and negative parts of u by

$$u^+ = \max\{u, 0\}, \quad u^- = \min\{u, 0\}.$$

Clearly $u = u^+ + u^-$ and $|u| = u^+ - u^-$.

Let $k \geqslant 0$ be an integer and $0 \leqslant \beta < 1$ be a constant. We say that a domain $\Omega \subset \mathbb{R}^n$ is of *class* $C^{k+\beta}$, or Ω has a $C^{k+\beta}$ *boundary* $\partial\Omega$ if for each point $x_0 \in \partial\Omega$ there exist a neighborhood U of x_0 and a function $\Phi \in C^{k,\beta}(\overline{U})$ such that

(1) the inverse function Φ^{-1} exists and $\Phi^{-1} \in C^{k,\beta}(\Phi(\overline{U}))$, and
(2) if we set $y = (y_1, \ldots, y_n) = \Phi(x)$, then $\Phi(\partial\Omega \cap U) \subset \{y \in \mathbb{R}^n : y_n = 0\}$ and $\Phi(\Omega \cap U) \subset \{y \in \mathbb{R}^n : y_n > 0\}$.

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We say that a domain Ω has the *interior ball property* at the point $x \in \partial\Omega$, if there exists a ball B with radius $r > 0$ for which $B \subset \Omega$ and $\overline{B} \cap \partial\Omega = \{x\}$. If Ω has the interior ball property at each point $x \in \partial\Omega$, then we say that Ω has the *interior ball property*. When Ω is of class C^2, it must have the interior ball property. However, if Ω is only of class $C^{1+\alpha}$ with $0 < \alpha < 1$, then it may not have the interior ball property. For example, the curve $x_2 = |x_1|^{1+\alpha}$ does not have the interior ball property at the point (0, 0).

Let Ω be a bounded domain of $\mathbb{R}^n$, k be a positive integer and $p \geqslant 1$. The function space $W^{k,p}(\Omega)$ is the standard Sobolev space, and $C_0^k(\Omega)$ is the function space composed of k-times continuously differentiable functions with compact support in Ω. We denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ by $W_0^{k,p}(\Omega)$, and use the notation $L^p(\Omega)$ instead of $L_p(\Omega)$.

When the domain Ω is fixed, to simplify notations and save space, we will write

$$|\cdot|_{i+\alpha} = |\cdot|_{i+\alpha,\,\overline{\Omega}} = \|\cdot\|_{C^{i+\alpha}(\overline{\Omega})}, \quad i = 0, 1, 2,$$

$$\|\cdot\|_{j,\,p} = \|\cdot\|_{j,\,p,\,\Omega} = \|\cdot\|_{W^{j,p}(\Omega)}, \quad j = 1, 2; \quad \|\cdot\|_p = \|\cdot\|_{p,\,\Omega} = \|\cdot\|_{L^p(\Omega)},$$

$$\int_\Omega f = \int_\Omega f(x) = \int_\Omega f(x)\mathrm{d}x \quad \text{for } f \in L^1(\Omega).$$

Let X be a Banach space, and $f, g \in X$. We caution that the symbol $\|f, g\|_X$ may denote $\|f\|_X + \|g\|_X$ in some places and $\max\{\|f\|_X, \|g\|_X\}$ in others; the usage will be clear from the context.

Throughout this book, unless it is clearly stated otherwise, we shall adopt the following conventions (standing assumptions):

- All functions are real-valued, and we usually refer to a vector-function (and also a matrix of functions) briefly as a function.
- The symbol $f_k \to f$ means that f_k converges strongly to f, while $f_k \rightharpoonup f$ means that f_k converges weakly to f.
- The constant $0 < \alpha < 1$ may be different in different places.
- We use C, C' and C_i to represent the generic constants.
- $\Omega \subset \mathbb{R}^n$ is a bounded domain and is of class C^2, $\boldsymbol{n}$ is the outward normal vector of $\partial\Omega$, and $\partial_{\boldsymbol{n}}$ is the outer normal derivative $\frac{\partial}{\partial \boldsymbol{n}}$.
- $1 < p < \infty$ as a constant.
- $a_{ij}D_{ij} := \sum_{i,j=1}^n a_{ij}D_{ij}$, and $b_iD_i := \sum_{i=1}^n b_iD_i$.
- $u \in C^\alpha(\Omega)$ means that $u \in C^\alpha_{\mathrm{loc}}(\Omega)$, i.e., $u \in C^\alpha(\overline{\Omega}_0)$ for any subdomain $\Omega_0 \Subset \Omega$.

Throughout this book, unless it is clearly stated otherwise, we will make the following assumptions:

(A) The second order linear operators

$$\mathscr{L} = -a_{ij}(x)D_{ij} + b_i(x)D_i + c(x)$$

and

$$\mathscr{L}_d = -D_j\big(a_{ij}(x)D_i\big) + c(x)$$

are *strongly elliptic*, i.e., $a_{ij}(x) = a_{ji}(x)$ in Ω for all $1 \leqslant i, j \leqslant n$, and there exist positive constants χ and Λ such that

$$\chi|y|^2 \leqslant a_{ij}(x)y_i y_j \leqslant \Lambda|y|^2, \quad \forall\, y \in \mathbb{R}^n, \quad x \in \Omega.$$

Operators $\mathscr{L}$ and $\mathscr{L}_d$ are called the *non divergence* and *divergence* type, respectively.

(B) $a_{ij} \in C(\overline{\Omega})$, $b_i,\ c \in L^\infty(\Omega)$.

(C) The boundary operator $\mathscr{B}u = a\partial_{\boldsymbol{n}}u + b(x)u$, where

(i) $a = 0$, $b(x) \equiv 1$, or
(ii) $a = 1$, $b(x) \geqslant 0$ and $b \in C(\partial\Omega)$.

Moreover, whenever we use the L^p theory, our default assumptions are that Ω is of class C^2, $b \in C^1(\partial\Omega)$ and the condition **(B)** holds.

The following assumption will be used in some places.

(B$_\alpha$) $a_{ij},\ b_i,\ c \in C^\alpha(\overline{\Omega})$.

Whenever, we use the Schauder theory, our default assumptions are that Ω is of class $C^{2+\alpha}$, $b \in C^{1+\alpha}(\partial\Omega)$ and **(B$_\alpha$)** holds.

We define

$$\omega(R) = \max_{i,j} \sup_{\substack{|x-y|\leqslant R\\ x,\,y\in\Omega}} |a_{ij}(x) - a_{ij}(y)| \to 0 \quad \text{as} \quad R \to 0.$$

The function $\omega(R)$ is called the *modulus of continuity* of a_{ij}.

1.2 Calculus in Banach Spaces

Let X and Y be two Banach spaces, $U \subset X$ be an open set, and the mapping $f : U \to Y$ be continuous. Denote by $\mathscr{B}(X, Y)$ the set of bounded linear operators from X to Y.

1.2.1 Fréchet Derivative

Definition 1.1 We say that f is *Fréchet differentiable* at the point $x_0 \in U$ if there exists $\mathcal{A} \in \mathscr{B}(X, Y)$ such that

$$\|f(x_0 + y) - f(x_0) - \mathcal{A}y\| = o(r) \quad \text{as} \quad \|y\| \leqslant r \to 0.$$

In this case, the operator $\mathcal{A}$ is called the *Fréchet derivative* of f at x_0.

We list some basic properties of the Fréchet derivative $\mathcal{A}$.

(1) If $\mathcal{A}$ exists, then it must be unique; it is denoted by $f_x(x_0)$, $Df(x_0)$ or $f'(x_0)$;
(2) If $Df(x): x \in U \to \mathscr{B}(X, Y)$ is a continuous map, we say that $f \in C^1(U)$. Inductively, we may define C^k mappings for $k = 1, 2, \ldots$. A mapping $f \in C^k(U)$ means that

$$D^k f : U \to \mathscr{B}(X, \mathscr{B}(X, \cdots, \mathscr{B}(X, Y \underbrace{) \cdots))}_{k} \quad \text{is continuous in } U,$$

where $D^k f = D(D^{k-1} f)$;
(3) *Chain rule*: Let X, Y and Z be three Banach spaces, $U \subset X$ and $V \subset Y$ be open sets, and $f : U \to Y$ and $g : V \to Z$ be two mappings. Suppose that $x_0 \in U$, $y_0 = f(x_0) \in V$, and f and g are Fréchet differentiable at x_0 and y_0, respectively. Then the composite mapping $g \circ f$ is also Fréchet differentiable at x_0, and $(g \circ f)'(x_0) = g'(y_0) f'(x_0)$.

Lemma 1.1 *Assume that the mapping $f : X \to Y$ is C^1 and compact in a neighborhood of x_0. Then $f'(x_0)$ is also compact.*

Proof On the contrary, suppose that $\mathcal{A} = f'(x_0)$ is not compact. Then there exist $\{x_i\}$ with $\|x_i\| = 1$ and $\varepsilon > 0$ such that

$$\|\mathcal{A}x_i - \mathcal{A}x_j\| \geqslant \varepsilon, \quad \forall\, i \neq j.$$

Take $\delta > 0$ sufficiently small such that

$$\|f(x_0 + \delta x_i) - f(x_0) - \delta \mathcal{A} x_i\| \leqslant \varepsilon\delta/4, \quad \forall\, i.$$

Without loss of generality, suppose $x_0 = 0$. Then, for $i \neq j$,

$$\begin{aligned} \varepsilon\delta/2 &\geqslant \|f(\delta x_i) - f(\delta x_j) - \delta\mathcal{A}x_i + \delta\mathcal{A}x_j\| \\ &\geqslant \|\delta\mathcal{A}x_i - \delta\mathcal{A}x_j\| - \|f(\delta x_i) - f(\delta x_j)\| \\ &\geqslant \varepsilon\delta - \|f(\delta x_i) - f(\delta x_j)\|, \end{aligned}$$

i.e.,

$$\|f(\delta x_i) - f(\delta x_j)\| \geqslant \varepsilon\delta/2, \quad \forall\, i \neq j.$$

This is a contradiction to the compactness of f. □

We end this subsection by recalling the *mean value formula*: let U be an open convex set and $f \in C^1(U)$. Then, for any $x, x' \in U$,

$$\begin{aligned} f(x') - f(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx' + (1-t)x)\mathrm{d}t \\ &= \int_0^1 f_x(tx' + (1-t)x)\mathrm{d}t(x' - x). \end{aligned} \tag{1.1}$$

1.2.2 Gâteaux Derivative

Definition 1.2 Let $f : U \subset X \to Y$, and $x_0 \in U$. If, for any $h \in X$ satisfying $x_0 + th \in U$ when t is sufficiently small, the limit

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t}$$

exists, then we say that f is *Gâteaux differentiable* at x_0, and call such a limit the *Gâteaux derivative* of f at x_0 in the direction h, and denote it by $f_G(x_0)h$. If f is Gâteaux differentiable at each point of U, we say that f is Gâteaux differentiable in U.

If $f_G(x_0) \in \mathscr{B}(X, Y)$, we say that f has a (bounded, linear) Gâteaux derivative at the point x_0.

For the Gâteaux derivative, we have a similar chain rule to that for the Fréchet derivative. The following result concerns the relationship between the Fréchet and Gâteaux derivatives.

Theorem 1.1 *Let X and Y be two Banach spaces, $U \subset X$ be an open set, $x_0 \in U$ and $f : U \to Y$.*

(1) *If f is Fréchet differentiable at x_0, then f admits a (bounded, linear) Gâteaux derivative and $f'(x_0) = f_G(x_0)$, i.e., the Fréchet and Gâteaux derivatives of f are equal at x_0;*
(2) *If f has a (bounded, linear) Gâteaux derivative $f_G(x)$ in a neighborhood of x_0 and $f_G(x)$ is continuous at x_0, then f is Fréchet differentiable at x_0, and the two derivatives are equal.*

Proof

(1) By the assumption, it is easy to see that for any $h \in X$, $h \neq 0$, when $|t|$ is sufficiently small, we have

$$f(x_0 + th) - f(x_0) - tf'(x_0)h = \omega(x_0, th),$$

where $\omega(x_0, h)$ satisfies

$$\lim_{\|h\|\to 0} \frac{\|\omega(x_0, h)\|}{\|h\|} = 0.$$

Therefore,

$$\frac{f(x_0 + th) - f(x_0)}{t} - f'(x_0)h = \frac{\omega(x_0, th)}{t}.$$

As

$$\lim_{t\to 0} \left\| \frac{\omega(x_0, th)}{t} \right\| = \lim_{t\to 0} \frac{\|\omega(x_0, th)\|}{\|th\|} \|h\| = 0,$$

it follows that

$$\lim_{t\to 0} \frac{f(x_0 + th) - f(x_0)}{t} = f'(x_0)h.$$

This shows that f is Gâteaux differentiable at x_0, and has the Gâteaux derivative $f'(x_0)$.

(2) Since the Gâteaux derivative $f_G(x)$ is continuous at x_0, for any given $\varepsilon > 0$, there is a constant $\delta > 0$ such that, when $\|h\| < \delta$,

$$\|f_G(x_0 + h) - f_G(x_0)\| < \varepsilon. \tag{1.2}$$

We claim that, when $0 < \|h\| < \delta$,

$$\|f(x_0 + h) - f(x_0) - f_G(x_0)h\| \leqslant \varepsilon\|h\|, \tag{1.3}$$

from which it follows that f is Fréchet differentiable at x_0, and $f'(x_0) = f_G(x_0)$.

To see the claim, fix h such that $0 < \|h\| < \delta$. We may assume that $f(x_0 + h) - f(x_0) - f_G(x_0)h \neq 0$; otherwise, (1.3) holds evidently. By the Hahn-Banach theorem, there exists $\phi \in Y^*$ such that $\|\phi\| = 1$ and

$$\phi\big(f(x_0 + h) - f(x_0) - f_G(x_0)h\big) = \|f(x_0 + h) - f(x_0) - f_G(x_0)h\|. \tag{1.4}$$

Considering the function

$$\varphi(t) = \phi\big(f(x_0 + th)\big), \quad 0 \leqslant t \leqslant 1,$$

it is clear that $\varphi'(t) = \phi\big(f_G(x_0+th)h\big)$. By the mean value formula, there exists $\theta \in [0, 1]$ such that $\varphi(1) - \varphi(0) = \varphi'(\theta)$, i.e.,

$$\phi\big(f(x_0 + h) - f(x_0)\big) = \phi\big(f_G(x_0 + \theta h)h\big). \tag{1.5}$$

It follows from (1.4), (1.5) and (1.2) that

$$\begin{aligned} \|f(x_0 + h) - f(x_0) - f_G(x_0)h\| &= \phi\big(f(x_0 + h) - f(x_0) - f_G(x_0)h\big) \\ &= \phi\big(f_G(x_0 + \theta h)h - f_G(x_0)h\big) \\ &\leqslant \|\phi\| \cdot \|f_G(x_0 + \theta h) - f_G(x_0)\| \cdot \|h\| \\ &= \|f_G(x_0 + \theta h) - f_G(x_0)\| \cdot \|h\| \\ &\leqslant \varepsilon\|h\|. \end{aligned}$$

This completes the proof. □

1.3 Unconditional Local Extremum

In this book, we shall be making substantial reference to extreme value problems for functionals. However, we will not develop the variational method systematically, but simply cover the essential topics so as to achieve self-containment.

Definition 1.3 Let X be a Banach space and $U \subset X$. We say that a functional $f : U \to \mathbb{R}$ is *lower semi-continuous* (*weakly lower semi-continuous*) in U if $U \ni x_i \to x \in U$ ($U \ni x_i \rightharpoonup x \in U$) implies $\liminf_{i\to\infty} f(x_i) \geqslant f(x)$.

Analogously, a functional $f : U \to \mathbb{R}$ is said to be *upper semi-continuous* (*weakly upper semi-continuous*) in U if $U \ni x_i \to x \in U$ ($U \ni x_i \rightharpoonup x \in U$) implies $\limsup_{i\to\infty} f(x_i) \leqslant f(x)$.

Definition 1.4 Let X be a Banach space. We say that a functional $f : U \subset X \to \mathbb{R}$ achieves the *unconditional local minimum* (*unconditional local maximum*) at the point $x_0 \in U$ if there is a neighborhood $\Sigma(x_0)$ of x_0 such that, for all $x \in U \cap \Sigma(x_0)$,

$$f(x) \geqslant f(x_0) \quad \big(f(x) \leqslant f(x_0)\big).$$

Definition 1.5 Let X be a Banach space, $U \subset X$, and $f : U \to \mathbb{R}$ be a functional that is bounded from below. A sequence $\{x_i\}_{i=1}^{\infty} \subset U$ is called a *minimizing sequence* if $\lim_{i\to\infty} f(x_i) = \inf_{x\in U} f(x)$.

Theorem 1.2 *Let X be a Banach space and $U \subset X$ be an open set. Suppose that the functional f achieves an unconditional extremum at the point $x_0 \in U$, and that f is Gâteaux differentiable at x_0. Then, for any $h \in U$, $f_G(x_0)h = 0$.*

Proof We may assume that f achieves an unconditional local minimum at $x_0 \in U$. Then there is a neighborhood $\Sigma(x_0)$ of x_0 such that

$$f(x) \geqslant f(x_0), \quad \forall\, x \in U \cap \Sigma(x_0).$$

Given $h \in X$, we note that the function $F_h(t) := f(x_0 + th)$ is well defined for $|t|$ sufficiently small, and achieves its minimum at $t = 0$. Hence $F_h'(0) = 0$, and

$$f_G(x_0)h = \lim_{t\to 0} \frac{f(x_0 + th) - f(x_0)}{t} = \lim_{t\to 0} \frac{F_h(t) - F_h(0)}{t} = F_h'(0) = 0.$$

The proof is complete. □

The following theorem is a generalization of the Weierstrass theorem:

Theorem 1.3 *Let X be a Banach space, the set $U \subset X$ be weakly compact. Suppose that the functional $f : U \to \mathbb{R}$ is weakly lower semi-continuous in U. Then f is bounded from below and achieves an infimum in U, i.e., there exists $x_0 \in U$ such that*

$$f(x_0) = \inf_{x\in U} f(x).$$

This x_0 is called a minimizer *of f in U.*

Proof If $c := \inf_{x\in U} f(x) = -\infty$, then there exists $\{x_n\}_{n=1}^{\infty} \subset U$ such that $f(x_n) < -n$. Since U is weakly compact, we may assume that $x_n \rightharpoonup x_0 \in U$. Owing to the weak lower semi-continuity of f, we have $f(x_0) \leqslant \liminf_{n\to\infty} f(x_n) = -\infty$. This contradiction tells us that f is bounded from below in U, and hence f admits an infimum. Let $\{x_i\}_{i=1}^{\infty} \subset U$ be a minimizing sequence, i.e., $f(x_i) \to c$ as $i \to \infty$. Since U is weakly compact, we may assume that $x_i \rightharpoonup x_0 \in U$. The weak lower semi-continuity of f gives

$$c \leqslant f(x_0) \leqslant \liminf_{i\to\infty} f(x_i) = \lim_{i\to\infty} f(x_i) = c,$$

i.e., $f(x_0) = c$. The proof is complete. □

Corollary 1.1 *Let X be a reflexive real Banach space, $K \subset X$ be a weakly closed set, and f be a weakly lower semi-continuous functional in K. Moreover, if K is unbounded, we assume further that f satisfies $\lim\limits_{x\in K,\, \|x\|\to\infty} f(x) = \infty$. Then f achieves an infimum in K, i.e., there exists a minimizer $x_0 \in K$ such that $f(x_0) = \inf_{x\in K} f(x)$.*

Proof When K is bounded, it must be weakly compact since X is reflexive, and the assertion follows from Theorem 1.3.

Now we suppose that K is unbounded. As $\lim\limits_{x\in K,\ \|x\|\to\infty} f(x) = \infty$, one can find $x^* \in K$ and a constant $r > 0$ such that $f(x^*) > 0$, and $f(x) > f(x^*)$ when $x \in K$ and $\|x\| > r$. As a closed ball of a reflexive Banach space is weakly compact and K is weakly closed, and so $U = K \cap \overline{B_r(0)}$ is weakly compact. Hence, by Theorem 1.3, there exists $x_0 \in U$ such that $f(x_0) = \inf_{x\in U} f(x)$. It is obvious that $x^* \in U$. Therefore,

$$f(x_0) \leqslant f(x^*) < f(x), \quad \forall\, x \in K \setminus U,$$

and hence $f(x_0) = \inf_{x\in K} f(x)$. This finishes the proof. □

Theorem 1.4 *Suppose that X is a reflexive real Banach space, and the functional f is bounded from below and weakly lower semi-continuous in X. If there exists a bounded minimizing sequence, then f achieves the minimum in X.*

Proof Since f is bounded from below, it has an infimum. Let $\{x_i\}_{i=1}^{\infty} \subset X$ be the bounded minimizing sequence, i.e., $\lim_{i\to\infty} f(x_i) = \inf_{x\in X} f(x)$ and there exists a positive constant C such that $\|x_i\| \leqslant C$ for all i. Taking into account the reflexivity of X, it follows that $\{x_i\}_{i=1}^{\infty}$ is a weakly sequentially compact set. We may assume that $x_i \rightharpoonup x_0 \in X$. Thanks to the weak lower semi-continuity of f, we have

$$\inf_{x\in X} f(x) \leqslant f(x_0) \leqslant \liminf_{i\to\infty} f(x_i) = \lim_{i\to\infty} f(x_i) = \inf_{x\in X} f(x).$$

Thus, $f(x_0) = \inf_{x\in X} f(x)$. The proof is complete. □

1.4 Applications

We will give two examples of applications to the study of quasi-linear boundary value problems. These results will be used in Chap. 7.

Example 1.1 Let $1 < p < \infty$, and define p^* as follows:

$$p^* = \frac{np}{n-p} \quad \text{if} \quad p < n, \qquad p^* = \infty \quad \text{if} \quad p \geqslant n.$$

Consider the boundary value problem

$$-\Delta_p u = |u|^{q-2} u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.6}$$

where $p > 1$, $1 \leqslant q < p^*$, $q \neq p$, and Δ_p is the *p-Laplacian* defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

A function $u \in W_0^{1,p}(\Omega)$ is said to be a *weak solution* of (1.6) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \phi = \int_\Omega |u|^{q-2}u\phi, \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

According to the Poincaré inequality, $\|\nabla u\|_p$ is the norm of $W_0^{1,p}(\Omega)$. The imbedding theorem indicates that $W_0^{1,p}(\Omega)$ is imbedded in $L^q(\Omega)$ compactly. Thus, there exists a positive constant C so that

$$\|u\|_q \leqslant C\|\nabla u\|_p, \quad \forall\, u \in W_0^{1,p}(\Omega). \tag{1.7}$$

Define

$$A(u) = \frac{1}{p}\int_\Omega |\nabla u|^p, \quad B(u) = \frac{1}{q}\int_\Omega |u|^q,$$
$$Y = \left\{u \in W_0^{1,p}(\Omega) : B(u) = 1\right\}.$$

Theorem 1.5 *The problem* (1.6) *admits at least one non-trivial and non-negative weak solution.*

Proof We first prove that the set Y is weakly closed. Suppose $u_m \in Y$ and $u_m \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Then the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$ and compact in $L^q(\Omega)$. We can find a subsequence $\{u_{m_i}\}_{i=1}^\infty \subset \{u_m\}_{m=1}^\infty$ so that $u_{m_i} \to u$ in $L^q(\Omega)$. Since $B(u_{m_i}) = 1$, we have $B(u) = 1$, i.e., $u \in Y$.

It is clear that $A(u)$ is weakly lower semi-continuous in Y (for the weak lower semi-continuity of norms, refer to Exercise 1.2) and $\lim\limits_{\|\nabla u\|_p \to \infty} A(u) = \infty$. By Corollary 1.1, there exists a minimizer $u_0 \in Y$ such that

$$A(u_0) = \inf_{u \in Y} A(u) := \sigma.$$

Thanks to (1.7), there is a constant $\alpha > 0$ such that $A(u) \geqslant \alpha$ for all $u \in Y$. Consequently, $\sigma \geqslant \alpha > 0$.

Note that both $A(u)$ and $B(u)$ are Fréchet differentiable at u_0, and

$$A'(u_0)\phi = \int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \phi, \quad \forall\, \phi \in W_0^{1,p}(\Omega),$$
$$B'(u_0)\phi = \int_\Omega |u_0|^{q-2}u_0\phi, \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

Also, $A(u)$ and $B(u)$ are both Gâteaux differentiable at u_0, and these Gâteaux derivatives are equal to the Fréchet derivatives. In view of the expression of $B(u)$ and the definition of the Gâteaux derivative, it follows that, for any $\mu > 0$, $\varepsilon \in \mathbb{R}$

and $\phi \in W_0^{1,p}(\Omega)$,

$$B(\mu u_0 + \varepsilon\mu\phi) = \mu^q B(u_0 + \varepsilon\phi)$$
$$= \mu^q\big[B(u_0) + \varepsilon B'(u_0)\phi + o(\varepsilon)\big]$$
$$= \mu^q\big[1 + \varepsilon B'(u_0)\phi + o(\varepsilon)\big].$$

Now, there exists $0 < \varepsilon_0(\phi) \ll 1$ such that $1 + \varepsilon B'(u_0)\phi + o(\varepsilon) > 0$ when $|\varepsilon| < \varepsilon_0(\phi)$. Furthermore, we can find an $\mu = \mu(\varepsilon) > 0$ so that $\mu^q\big[1 + \varepsilon B'(u_0)\phi + o(\varepsilon)\big] = 1$, i.e., $B(\mu u_0 + \varepsilon\mu\phi) = 1$.

For any fixed $\phi \in W_0^{1,p}(\Omega)$, constants $|\varepsilon| < \varepsilon_0(\phi)$ and $\mu = \mu(\varepsilon) > 0$ determined as above, we have

$$\sigma \leqslant A(\mu u_0 + \varepsilon\mu\phi) = \mu^p A(u_0 + \varepsilon\phi)$$
$$= \Big(\frac{1}{1 + \varepsilon B'(u_0)\phi + o(\varepsilon)}\Big)^{p/q} A(u_0 + \varepsilon\phi)$$
$$= \Big(1 - \varepsilon\frac{p}{q}B'(u_0)\phi + o(\varepsilon)\Big)\big[A(u_0) + \varepsilon A'(u_0)\phi + o(\varepsilon)\big]$$
$$= \Big(1 - \varepsilon\frac{p}{q}B'(u_0)\phi + o(\varepsilon)\Big)\big[\sigma + \varepsilon A'(u_0)\phi + o(\varepsilon)\big]$$
$$= \sigma + \varepsilon\Big(A'(u_0)\phi - \sigma\frac{p}{q}B'(u_0)\phi\Big) + o(\varepsilon).$$

It follows that

$$A'(u_0)\phi = \sigma\frac{p}{q}B'(u_0)\phi.$$

Noting that $A(|u|) = A(u)$, $B(|u|) = B(u)$ and $|u_0| \in W_0^{1,p}(\Omega)$, we have

$$|u_0| \in Y, \quad A(|u_0|) = \sigma = \inf_{u \in Y} A(u).$$

Similar to the above, we can prove $A'(|u_0|)\phi = \sigma\frac{p}{q}B'(|u_0|)\phi$. Clearly, the function $u = \lambda|u_0|$, with $\lambda = (q/(\sigma p))^{1/(p-q)}$, is in $W_0^{1,p}(\Omega)$ and satisfies

$$A'(u)\phi = B'(u)\phi, \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

This shows that u is a non-trivial and non-negative solution of (1.6). □

Remark 1.1 The author of [144] proved Theorem 1.5 by a different method, and Theorem II of [144] asserts that the solution u is in $L^\infty(\Omega)$, and satisfies $\|u\|_\infty \leqslant e^d$, where

$$d = \frac{p^* - p}{p(p_* - q)}\Big[q \ln \|u\|_q + \frac{pp_*}{p_* - p} \ln\big[K(q - k^-)\big] + p_*\Big(\frac{p}{p_* - p}\Big)^2 \ln \frac{p_*}{p}\Big],$$

$$k = \frac{p_*(q - p)}{p_* - p}, \quad p_* = \begin{cases} p^* = np/(n - p) & \text{if } \ p < n, \\ 2\max\{p, q\} & \text{if } \ p = n, \end{cases}$$

and K is the imbedding constant from $W_0^{1,p}(\Omega)$ to $L^{p_*}(\Omega)$, $k^- = \min\{k, 0\}$.

Making use of Theorems 7.2 and 7.3 in Chap. 7, we can see that the non-trivial and non-negative solution of (1.6) is in $C^{1+\alpha}(\overline{\Omega})$ and must be positive.

Example 1.2 Let $1 < p < \infty$, $p' = p/(p - 1)$ and $f \in L^{p'}(\Omega)$. Consider the boundary value problem

$$-\Delta_p u + g(x, u) = f(x) \quad \text{in } \ \Omega, \quad u = 0 \quad \text{on } \ \partial\Omega. \tag{1.8}$$

Denote the primitive function of g by

$$G(x, u) = \int_0^u g(x, s)\mathrm{d}s.$$

Suppose that

(G1) The function $G : \Omega \times \mathbb{R} \to [0, \infty)$ is measurable with respect to x, and is lower semi-continuous in u;

(G2) The set

$$K := \big\{u \in W_0^{1,p}(\Omega) : G(x, u) \in L^1(\Omega)\big\} \neq \emptyset.$$

A function $u \in W_0^{1,p}(\Omega)$ is said to be a *weak solution* of (1.8) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi + \int_\Omega g(x, u)\phi = \int_\Omega f\phi, \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

Theorem 1.6 *Under the above assumptions* **(G1)** *and* **(G2)***, the problem* (1.8) *has at least one weak solution* $u \in W_0^{1,p}(\Omega)$.

Proof Define a functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \int_\Omega G(x, u) - \int_\Omega fu.$$

By virtue of the assumption **(G1)**, the Sobolev inequality and Young inequality, it is easy to see that $J(u)$ is bounded from below in K. Denote its infimum by M and a minimizing sequence by $\{u_i\}_{i=1}^\infty \subset K$. As above, there is a positive constant C, independent of i, such that $\int_\Omega |\nabla u_i|^p \leqslant C$. As $W_0^{1,p}(\Omega)$ is reflexive, there exist a subsequence of $\{u_i\}_{i=1}^\infty$, denoted by itself, and a function $u \in W_0^{1,p}(\Omega)$, such that $\nabla u_i \rightharpoonup \nabla u$ and $u_i \to u$ in $L^p(\Omega)$.

Using the lower semi-continuity of $G(x, u)$ in u and Fatou's lemma, it follows that

$$\int_\Omega G(x, u) \leqslant \liminf_{i\to\infty} \int_\Omega G(x, u_i).$$

Since $G(x, u) \geqslant 0$, the above inequality implies that $u \in K$. Applying the weak lower semi-continuity of the functional $I[u] = \int_\Omega |\nabla u|^p$, we get $J(u) \leqslant M$. Thus, $J(u) = M$, i.e., $J(u) = \inf_{y\in K} J(y)$.

In view of Theorem 1.2, it can be deduced that, for any given $\phi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi + \int_\Omega g(x, u)\phi = \int_\Omega f\phi.$$

This shows that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.8). □

Notes

The materials in Sects. 1.2 and 1.3 are standard and can be found in textbooks on nonlinear functional analysis such as [78]. The contents of Sect. 1.4 are standard applications of critical point theory and the weak convergence method.

Exercises

1.1 Prove the mean value formula (1.1).
1.2 Prove that the norm is weakly lower semi-continuous.

Chapter 2
Eigenvalue Problems of Second Order Linear Elliptic Operators

Eigenvalue problems have a wide range of applications. In particular, the existence of positive solutions to second order semi-linear and quasi-linear elliptic equations and systems depends critically on the principal eigenvalue (the first or smallest eigenvalue) of a corresponding eigenvalue problem. In this chapter, we introduce the theory of eigenvalue problems for second order linear elliptic operators. These results will be used extensively in the later chapters. In the last chapter, we will also introduce the eigenvalue problem for the p-Laplace operator.

We first discuss the *eigenvalue problem* of a second order linear elliptic operator in general form:

$$\mathscr{L}u = \lambda u \quad \text{in} \quad \Omega, \quad \mathscr{B}u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where the operators $\mathscr{L}$ and $\mathscr{B}$ satisfy the conditions **(A)**–**(C)** (see pp. 2, 3), and λ is a (real or complex) number. As the operator $(\mathscr{L}, \mathscr{B})$ is asymmetrical, its eigenvalue structure is very complex. However, in applications, we are usually only concerned with the principal eigenvalue. Hence we will only focus on that here.

After discussing the general form, we shall focus on the special case of the second order linear elliptic operator in divergence form:

$$\mathscr{L}_d u = \lambda u \quad \text{in} \quad \Omega, \quad \mathscr{B}_d u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

where $\mathscr{L}_d$ satisfies the condition **(A)** and $a_{ij},\ c \in L^\infty(\Omega)$, and $\mathscr{B}_d$ is

$$\mathscr{B}_d u = u, \tag{2.3}$$

or

$$\mathscr{B}_d u = a_{ij}(x) D_i u \cos(\boldsymbol{n}, x_j) + b(x)u, \quad b(x) \geqslant 0. \tag{2.4}$$

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Here, we note that the operator $(\mathscr{L}_d, \mathscr{B}_d)$ is symmetrical, and its eigenvalue structure can be fully explicated.

A number λ is called an *eigenvalue* of (2.1) or (2.2) if the problem (2.1) or (2.2) has a non-trivial solution for such λ, and the corresponding non-trivial solution u is called the corresponding *eigenfunction* to λ. In this case, the pair (λ, u) is called the *eigenpair* of (2.1) or (2.2). If the corresponding eigenfunction is positive or negative, such an eigenvalue is called the *principal eigenvalue*. Let λ be a principal eigenvalue and ϕ be the corresponding positive eigenfunction. The pair (λ, ϕ) is called the *principal eigenpair*.

2.1 Principal Eigenvalue of the Non-divergence Operator

In this section we will introduce the existence and uniqueness of principal eigenvalue of the eigenvalue problem (2.1), and equivalent forms of the maximum principle.

2.1.1 The Existence and Uniqueness of Principal Eigenvalue

We first recall the Krein-Rutman theorem, which concerns the existence of positive eigenfunctions.

Let E be a Banach space. A set $P \subset E$ is called a *cone* if it has the following properties:

(1) $x, y \in P$ implies $x + y \in P$;
(2) for any $x \in P$ and $\lambda \geqslant 0$, we have $\lambda x \in P$;
(3) if $x \in P$ and $x \neq 0$, then $-x \notin P$.

Suppose that P is a closed cone in E. If $\overline{P - P} = E$, i.e., the set $\{u - v : u, v \in P\}$ is dense in E, then P is said to be a *total cone*. If a cone has a nonempty interior P°, then it is said to be a *solid cone*.

Let $\mathcal{A}$ be a linear operator. If $\mathcal{A}(P) \subset P$, we say that $\mathcal{A}$ is *positive* with respect to P. If $\mathcal{A}(P \setminus \{0\}) \subset P^\circ$, we say that $\mathcal{A}$ is *strongly positive* with respect to P.

Let $\mathcal{A}$ be a bounded linear operator and $\sigma(\mathcal{A})$ be the *spectrum* of $\mathcal{A}$. The number

$$r(\mathcal{A}) = \sup\left\{|\lambda| : \lambda \in \sigma(\mathcal{A})\right\}$$

is called the *spectral radius* of $\mathcal{A}$. Note that when $\mathcal{A}$ is a compact linear operator then

$$r(\mathcal{A}) = \sup\left\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\right\}.$$

Theorem 2.1 (Krein-Rutman Theorem (the Weak Version) [96, 46]) *Let E be a Banach space and P a total cone in E. Let $\mathcal{A} : E \to E$ be a positive compact linear operator with respect to P, and $r(\mathcal{A}) > 0$. Then $r(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, with the corresponding eigenvector $u \in P$.*

Theorem 2.2 (Krein-Rutman Theorem (the Strong Version) [96, 46, 203]) *Let E be a Banach space and P a solid cone in E. Let $\mathcal{A} : E \to E$ be a strongly positive compact linear operator with respect to P, and $\mathcal{A}^*$ be its adjoint. Then,*

(1) *$r(\mathcal{A}) > 0$ is a geometrically simple eigenvalue of $\mathcal{A}$ pertaining to an eigenvector $u \in P^\circ$;*
(2) *$r(\mathcal{A})$ is algebraically simple;*
(3) *for any eigenvalue μ of $\mathcal{A}$, $\mu \neq r(\mathcal{A})$ implies $|\mu| < r(\mathcal{A})$;*
(4) *eigenvectors corresponding to the other eigenvalues are not in P;*
(5) *$r(\mathcal{A})$ is an algebraically simple eigenvalue of $\mathcal{A}^*$, with an eigenvector u_0^* which is a strictly positive functional: $\langle u_0^*, u\rangle > 0$ for any $u \in P^\circ$; moreover, any eigenvector of $\mathcal{A}^*$ in $(P \setminus \{0\})^*$ corresponds to $r(\mathcal{A})$.*

In the applications to differential equations, E is usually a function space, $\mathcal{A}$ the inverse of a linear differential operator, and P a positive cone consisting of non-negative functions of E.

We first give a technical lemma.

Lemma 2.1 *Let Ω be of class C^1, $u, v \in C^1(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$, $v|_{\partial\Omega} \geqslant 0$, $u > 0$ in Ω, and $\partial_{\boldsymbol{n}} u|_{\partial\Omega} < 0$. Then there exists a constant $\varepsilon > 0$ such that $u + \varepsilon v > 0$ in Ω.*

Proof Since $u, v \in C^1(\overline{\Omega})$ and $\partial_{\boldsymbol{n}} u|_{\partial\Omega} < 0$, we have $\partial_{\boldsymbol{n}}(u + \varepsilon_1 v)|_{\partial\Omega} < 0$ when $0 < \varepsilon_1 \ll 1$. Noting that $(u + \varepsilon_1 v)|_{\partial\Omega} \geqslant 0$, there is a subset $\Omega_0 \Subset \Omega$ such that $u + \varepsilon_1 v > 0$ in $\Omega \setminus \Omega_0$. As $u > 0$ in $\overline{\Omega}_0$, we can find $\varepsilon_2 > 0$ such that $u + \varepsilon_2 v > 0$ in $\overline{\Omega}_0$. Take $\varepsilon = \min\left\{\varepsilon_1, \varepsilon_2\right\}$. Then we reach the desired conclusion. □

Now we investigate the *principal eigenvalue* of (2.1). We first look at the Dirichlet boundary condition: $\mathscr{B}u = u$.

Take $E = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. We first assume that $c(x) > 0$. For any given $u \in E$, the linear problem

$$\mathscr{L}v = u(x) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

admits a unique solution $v \in C^{1+\alpha}(\overline{\Omega}) \cap E$. Define $\mathcal{A}u = v$. Since the imbedding $C^{1+\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ is compact, we see that $\mathcal{A}$ is a linear compact operator. Define

$$P = \text{closure}\left\{u : u \in E,\ u|_{\Omega} > 0,\ \partial_{\boldsymbol{n}} u|_{\partial\Omega} < 0\right\}.$$

Then P is a closed cone in E, and the interior $P^\circ \neq \emptyset$. In fact, by virtue of Lemma 2.1, it is easy to prove that

$$P^\circ = \big\{u : u \in E,\ u|_\Omega > 0,\ \partial_{\boldsymbol{n}} u|_{\partial\Omega} < 0\big\}.$$

According to the strong maximum principle and Hopf boundary lemma, it is clear that $\mathcal{A}u \in P^\circ$ for all $u \in P \setminus \{0\}$, i.e., $\mathcal{A}$ is strongly positive with respect to P. Theorem 2.2 indicates that there exists a unique function $v \in P^\circ$ with $\|v\| = 1$ such that $\mathcal{A}v = r(\mathcal{A})v$, while the other eigenvalues μ of $\mathcal{A}$ satisfy $|\mu| < r(\mathcal{A})$. Therefore,

$$v = \mathscr{L}(\mathcal{A}v) = r(\mathcal{A})\mathscr{L}v, \quad \text{i.e.,} \quad \mathscr{L}v = \frac{1}{r(\mathcal{A})}v,$$

and hence $\lambda_1 = 1/r(\mathcal{A}) > 0$ is an eigenvalue of
(2.1), and the corresponding eigenfunction $v > 0$ in Ω.

From the above investigation we see that eigenvalues of (2.1) and eigenvalues of $\mathcal{A}$ have a one-to-one, reciprocal relationship (note that 0 is not an eigenvalue of either). As $c(x) > 0$, by the maximum principle, it is easy to deduce that the real eigenvalues of (2.1) are positive, and thus so are the real eigenvalues of $\mathcal{A}$. Since the other eigenvalues μ of $\mathcal{A}$ satisfy $|\mu| < r(\mathcal{A})$, we see that the other eigenvalues λ of (2.1) satisfy $|\lambda| > \lambda_1$. As $r(\mathcal{A})$ is a simple eigenvalue of $\mathcal{A}$, so is λ_1 for (2.1). Moreover, the eigenfunction corresponding to λ_1 has no zero in Ω, while the eigenfunctions corresponding to the other eigenvalues must change sign if they are real. It can also be proved that λ_1 has the smallest real part among all eigenvalues ([54, Theorem 1.2]).

For the general case, take a constant $C > 0$ so that $C + c(x) > 0$ in $\overline{\Omega}$. Rewrite the eigenvalue problem (2.1) as

$$\mathscr{L}u + Cu = (\lambda + C)u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Summing up the above discussion we have the following theorem.

Theorem 2.3 (Existence and Uniqueness of Principal Eigenvalue) *The eigenvalue problem* (2.1) *has a unique principal eigenvalue which is simple, real and has the smallest real part among all eigenvalues. Moreover, the eigenfunctions corresponding to the other eigenvalues must change sign in Ω when they are real. By the spectral theory for compact operators,* (2.1) *has at most countably many eigenvalues.*

When the boundary condition is $\partial_{\boldsymbol{n}} u + b(x)u = 0$, we have the analogous conclusions.

If Ω is of class $C^{2+\alpha}$ and the coefficients of $\mathscr{L}$ are Hölder continuous, i.e., the condition $(\mathbf{B}_\alpha)$ (see p.3) holds, then the eigenfunction belongs to $C^{2+\alpha}(\overline{\Omega})$ for the boundary condition $\mathscr{B}u = u$. If, in addition, $b \in C^{1+\alpha}(\overline{\Omega})$, then the eigenfunction belongs to $C^{2+\alpha}(\overline{\Omega})$ for the boundary condition $\mathscr{B}u = \partial_{\boldsymbol{n}} u + bu$.

Theorem 2.4 *Suppose that* $b \in C(\partial\Omega)$, *and* b, $c \geqslant 0$. *Let* $\lambda_1(\mathscr{L}, \mathscr{B})$ *be the* principal eigenvalue *of* (2.1). *If either* $c(x) \not\equiv 0$ *or* $b(x) \not\equiv 0$, *then* $\lambda_1(\mathscr{L}, \mathscr{B}) > 0$. *If* $c(x) \equiv b(x) \equiv 0$, *then* $\lambda_1(\mathscr{L}, \mathscr{B}) = 0$.

For the existence of principal eigenvalue, we have the following more general results.

Theorem 2.5 (Existence and Uniqueness of Principal Eigenvalue [9, Theorem 12.1], [11, Theorem 2.2]) *Let* $\mathscr{L}$ *be strongly elliptic in* Ω *with* $a_{ij} \in C(\overline{\Omega})$, b_i, $c \in L^\infty(\Omega)$, *and* Ω *be of* C^2. *Assume that* Γ_0 *and* Γ_1 *are two disjoint open and closed subsets of* $\partial\Omega$ *with* $\Gamma_0 \cup \Gamma_1 = \partial\Omega$. *Consider the eigenvalue problem*

$$\mathscr{L}u = \lambda u \quad in \ \ \Omega, \quad \mathscr{B}_* u = 0 \quad on \ \ \partial\Omega, \tag{2.5}$$

where

$$\mathscr{B}_* u = \begin{cases} u & on \ \ \Gamma_0, \\ \partial_{\boldsymbol{n}} u + b(x)u & on \ \ \Gamma_1, \end{cases} \tag{2.6}$$

and $b \in C(\Gamma_1)$ *with* $b(x) \geqslant 0$. *Then,* (2.5) *has a unique simple and real eigenvalue, denoted by* λ_1, *its corresponding eigenfunction is positive in* Ω. *Thus,* λ_1 *is the* principal eigenvalue. *Moreover,* λ_1 *is the eigenvalue having the smallest real part.*

Let's consider the case of bounded coefficients.

Theorem 2.6 (Existence and Uniqueness of Principal Eigenvalue [54, Theorem 2.7]) *Let* $a_{ij} \in C(\Omega) \cap L^\infty(\Omega)$, $b_i, c \in L^\infty(\Omega)$. *Suppose that* $\mathscr{L}$ *is strongly elliptic in* Ω, *and* Ω *has Lipschitz boundary. Then the following conclusions hold for* $p \geqslant n$.

(1) *There exist a real number* λ_1 *and a function* $\varphi_1 \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$, $\varphi_1 > 0$ *in* Ω, *such that*

$$\mathscr{L}\varphi_1 = \lambda_1\varphi_1 \quad in \ \ \Omega, \quad \varphi_1 = 0 \quad on \ \ \partial\Omega;$$

(2) *If* $\psi \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ *with* $\psi > 0$ *in* Ω, $\lambda \in \mathbb{C}$ *and satisfy*

$$\mathscr{L}\psi = \lambda\psi \quad in \ \ \Omega, \quad \psi = 0 \quad on \ \ \partial\Omega,$$

then $\lambda = \lambda_1$, *and* ψ *is a multiple of* φ_1;

(3) *If* $\psi \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ (*possibly complex-valued*) *with* $\psi \not\equiv 0$, $\lambda \in \mathbb{C}$ *and satisfy*

$$\mathscr{L}\psi = \lambda\psi \quad in \ \ \Omega, \quad \psi = 0 \quad on \ \ \partial\Omega,$$

then $\lambda \neq \lambda_1$ *implies* ${\rm Re}\lambda > \lambda_1$.

These show that λ_1 *is the principal eigenvalue and it exists uniquely.*

Before ending this section, we shall mention the *eigenvalue problem with signed weight function*

$$\mathscr{L}u = \lambda m(x)u \quad \text{in} \ \ \Omega, \quad \mathscr{B}u = 0 \quad \text{on} \ \ \partial\Omega, \tag{2.7}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^n$.

Similar to the above, using Theorem 2.1 we can prove the following theorem.

Theorem 2.7 *Let $c(x) \geq 0$, $m \in L^\infty(\Omega)$ and $m(x) > 0$ in Ω. Then the conclusion of Theorem 2.3 holds for (2.7). Moreover, if either $c(x) \not\equiv 0$ or $b(x) \not\equiv 0$, then the principal eigenvalue of (2.7) is positive.*

2.1.2 Equivalent Forms of the Maximum Principle

We have already mentioned above that the principal eigenvalue of corresponding eigenvalue problems plays an important role in the study of elliptic partial differential equations. To underscore this, in this section we study the relations between the principal eigenvalue, maximum principle and positive strict upper solution for second order linear elliptic equations. We first consider the case with smooth coefficients and the general boundary operator $\mathscr{B}_*$, i.e., the coefficients of $\mathscr{L}$ satisfy the condition $(\mathbf{B}_\alpha)$, and the operator $\mathscr{B}_*$ is defined by (2.6) with $b \in C^{1+\alpha}(\partial\Omega)$.

Definition 2.1 An operator $(\mathscr{L}, \mathscr{B}_*)$ is said to have the *strong maximum principle property* if, for any function $u \in C(\Omega \cup \Gamma_0) \cap C^1(\Omega \cup \Gamma_1) \cap C^2(\Omega)$, from

$$\mathscr{L}u \geqslant 0, \ \ u \not\equiv 0 \quad \text{in} \ \ \Omega, \quad \mathscr{B}_*u \geqslant 0 \quad \text{on} \ \ \partial\Omega,$$

one can conclude $u > 0$ in Ω.

Definition 2.2 A function $u \in C(\Omega \cup \Gamma_0) \cap C^1(\Omega \cup \Gamma_1) \cap C^2(\Omega)$ is said to be an *upper solution* (a *lower solution*) of the operator $(\mathscr{L}, \mathscr{B}_*)$ if

$$\mathscr{L}u \geqslant (\leqslant)\ 0 \quad \text{in} \ \ \Omega, \quad \mathscr{B}_*u \geqslant (\leqslant)\ 0 \quad \text{on} \ \ \partial\Omega.$$

It is called a *strict upper solution* (*strict lower solution*) if it is an upper solution (a lower solution) but not a solution.

Theorem 2.8 (Equivalent Forms of the Maximum Principle [54, Theorem 2.4]) *Let $\mathscr{L}$ be strongly elliptic in Ω, the condition $(\mathbf{B}_\alpha)$ hold, and let the boundary operator $\mathscr{B}_*$ be given by (2.6). Suppose that $\partial\Omega \in C^{2+\alpha}$ and $b \in C^{1+\alpha}(\Gamma_1)$. Then the following statements are equivalent*:

(1) *$(\mathscr{L}, \mathscr{B}_*)$ has the strong maximum principle property when restricted to the function space $C^1(\overline{\Omega}) \cap C^2(\Omega)$;*