SCALAR AND ASYMPTOTIC SCALAR DERIVATIVES

Theory and Applications

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Aims and Scope

Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics and other sciences.

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SCALAR AND ASYMPTOTIC SCALAR DERIVATIVES

Theory and Applications

By

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The sage gives without reservation.
He offers all to others, and his life
is more abundant. He helps all men alike,
and his life is more exuberant.

(Lao Zi: *Truth and Nature*)

Preface

This book is devoted to the study of scalar and asymptotic scalar derivatives and their applications to the study of some problems considered in nonlinear analysis, in geometry, and in applied mathematics.

The notion of a scalar derivative is due to S. Z. Németh, and the notion of an asymptotic scalar derivative is due to G. Isac. Both notions are recent, never considered in a book, and have interesting applications. About applications, we cite applications to the study of complementarity problems, to the study of fixed points of nonlinear mappings, to spectral nonlinear analysis, and to the study of some interesting problems considered in differential geometry and other applications.

A new characterization of monotonicity of nonlinear mappings is another remarkable application of scalar derivatives.

A relation between scalar derivatives and asymptotic scalar derivatives, realized by an inversion operator is also presented in this book. This relation has important consequences in the theory of scalar derivatives, and in some applications. For example, this relation permitted us a new development of the method of exceptional family of elements, introduced and used by G. Isac in complementarity theory.

Now, we present a brief description of the contents of this book.

Chapter 1 is dedicated to the study of scalar derivatives in Euclidean spaces. In this chapter we explain the reason for introducing scalar derivatives as good mathematical tools for characterizing important properties of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. In order to avoid some difficulties, we consider only upper and lower scalar derivatives which are extensions to vector functions of Dini derivatives. We consider also the case when lower and upper scalar derivatives coincide. This is a strong restriction and we show that for $n = 2$ the existence of a single-valued scalar derivative is strongly related to complex differentiability. The lower and upper scalar derivatives are also used to characterize convexity like notions.

Chapter 2 essentially has two parts. In the first part we present the notion of the asymptotic derivative and some results related to this notion and in the second part we introduce the notion of the asymptotic scalar derivative. The results presented in the first part are necessary for understanding the notions given in the second part. It is known that the notion of the asymptotic derivative was introduced by the Russian school, in particular by M. A. Krasnoselskii, under the name of asymptotic linearity. The main goal of this chapter is to present the notion of the asymptotic scalar derivative and some of its applications.

Chapter 3 presents the scalar derivatives in Hilbert spaces and several results and properties are given. We note that in this chapter we give the definitions of scalar derivatives of rank p, named briefly for $p = 2$, scalar derivatives. We also put in evidence the fact that the case $p = 1$ is strongly related to the notion of submonotone mapping, introduced in 1981 by J. E. Spingarn and studied in 1997 by P. Georgiev. Several new results related to computation of the scalar derivative and some interesting relations with skew-adjoint operators are also presented. The scalar derivatives are used to characterize the monotonicity of mappings in Hilbert spaces. Many of the formulae presented in this chapter arise from applications such as *fixed point* theorems, surjectivity theorems, integral equations, and complementarity problems, among others.

Chapter 4 contains the extension of the theory of scalar derivatives to Banach spaces. This extension is based on the notion of the semi-inner product in Lumer's sense. The notion of scalar derivatives defined in this case is applied to fixed point theory, to the study of solvability of integral equations, of variational inequalities, and of complementarity problems.

Chapter 5 is dedicated to a generalization of the notion of Kachurovskii–Minty–Browder monotonicity to Riemannian manifolds and to realize this we introduce the notion of the geodesic monotone vector field. The geodesic convexity for mappings is also considered. For a global example of monotone vector fields we consider Hadamard manifolds (complete, simply connected Riemannian manifolds with nonpositive sectional curvature). Analyzing the existence of geodesic monotone vector fields, we prove that there are no strictly geodesic monotone vector fields on a Riemannian manifold that contain a closed geodesic. We note that many results presented in this chapter are based on a generalization to Riemannian manifolds of scalar derivatives studied in the previous chapters. The nongradient type monotonicity on Riemannian manifolds is considered for the first time in a book.

This book is the first book dedicated to the study of scalar and asymptotic scalar derivatives and certainly new developments related to these notions are possible.

It is impossible to finish this preface without giving many thanks to the people who spent their time developing the open source tools (operating system, window manager, and software) that were essential for writing this book,

greately reducing the time and energy spent in word processing. These open source tools are: the Linux and FreeBSD operating systems, the Ratpoison window manager, the LaTeX word processing language, and the VIM and Bluefish editors.

We are grateful to the reviewers for their valuable comments and suggestions. Taking them into consideration has greately improved the quality and presentation of the book.

To conclude, we would like to say that we very much appreciated the excellent assistance offered to us by the staff of Springer Publishers.

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Contents

Chapter 1

Scalar Derivatives in Euclidean Spaces

1.1 Scalar Derivatives of Mappings in Euclidean Spaces

The behaviour of the scalar product $\langle f(x) - f(y),\, x - y\rangle$ (with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\langle\, .\,,\, .\,\rangle$ the usual scalar product in $\mathbb{R}^n$) when x and y run over $\mathbb{R}^n$ is a good tool in characterizing important properties of f. If f is bounded, then this product converges to 0 for $x \to y$. Therefore it cannot be used in obtaining a local characterization. Hence it is natural to consider at y limits of the expressions of the form $\langle f(x) - f(y),\, x - y\rangle / \langle x - y,\, x - y\rangle$ for $x \to y$. Thus we arrive naturally at a notion that we call the scalar derivative. It is in general a multivalued mapping from $\mathbb{R}^n$ to $\mathbb{R}$ even if f is linear.

In order to avoid the difficulties in considering multifunctions we only consider so-called upper and lower scalar derivatives, which are extensions to vector functions of the Dini derivatives.

We consider mostly the case when lower and upper scalar derivatives coincide. This restriction is a very strong one. In Section 1.1.3 it is shown that for $n = 2$ the existence of a single-valued scalar derivative is strongly related to the complex differentiability. In Section 1.1.4 we consider various examples and counterexamples. Lower and upper scalar derivatives can be used in characterizing the monotone operators in the way this is done in Section 1.1.5.

Convex functionals have as gradients monotone operators. Hence the scalar derivative can also be used to characterize convexity like notions. Thus Propositions 2.1 and 2.2 in Karamardian and Schaible [1990] together with the results in our Section 1.1.5 give some characterizations of convex and strictly convex functionals.

We have defined the notion of scalar derivative having in mind Minty's monotonicity notion [Minty, 1962]. To simplify the notations, in this chapter a monotone mapping (strictly monotone mapping) f will be called increasing (strictly increasing). If $-f$ is monotone (strictly monotone), then f will be called decreasing (strictly decreasing).

1.1.1 Some Basic Results Concerning Skew-Adjoint Operators

DEFINITION 1.1 *Consider the operator* $f : \mathbb{R}^n \to \mathbb{R}^n$. *It is called* increasing (decreasing) *if for any x and y in $\mathbb{R}^n$ one has*

$$\langle f(x) - f(y), x - y \rangle \geq 0 \ \ (\leq 0).$$

If

$$\langle f(x) - f(y), x - y \rangle > 0 \ \ (< 0)$$

whenever $x \neq y$, then f is called strictly increasing (strictly decreasing).

DEFINITION 1.2 *The linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is called* skew-adjoint *if for any x and y in $\mathbb{R}^n$ the relation $\langle Ax, y \rangle + \langle Ay, x \rangle = 0$ holds.*

THEOREM 1.3 *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the following statements are equivalent.*

1. *A is skew-adjoint.*
2. $\langle Ax - Ay, x - y \rangle = 0$ *for any $x, y \in \mathbb{R}^n$.*
3. *Taking an arbitrary orthonormal basis in $\mathbb{R}^n$, A can be represented by a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ such that $a_{ij} = -a_{ji}\ \forall i, j \in \{1, 2, \ldots, n\}$.*

Proof. $1 \Rightarrow 2$ Take x and y arbitrarily in $\mathbb{R}^n$. By the definition of the skew-adjoint operator A we have $\langle Ax, y \rangle + \langle Ay, x \rangle = 0$. Put $y = x$. Then $\langle Ax, x \rangle = 0$ for arbitrary x in $\mathbb{R}^n$. Whence we also have $\langle Ax - Ay, x - y \rangle = 0$ by the linearity of A. The implication $2 \Rightarrow 1$ can be shown similarly.

The equivalence $1 \Leftrightarrow 3$ is obvious. □

REMARK 1.1

1. *There exist injective skew-adjoint operators. For instance, the operators represented by the matrices*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \textit{for } n = 2$$

and

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \textit{for } n = 4$$

are injective.

2. *If n is odd, then there is no injective skew-adjoint operator in $\mathbb{R}^n$. Indeed let A be the matrix corresponding to an skew-adjoint operator. Let the superscript T denote transposition. Then $A^T = -A$ and hence* $\det A = -\det A$ this means that $\det A = 0$.

THEOREM 1.4 *Consider the operator $F : \mathbb{R}^n \to \mathbb{R}^n$. The following assertions are equivalent.*

1. $\langle F(x) - F(y),\, x - y\rangle = 0$, $\forall x,\, y \in \mathbb{R}^n$.

2. *F is an affine operator with a skew-adjoint linear term.*

Proof. Suppose that 1 holds. Put $f(x) = F(x) - F(0)$ for x in $\mathbb{R}^n$. Then $f(0) = 0$ and $\langle f(x) - f(y),\, x - y\rangle = 0\ \forall x,\, y \in \mathbb{R}^n$. Let x be arbitrary in $\mathbb{R}^n$ and $y = 0$. Then $\langle f(x),\, x\rangle = 0\ \forall x \in \mathbb{R}^n$. The above relation also yields

$$\langle f(x),\, x\rangle - \langle f(x),\, y\rangle - \langle f(y),\, x\rangle + \langle f(y),\, y\rangle = 0,\ \ \forall x,\, y \in \mathbb{R}^n$$

and hence

$$\langle f(x),\, y\rangle + \langle f(y),\, x\rangle = 0,\ \ \forall x,\, y \in \mathbb{R}^n.$$

Put $x = \lambda x_1 + \mu x_2$ with arbitrary x_1 and x_2 in $\mathbb{R}^n$. Then

$$\begin{aligned}\langle f(\lambda x_1 + \mu x_2),\, y\rangle &= -\langle f(y),\, \lambda x_1 + \mu x_2\rangle = -\lambda\langle f(y),\, x_1\rangle - \mu\langle f(y),\, x_2\rangle \\ &= \lambda\langle f(x_1),\, y\rangle + \mu\langle f(x_2),\, y\rangle,\end{aligned}$$

wherefrom

$$\langle f(\lambda x_1 + \mu x_2) - \lambda f(x_1) - \mu f(x_2),\, y\rangle = 0$$

for any x_1, x_2 and y in $\mathbb{R}^n$ and any $\lambda,\ \mu$ in $\mathbb{R}$, wherefrom we have the linearity of f. Because $\langle f(x)-f(y),\, x-y\rangle = 0$, for any $x,\ y$ in $\mathbb{R}^n$, f is also skew-adjoint. Thus $F(x) = f(x) + F(0)$ and hence it is indeed affine with a skew-adjoint linear term.

The implication $2 \Rightarrow 1$ is obvious. $\square$

1.1.2 The Scalar Derivative and its Fundamental Properties

DEFINITION 1.5 *Consider the operator $f : \mathbb{R}^n \to \mathbb{R}^n$. If the limit*

$$\lim_{x\to x_0} \frac{\langle f(x) - f(x_0), x - x_0\rangle}{\|x - x_0\|^2} =: f^{\#}(x_0) \in \mathbb{R}$$

exists (here $\|x-x_0\|^2 = \langle x-x_0,\, x-x_0\rangle$), then it is called the scalar derivative *of the operator f in x_0. In this case f is said to be* scalarly differentiable *at x_0. If $f^{\#}(x)$ exists for every x in $\mathbb{R}^n$, then f is said to be* scalarly differentiable *on $\mathbb{R}^n$, with the* scalar derivative $f^{\#}$.

It follows from this definition that both the set of operators scalarly differentiable in x_0, and the set of operators scalarly differentiable on $\mathbb{R}^n$ form linear spaces.

DEFINITION 1.6 *Consider the operator* $f : \mathbb{R}^n \to \mathbb{R}^n$. *The limit*

$$\underline{f}^{\#}(x_0) := \liminf_{x \to x_0} \frac{\langle f(x) - f(x_0),\, x - x_0 \rangle}{\|x - x_0\|^2}$$

is called the lower scalar derivative *of* f *at* x_0. *Taking* $\limsup$ *in place of* $\liminf$ *we can define the* upper scalar derivative $\overline{f}^{\#}(x_0)$ *of* f *at* x_0 *similarly.*

THEOREM 1.7 *The linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is scalarly differentiable on* $\mathbb{R}^n$ *if and only if it is of the form* $A = B + cI_n$ *with* B *skew-adjoint linear operator,* I_n *the identity of* $\mathbb{R}^n$, *and* c *a real number.*

Proof. Let us suppose that A is scalarly differentiable in $x_0 \in \mathbb{R}^n$. Then

$$A^{\#}(x_0) = \liminf_{x \to x_0} \frac{\langle Ax - Ax_0,\, x - x_0 \rangle}{\|x - x_0\|^2} = \liminf_{h \to 0} \frac{\langle Ah,\, h \rangle}{\|h\|^2} = A^{\#}(0).$$

Take $h = \lambda x$ with $x \in \mathbb{R}^n$ and $\lambda > 0$. Then

$$A^{\#}(0) = \liminf_{\lambda \downarrow 0} \frac{\langle A\lambda x,\, \lambda x \rangle}{\|\lambda x\|^2} = \frac{\langle Ax,\, x \rangle}{\|x\|^2}.$$

That is, $\langle Ax,\, x \rangle / \|x\|^2 = c = A^{\#}(0)$. Accordingly,

$$\langle (A - cI_n)x,\, x \rangle = 0,\ \forall x \in \mathbb{R}^n.$$

This means that $B = A - cI_n$ is a skew-adjoint linear operator and hence A has the representation given in the theorem. Obviously, every $A = B + cI_n$ with B a skew-adjoint linear operator has the scalar derivative c at every point of $\mathbb{R}^n$. □

THEOREM 1.8 *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, \cdots, f_n)$ *is scalarly differentiable in* x_0. *Then for every* $i \in \{1, \ldots, n\}$ *there exists the partial derivative*

$$\frac{\partial f_i(x_0)}{\partial x^i} \text{ and } \frac{\partial f_1(x_0)}{\partial x^1} = \cdots = \frac{\partial f_n(x_0)}{\partial x^n} = f^{\#}(x_0).$$

Proof. If we consider $x = (x_0^1, \ldots, x^i, \ldots, x_0^n)$ and $x_0 = (x_0^1, \ldots, x_0^n)$, by letting $x \to x_0$, we obtain that

$$\frac{\partial f_i(x_0)}{\partial x^i} = f^{\#}(x_0).$$

□

THEOREM 1.9 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in x_0 and scalarly differentiable in x_0. Then we have for the differential $df(x_0)$ of f at x_0 the relation*

$$df(x_0) = B + f^{\#}(x_0) I_n,$$

with $B : \mathbb{R}^n \to \mathbb{R}^n$ linear and skew-adjoint .

Proof. Let $t \in \mathbb{R}^n$ be given. Then

$$f^{\#}(x_0) = \frac{1}{\|t\|^2} \liminf_{\lambda \downarrow 0} \left\langle \frac{f(x_0 + \lambda t) - f(x_0)}{\lambda}, \quad t \right\rangle = \frac{1}{\|t\|^2} \langle df(x_0)(t),\, t \rangle,$$

wherefrom $\langle (df(x_0) - f^{\#}(x_0) I_n)(t),\, t \rangle = 0$, $\forall t \in \mathbb{R}^n$, that is,

$$B = df(x_0) - f^{\#}(x_0) I_n$$

is linear and skew-adjoint. □

REMARK 1.2

1. *The theorem holds for the Gateaux differential $\delta f(x_0)$ in place of $df(x_0)$. The differentiability condition is often used next and hence we state the theorem for this stronger condition.*

2. *If we denote by $f'(x_0)$ the Jacobi matrix of f at x_0 in some coordinate representation and the symbols B and I_n stand for matrices of the corresponding operators, then our relation becomes*

$$f'(x_0) = B + f^{\#}(x_0) I_n.$$

THEOREM 1.10 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ is differentiable in x_0. Then the following statements are equivalent.*

1. *f is scalarly differentiable in x_0;*

2. (a) $\dfrac{\partial f_1(x_0)}{\partial x^1} = \cdots = \dfrac{\partial f_n(x_0)}{\partial x^n}$;

   (b) $\dfrac{\partial f_i(x_0)}{\partial x^j} = -\dfrac{\partial f_j(x_0)}{\partial x^i}$, $\forall i, j \in \{1, \ldots, n\}$, $i \neq j$.

Condition 2 is called the *Cauchy–Riemann relation* at x_0.

Proof. $1 \Rightarrow 2$ By Remark 2 one has

$$f'(x_0) = B + f^{\#}(x_0) I_n,$$

where B is a skew-symmetric matrix and $f'(x_0)$ is the Jacobi matrix of f at x_0. Because from the above relation

$$B = \begin{pmatrix} \frac{\partial f_1(x_0)}{\partial x^1} - f^{\#}(x_0) & \frac{\partial f_1(x_0)}{\partial x^2} & \cdots & \frac{\partial f_1(x_0)}{\partial x^n} \\ \frac{\partial f_2(x_0)}{\partial x^1} & \frac{\partial f_2(x_0)}{\partial x_2} - f^{\#}(x_0) & \cdots & \frac{\partial f_2(x_0)}{\partial x^n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n(x_0)}{\partial x^1} & \frac{\partial f_n(x_0)}{\partial x^2} & \cdots & \frac{\partial f_n(x_0)}{\partial x^n} - f^{\#}(x_0) \end{pmatrix}$$

and because B is a skew-symmetric matrix, we must have the relations at 2 of the theorem.

$2 \Rightarrow 1$ Consider the Taylor expansions of $f_1, \dots, f_n$ around x_0:

$$\begin{aligned} f_1(x) &= f_1(x_0) + \sum_{i=1}^{n} \frac{\partial f_1(x_0)}{\partial x^i}(x^i - x_0^i) + u_1(x)\|x - x_0\| \\ &\vdots \\ f_n(x) &= f_n(x_0) + \sum_{i=1}^{n} \frac{\partial f_n(x_0)}{\partial x^i}(x^i - x_0^i) + u_n(x)\|x - x_0\|, \end{aligned}$$

where $\liminf_{x \to x_0} u_i(x) = 0$, $\forall i \in \{1, \dots, n\}$. Usage of the above formulae gives

$$\frac{\langle f(x) - f(x_0),\, x - x_0 \rangle}{\|x - x_0\|^2} = \frac{1}{\|x - x_0\|^2} \left[\sum_{i,j=1}^{n} \frac{\partial f_i(x_0)}{\partial x^j}(x^i - x_0^i)(x^j - x_0^j) + \sum_{i=1}^{n} u_i(x)\|x - x_0\|(x^i - x_0^i) \right].$$

By the relations (a) and (b) one obtains

$$\frac{\langle f(x) - f(x_0),\, x - x_0 \rangle}{\|x - x_0\|^2} = \frac{1}{\|x - x_0\|^2} \left[\frac{\partial f_1(x_0)}{\partial x^1}\|x - x_0\|^2 + \sum_{i=1}^{n} u_i(x)\|x - x_0\|(x^i - x_0^i) \right] = \frac{\partial f_1(x_0)}{\partial x^1} + \sum_{i=1}^{n} \frac{u_i(x)(x^i - x_0^i)}{\|x - x_0\|},$$

wherefrom, because $-1 \le x^i - x_0^i/\|x - x_0\| \le 1$ and $\liminf_{x \to x_0} u_i(x) = 0$, it follows that

$$f^{\#}(x_0) = \liminf_{x \to x_0} \frac{\langle f(x) - f(x_0),\, x - x_0 \rangle}{\|x - x_0\|^2} = \frac{\partial f_1(x_0)}{\partial x^1} = \cdots = \frac{\partial f_n(x_0)}{\partial x^n}.$$

□

1.1.3 Case $n = 2$. The Relation of the Scalar Derivative with the Complex Derivative

We identify in this chapter the complex numbers with points in $\mathbb{R}^2$. The scalar product of these numbers means the scalar product of the vectors representing them in $\mathbb{R}^2$.

THEOREM 1.11 *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function. The following statements are equivalent.*

1. *f is differentiable in z_0 as a complex function.*

2. *f is differentiable in z_0 as a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ and is scalarly differentiable in this point.*

Proof. Follows directly from Theorem 1.10. □

The differentiability condition of f at z_0 in 2. is essential. In examples 2 and 3. of Section 1.1.4 we construct two discontinuous mappings at 0, which are scalarly differentiable in this point.

REMARK 1.3

1. *Let G be an open subset of $\mathbb{C}$. Then f is holomorphic on G if and only if it is differentiable as a vector function and scalarly differentiable on G. As is well known, the set of holomorphic functions on $\mathbb{C}$ is closed with respect to the compositions of functions.*

2. *The above remark justifies the following generalization of a holomorphic function.*

DEFINITION 1.12 *Let G be open in $\mathbb{R}^n$. The mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called* $\mathbb{R}$-holomorphic *on G if and only if it is differentiable and scalarly differentiable on G. The set of $\mathbb{R}$-holomorphic mappings on G is denoted by $\mathcal{H}(G)$.*

THEOREM 1.13 *For the complex function $f : \mathbb{C} \to \mathbb{C}$ the following statements are equivalent*

1. *f is differentiable in $z_0 \in \mathbb{C}$ as a complex function.*

2. *f and $\mathrm{i}f$ are scalarly differentiable in z_0.*

Proof. Let us denote $f = u + \mathrm{i}v$, $z = x + \mathrm{i}y$, $z_0 = x_0 + \mathrm{i}y_0$. Then

$$\begin{aligned}\frac{f(z)-f(z_0)}{z-z_0} &= \frac{u(z)-u(z_0)+\mathrm{i}(v(z)-v(z_0))}{x-x_0+\mathrm{i}(y-y_0)} = \\ &= \frac{[u(z)-u(z_0)+\mathrm{i}(v(z)-v(z_0))][x-x_0-\mathrm{i}(y-y_0)]}{(x-x_0)^2+(y-y_0)^2} \\ &= \frac{[u(z)-u(z_0)](x-x_0)+[v(z)-v(z_0)](y-y_0)}{(x-x_0)^2+(y-y_0)^2} \\ &\quad + \mathrm{i}\frac{[v(z)-v(z_0)](x-x_0)-[u(z)-u(z_0)](y-y_0)}{(x-x_0)^2+(y-y_0)^2} \\ &= \frac{\langle f(z)-f(z_0),\, z-z_0\rangle}{|z-z_0|^2} - \mathrm{i}\frac{\langle (\mathrm{i}f)(z)-(\mathrm{i}f)(z_0),\, z-z_0\rangle}{|z-z_0|^2}.\end{aligned}$$

From the obtained relation it follows that

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists; that is, f is differentiable in z_0 as a complex function if and only if the limits

$$\lim_{z\to z_0}\frac{\langle f(z)-f(z_0),\, z-z_0\rangle}{|z-z_0|^2}$$

and

$$\lim_{z\to z_0}\frac{\langle (\mathrm{i}f)(z)-(\mathrm{i}f)(z_0),\, z-z_0\rangle}{|z-z_0|^2}$$

exist. □

REMARK 1.4 *The function f is holomorphic on the open set $G \subset C$ if and only if f and $\mathrm{i}f$ are scalarly differentiable on G.*

THEOREM 1.14 *If $f : \mathbb{C} \to \mathbb{C}$ is differentiable in z_0 as a complex function then f and $\mathrm{i}f$ are scalarly differentiable in z_0 and the relation*

$$f'(z_0) = f^{\#}(z_0) - \mathrm{i}(\mathrm{i}f)^{\#}(z_0)$$

holds. This relation is equivalent with the relations

$$\begin{cases} \mathrm{Re}\, f'(z_0) = f^{\#}(z_0), \\ \mathrm{Im}\, f'(z_0) = -(\mathrm{i}f)^{\#}(z_0). \end{cases}$$

Proof. It is indeed contained in the proof of Theorem 1.13. □

1.1.4 Miscellanea Concerning Scalar Differentiability

Examples and counterexamples

1. Let $n \in \mathbb{N}$; $n > 2$. Then the set $\mathcal{H}(\mathbb{R}^n)$ of the holomorphic functions on $\mathbb{R}^n$ is not closed under compositions of functions.

   Indeed consider $A : \mathbb{R}^n \to \mathbb{R}^n$ represented by the matrix $A = (a_{ij})_{i,j} = 1, \ldots, n$, where

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i < j, \\ 0 & \text{if} \quad i = j, \\ -1 & \text{if} \quad i > j. \end{cases}$$

   Obviously, A is a skew-adjoint operator. Hence A is holomorphic on $\mathbb{R}^n$. Consider $A^2 = A \circ A$ and assume that it is holomorphic. Then by Theorem 1.7 it must be of the form $A^2 = B + cI_n$ with B a skew-adjoint linear operator, c a real number, and I_n the identical map. Let us denote the matrix representing A^2 by $D = (d_{ij})_{i,j=1,\ldots,n}$, then $d_{12} + d_{21} = 0$. That is

$$(a_{11}a_{12} + \cdots + a_{1n}a_{n2}) + (a_{21}a_{11} + \cdots + a_{2n}a_{n1}) = 0.$$

   From the definition of A it follows that $2(n-2) = 0$ and hence $n = 2$, contradicting the hypothesis on n.

   From this example and the results in Section 1.1.3 the next assertion follows.

THEOREM 1.15 *The set of scalarly differentiable linear mappings in* $\mathbb{R}^n$ *is closed under composition if and only if* $n \leq 2$.

From the definition of the scalar derivative the next assertion follows easily.

LEMMA 1.16 *Let* $0 = (0, \ldots, 0) \in \mathbb{R}^n$ *and let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a mapping having the properties:*

(a) $f(0) = 0$.

(b) $\langle f(x), x \rangle = 0$, $\forall x \in \mathbb{R}^n$.

Then f is scalarly differentiable in 0 and $f^{\#}(0) = 0$.

Usage of this lemma allows us to construct the following two examples of discontinuous mappings at 0, which are scalarly differentiable in this point.

2. $f : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f(x,y) = \begin{cases} \left(\dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2}\right) & \text{if} \quad x^2+y^2 \neq 0 \\ (0,0) & \text{if} \quad x = y = 0. \end{cases}$$

3. $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$, with

$$f(x) = \begin{cases} \dfrac{1}{\|x\|^2} Ax & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

where A is a nonzero, linear, skew-adjoint operator.

In fact, Example 3 generalizes Example 2. Both mappings satisfy the conditions of the above lemma and hence they are scalarly differentiable in 0. Let us show that f in 3. is not continuous at 0. Because $A \neq 0$, there exists some t in $\mathbb{R}^n$ with $At \neq 0$. Put $x = \lambda t$, $\lambda > 0$. Then the relation

$$\liminf_{\lambda \downarrow 0} \frac{A(\lambda t)}{\lambda^2 \|t\|^2} = \liminf_{\lambda \downarrow 0} \frac{1}{\lambda} \frac{At}{\|t\|^2} \neq 0$$

shows that f is not continuous at 0.

4. Example of a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is continuous at 0, scalarly differentiable in this point, but not differentiable as a vector function.

Consider f given by

$$f(x,y) = \begin{cases} \left(\dfrac{xy^2}{x^2+y^2}, \dfrac{-x^2y}{x^2+y^2}\right) & \text{if} \quad x^2+y^2 \neq 0, \\ (0,0) & \text{if} \quad x = y = 0. \end{cases}$$

Then f fulfills the conditions of the lemma; hence it is scalarly differentiable in 0.

The continuity of the two components of f is a standard exercise in calculus.

If f is differentiable in 0, then its components f_1 and f_2 are differentiable real-valued functions. Because

$$\frac{\partial f_1(0,0)}{\partial x} = \frac{\partial f_1(0,0)}{\partial y} = 0,$$

then $df_1(0,0) = 0$. Hence we cannot have

$$\liminf_{\substack{x\to 0\\ y\to 0}} \frac{|f_1(x,y) - f_1(0,0) - df_1(0,0)(x,y)|}{\sqrt{x^2+y^2}} \neq 0.$$

Taking for instance $x = y > 0$ the above limit will be $1/(2\sqrt{2})$.

5. Example of a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is continuous at 0, possesses partial derivatives at 0, does not satisfy the Cauchy–Riemann conditions at 0, but is scalarly differentiable in 0.

   Take
$$\begin{cases} \left(\dfrac{-y^3}{x^2+y^2}, \dfrac{xy^2}{x^2+y^2}\right) & \text{if} \quad x^2+y^2 \neq 0, \\ \\ (0,0) & \text{if} \quad x = y = 0. \end{cases}$$
   The continuity of f at 0 can be verified passing to polar coordinates. By direct verification
$$\frac{\partial f_1(0,0)}{\partial x} = \frac{\partial f_2(0,0)}{\partial x} = \frac{\partial f_2(0,0)}{\partial y} = 0,$$
$$\frac{\partial f_1(0,0)}{\partial y} = -1;$$
   that is, the Cauchy–Riemann conditions do not hold at 0.

   The scalar differentiability of f at 0 follows from the fact that it satisfies the conditions of the lemma.

6. Example of a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is continuous at 0, satisfies the Cauchy–Riemann conditions at 0, but is not scalarly differentiable at this point.

   Take
$$\begin{cases} \left(\dfrac{x^2y}{x^2+y^2}, \dfrac{x^2y}{x^2+y^2}\right) & \text{if} \quad x^2+y^2 \neq 0, \\ \\ (0,0) & \text{if} \quad x = y = 0. \end{cases}$$
   As in the above examples, the components of f are continuous at 0. Furthermore,
$$\frac{\partial f_1(0,0)}{\partial x} = \frac{\partial f_1(0,0)}{\partial y} = \frac{\partial f_2(0,0)}{\partial x} = \frac{\partial f_2(0,0)}{\partial y} = 0,$$
   that is, the Cauchy–Riemann conditions are satisfied at 0.

Assume that f is scalarly differentiable in 0. Then the limit

$$\liminf_{\substack{x\to 0\\ y\to 0}} \frac{f_1(x,y)x + f_2(x,y)y}{x^2+y^2} = \liminf_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y(x+y)}{(x^2+y^2)^2}$$

must exist. Put $x = 0$ and $y \to 0$. Then this limit will be 0. Put $x = y \neq 0$, $x \to 0$. Then this limit will be $1/2$. That is, the limit does not exist.

7. Example of a nonlinear $\mathbb{R}$-holomorphic mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ for arbitrary n $(n > 2)$.

   By direct verification it can be seen that

$$f(x^1, x^2, \ldots, x^n) = \left((x^1)^2 - (x^2)^2 - \cdots - (x^n)^2, 2x^1x^2, \ldots, 2x^1x^n\right)$$

   has scalar derivative $f^{\#}(x^1, x^2, \ldots, x^n) = 2x^1$ and satisfies the Cauchy–Riemann conditions at every point.

REMARK 1.5 *In* Ahlfors [1981] *it is proved that there are no nonlinear mappings of other type as that in* 7. *which satisfies the Cauchy–Riemann equations at every point (i.e., there are no nonlinear* $\mathbb{R}$*-holomorphic mappings of other type). Because* $\mathcal{H}(\mathbb{R}^n)$ *for* $n > 2$ *is not closed with respect to the composition (see Example* 1 *and Theorem* 1.15*), we cannot derive other holomorphic mappings in this way.*

1.1.5 Characterization of Monotonicity by Scalar Derivatives

By using the notion of an upper (lower) scalar derivative we obtain the following assertion

THEOREM 1.17 *Let* G *be an open convex set in* $\mathbb{R}^n$. *Then the following statements are equivalent.*

1. $f : G \to \mathbb{R}^n$ *is an increasing (decreasing) mapping.*
2. $\underline{f}^{\#}(x) \geq 0$ $(\overline{f}^{\#}(x) \leq 0)$ *for each* x *in* G.

Proof.. The implication $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 1$ Take $\varepsilon > 0$ arbitrarily and put $g = f + \varepsilon I_n$. Then

$$\underline{g}^{\#}(x) = \underline{f}^{\#}(x) + \varepsilon > 0, \ \forall x \in G.$$

Take a, b in G; $a \neq b$. For x in the line segment $[a, b]$ determined by a and b, one has by hypothesis:

$$\liminf_{y\to x} \frac{\langle g(y) - g(x),\, y - x\rangle}{\|y - x\|^2} > 0,$$

and hence there exists $\delta(x) > 0$ such that for any y in $I_x =]x - \delta(x)(b-a), x + \delta(x)(b-a)[\subset G$, $\langle g(y) - g(x),\, y - x\rangle > 0$ holds as far as $y \neq x$. Obviously,

$$[a, b] \subset \bigcup_{x \in [a,b]} I_x;$$

that is, $\{I_x : x \in [a, b]\}$ is an open cover of the compact set $[a, b]$. Hence

$$[a, b] \subset I_{y_1} \cup I_{y_2} \cup \cdots \cup I_{y_{m-1}}$$

for an appropriate set $y_1, \ldots, y_{m-1}$ of points in $]a, b[$. We can suppose that $y_1, \ldots, y_{m-1}$ are ordered from a to b. Hence $a = y_0 \in I_{y_1}$, $b = y_m \in I_{y_{m-1}}$. We can also consider that no interval I_{y_i} is contained in any other. Take $\xi_i \in I_{y_{i-1}} \cap I_{y_i} \cap]y_{i-1}, y_i[$. Then by the construction of these intervals

$$\langle g(\xi_i) - g(y_{i-1}),\, \xi_i - y_{i-1}\rangle > 0,$$

$$\langle g(y_i) - g(\xi_i),\, y_i - \xi_i\rangle > 0$$

and because ξ_i is in $]y_{i-1}, y_i[$,

$$y_i - \xi_i = \alpha(y_i - y_{i-1}),$$

$$\xi_i - y_{i-1} = \beta(y_i - y_{i-1}),$$

for appropriate positive α and β. Hence

$$\langle g(\xi_i) - g(y_{i-1}),\, y_i - y_{i-1}\rangle > 0,$$

$$\langle g(y_i) - g(\xi_i),\, y_i - y_{i-1}\rangle,$$

wherefrom

$$\langle g(y_i) - g(y_{i-1}),\, y_i - y_{i-1}\rangle > 0.$$

But $y_i - y_{i-1} = \lambda_i(b - a)$ for some positive λ_i, and then we must also have

$$\langle g(y_i) - g(y_{i-1}),\, b - a\rangle > 0.$$

By summing the above relations from $i = 1$ to $i = m$, we obtain

$$\langle g(b) - g(a),\, b - a\rangle > 0.$$

Rewriting this relation using the definition of g we have

$$\langle f(b) - f(a),\, b - a\rangle + \varepsilon\|b - a\|^2 > 0.$$

By letting $\varepsilon \to 0$ we conclude that

$$\langle f(b) - f(a),\, b - a\rangle \geq 0.$$

The case $\overline{f}^{\#}(x) \leq 0, \forall x \in G$ can be handled similarly. □

THEOREM 1.18 *Let G be an open convex set in $\mathbb{R}^n$ and suppose that $f : G \to \mathbb{R}^n$ satisfies*

$$\underline{f}^{\#}(x) > 0 \ \ (\overline{f}^{\#}(x) < 0), \ \forall x \in G.$$

Then f is strictly increasing (strictly decreasing) on G.

The proof of this theorem is in fact contained in the proof of Theorem 1.17.

COROLLARY 1.19 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given. The following statements are equivalent.*

1. $f^{\#}(x) = 0, \forall x \in \mathbb{R}^n$.
2. *f is an affine mapping with a skew-adjoint linear part.*

Proof. The implication $2 \Rightarrow 1$ is trivial.

To show that $1 \Rightarrow 2$ we apply Theorem 1.17 to conclude that

$$\langle f(y) - f(x), \, y - x\rangle = 0,$$

$\forall x, \, y \in \mathbb{R}^n$, and then by usage of Theorem 1.4 we conclude Assertion 2. □

THEOREM 1.20 *Let G be convex and open in $\mathbb{R}^n$ and let $f : G \to \mathbb{R}^n$ be a Gateaux differentiable mapping on G. Then the following statements are equivalent.*

1. *f is increasing (decreasing) on G.*
2. *The Gateaux differential of f is positive (negative) semi-definite in every point of G.*

Proof. $1 \Rightarrow 2$ Suppose that $\langle f(y) - f(x), \, y - x\rangle \geq 0 \; \forall x, \, y \in \mathbb{R}^n$. Take $y = x + \lambda t$ with $\lambda \in \mathbb{R}$, $\lambda > 0$, and $t \in \mathbb{R}^n$ arbitrarily. Then

$$\left\langle \frac{f(x+\lambda t) - f(x)}{\lambda}, t \right\rangle \geq 0$$

and letting $\lambda \to 0$ we obtain for the Gateaux differential $\delta f(x)$ that $\langle \delta f(x)t, \, t\rangle \geq 0$.

$2 \Rightarrow 1$ Suppose that the Gateaux differential is positive semi-definite at each point of G. Take a, b in G, $a \neq b$ and $x \in [a, b]$. Then $\langle \delta f(x)t, \, t\rangle \geq 0$ for every $t \in \mathbb{R}^n$; that is,

$$\liminf_{\lambda \downarrow 0} \left\langle \frac{f(x+\lambda t) - f(x)}{\lambda}, t \right\rangle \geq 0, \, \forall t \in \mathbb{R}^n.$$