Research in Mathematics Education

Series Editors: Jinfa Cai · James A. Middleton

Paul Christian Dawkins
Amy J. Hackenberg
Anderson Norton *Editors*

Piaget's Genetic Epistemology for Mathematics Education Research

Springer

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Series Editors
Jinfa Cai, Newark, DE, USA
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Editors

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To our teachers, most prominently the children who teach us their mathematics as well as Les Steffe and Pat Thompson who paved the way for this flourishing work.

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About the Authors

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Steven Boyce is an associate professor of Mathematics Education at Portland State University in Portland, Oregon. His passion is modeling students' mathematical thinking, in particular connections between adolescents' and adults' coordinations of numerical and quantitative units across contexts. Dr. Boyce is the author of dozens of articles and conference proceedings pertaining to units coordination, fractions, pre-calculus, calculus, and teacher education. He is currently serving as guest editor of a special issue on Units Construction and Coordination across the Curriculum, PK-20, to appear in the journal *Investigations in Mathematics Learning* in Fall 2024.

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Hwa Young Lee received her PhD in Mathematics Education from the University of Georgia and is an associate professor in the Department of Mathematics at Texas State University. Her main research interest is in investigating students' mathematical thinking—specifically, students' constructions of frames of reference, coordinate systems, and graphs—and in learning how teachers can facilitate and support their students' mathematical thinking.

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Kevin C. Moore Born and raised in Ohio, I attended The University of Akron from 2001 to 2006. I worked as a Graduate Assistant in the Department of Mathematics and grew curious about my students' mathematical thinking when teaching as part of the assistantship duties. This curiosity landed me at Arizona State University under the guidance of Professor Marilyn P. Carlson. I immediately grew interested in the constructivist movement in mathematics education, and specifically the ability to take a scientific-inquiry approach to modeling students' mathematical thinking. Since this initial interest, I have rooted myself with other researchers who participate in this progressive research program in the hopes of better understanding students' mathematical thinking, improving the teaching and learning of mathematics, and opposing outcome-based forces in education.

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Interdisciplinary Research on Number (IRON) with the epistemologist, Ernst von Glasersfeld, and the philosopher, John Richards, starting circa 1977 and lasting until circa 2000. From that point on, he formed the research program Ontogenetic Analysis of Algebraic Knowing (OAK) which lasted until he retired in 2017.

Irma E. Stevens is an assistant professor in the Department of Mathematics and Applied Mathematical Sciences at the University of Rhode Island. She earned her PhD in Mathematics Education and MA in Mathematics at the University of Georgia and her BS in Mathematics at the University of North Carolina at Charlotte. Her research interests involve teacher decision making and student learning at the secondary and postsecondary level. Her current projects include understanding undergraduate students' reasoning with dynamic quantities and formulas. Her work uses technology that has resulted in the construction of materials used for research, professional development, and classroom instruction.

Michael A. Tallman received his BSc and MA in Mathematics from the University of Northern Colorado and his PhD in Mathematics Education from Arizona State University. Dr. Tallman has taught mathematics at the secondary and post-secondary levels. His primary research is in the area of mathematical knowledge for teaching secondary and post-secondary mathematics. Dr. Tallman's work informs the design of teacher preparation programs and professional development initiatives through an investigation of the factors that affect the nature and quality of the mathematical knowledge teachers leverage during instruction. In particular, his research examines how various factors like curricula, emotional regulation, identity, and teachers' construction and appraisal of instructional constraints mediate the enactment of their subject matter knowledge.

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Part I
Introduction to Piaget's Genetic Epistemology and the Tradition of Use Featured in This Book

Chapter 1
Introduction to Piaget's Genetic Epistemology

Paul Christian Dawkins, Amy J. Hackenberg, and Andy Norton

Introduction

Piaget is known for his work in developmental psychology, but he began his career as a biologist whose primary interests evolved into epistemology; that is, theories of knowledge and knowing. While studying snails, he was introduced to Bergson's (1998) idea of creative evolution, in response to which he later said,

> The problem of knowing (properly called the epistemological problem) suddenly appeared to me in an entirely new perspective and as an absorbing topic of study. It made me decide to consecrate my life to the biological explanation of knowledge. (Vidal, 1994, p. 52)

Piaget began to study children's psychological development as a means of investigating the biological origins of logic and mathematics. In this pursuit, he followed the biogenetic law that "ontology recapitulates phylogeny": the development of the individual follows a similar trajectory as the development of humankind.

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We can sum up the constructivist epistemology[1] with the mantra, “All knowledge is constructed” (Glasersfeld, 1995, p. 160). Here, we focus on Piaget’s genetic epistemology, which implies more than its constructivist tenets. We frame Piaget’s genetic epistemology in terms of his lifelong pursuit to understand the power and origins of logico-mathematical operations. It directs us back to the biological origins of our logico-mathematical operations and to the structures that organize them so that they, in turn, might organize the world.

To say “all knowledge is constructed” is to challenge the notion that knowledge comes in from outside a person in some simple or reliable way. Certainly, humans build knowledge based on their experiences in their environments, but those experiences are unique to each individual. In the same way, the physical form that an organism takes is largely driven by its internal structure (e.g., DNA), humans’ construction of knowledge operates through internal structures in a complex interplay with (their sensorimotor experience of) their environment. This is primarily an epistemological claim about the construction of knowledge, not an ontological claim about the extent to which the knowledge so constructed is, in some sense, an “accurate” depiction of the world. A key feature of this epistemology is that it can explain learning without a strong notion of correspondence to reality.

Modern biology shows that different kinds of animals experience the world very differently from humans. By framing human psychology as fundamentally biological, we gain at least two important perspectives. First, we learn to respect how much our experience of the world is structured by our bodies and by our cognitive processes. Second, we may consider how much those experiences have changed over the course of our lifetimes. Children at different stages of development may experience different worlds and reason in markedly different ways from us as adults. We may question how much of our current experience depends intrinsically on earlier constructions that we cannot recall ever having done without.

In his genetic epistemology, Piaget adopted a Kantian perspective, but empirical results from his research with children challenged core assumptions in Kant’s philosophy. Kant’s (1781) *Critique of Pure Reason* blended empiricism with rationalism by accepting a few principal cognitive structures as innate. These innate structures included space, time, and number, which enable us to organize our experiences in the world. They also explain how we might take experiences as shared. If we all construct reality within the same God-given space–time framework, we can expect some commonalities among those realities. However, Piaget demonstrated that children construct these foundational structures too, during their first few years of life. In other words, Piaget’s research countermanded Kant’s assumptions. Few, if any of us, can recall the years we spent playfully organizing our worlds, so we take those constructions (e.g., space, time) for granted.

[1] We have found Noddings (1990) explanation of constructivist epistemology, especially illuminating.

Consider Kant's (1781) famous statement: "The concept of Euclidean space is by no means of empirical origin, but is the inevitable necessity of thought." From our perspective now, this statement is an error. The mere possibility of non-Euclidean geometries, such as the one Gauss invented during the nineteenth century, refutes Kant's claim. Another century later, Piaget and Inhelder (1967) demonstrated that children construct space during their first years of life, on the basis of their own sensorimotor activity in the world. Thereafter, the objects children experience have a home to persist in even when they are out of sight (i.e., object permanence).

Piaget demonstrated that children construct number, too. Steffe has elaborated on this construction through learning levels that he referred to as children's number sequences. At every stage, development depends upon the coordination of actions—first sensorimotor, then internalized as mental actions, and finally organized within structures that render them logico-mathematical operations. These structures, both spatial and numerical, serve the role Kant envisioned for them, but they are the result of years of labor. Once we have constructed them, it becomes difficult to imagine a world without them.

What then allows us to build up concepts like space, time, and number if we do not, as Kant claimed, begin with certain structures already in place? The heart of Piaget's answer to this is the organization of our own activity. Children act in their experiential worlds, and their organization of those actions provides the basis for the organization of their experiential worlds.

Piaget's genetic epistemology emphasizes the unique status of logico-mathematical operations within human knowledge. It also affords a different account of mathematical objects themselves. If knowledge corresponds to reality in a strong sense, then it raises questions about the nature and source of abstract concepts such as number, line, function, and set. The philosophical stance known as Platonism classically solves this problem by asserting the real existence of abstract entities (an ontological claim). This allows us to somehow learn abstract concepts under the assumption that they come in from the outside world (an epistemological claim). Since constructivism provides an alternative account of how concepts form, it provides a resolution of this epistemological issue that can remain ontologically neutral. It thus provides an alternative to Platonism in explaining the power of logic and mathematics. This power owes to the structures that we construct through the coordination of our own mental actions rather than structures imposed upon us by the worlds we ourselves organize through those very same structures.

> In all, the chief apparent advantage of Platonism, which is to account for the objective robustness of logico-mathematical entities and structures, is guaranteed in the same way by the concept of the general and internal co-ordinations of actions and operations. That hypothesis that ideal entities are external is thus unnecessary to guarantee the independence of structures in view of the free will of individual subjects. (Beth & Piaget, 1966, p. 294)

Why Is Piaget's Genetic Epistemology Useful?

For Glasersfeld (1995), a way of knowing is valuable to the extent it is useful. Another way to say this is that people construct ways of knowing to serve purposes. We apply this orientation repeatedly in this book to articulate why Piaget's genetic epistemology—and the research tradition in Mathematics Education that has been built from it—is useful. One reason it is useful is that it acknowledges that people build up knowledge to organize their experiential worlds and pursue goals within those worlds, not to describe an observer-independent world. As Glasersfeld (1995) pointed out, Piaget was not the first to take this position on knowledge, but he was the first to take a developmental approach (p. 13). As we have introduced above, Piaget viewed the construction of knowing in an individual (1) to be a process of construction over a lifetime and (2) to reflect the construction of knowing in humans as a species. The first point means that no person's ways of knowing are ever complete—they are always evolving. The second point means that understanding the nature of knowing requires studying its ontogenesis—its development in humans across their lives.

In our experience, as researchers and teachers in mathematics education, Piaget's views on knowing provide the basis for generating rich tools for describing and accounting for students' mathematics. Piaget's views also enlarge what is considered mathematical—and, therefore, who is considered a mathematical thinker. Piaget's views on knowing imply that a great variety of ways of knowing and thinking can be admitted into mathematical knowledge, including children's mathematics (Steffe & Olive, 2010). So, researchers can co-construct with participants, including children, ways of thinking that can be understood as mathematical beyond what has traditionally been viewed as mathematical. These ways of thinking are models of participants' mathematical knowledge that researchers can use to support future interactions with other participants (see Chaps. 9 and 14). As a consequence, students whose ways of thinking differ from what has traditionally been considered standard mathematical ideas can be legitimated, and these students can be seen more fully as mathematical thinkers (e.g., Hackenberg, 2013; Hackenberg & Sevinc, 2021; Norton & Boyce, 2015).

Piaget is known for developmental stages (e.g., concrete operations, formal operations) that have been critiqued as being rigid and nonrepresentative of all people. We would like to address that directly using children's number sequences (Steffe et al., 1983; Steffe & Cobb, 1988) as an example. In this research, Steffe and colleagues studied how young children construct whole numbers by studying how they count and how the nature of counting changes with successive constructions. They found that children constructed approximately four number sequences, and these occur in order because later number sequences involve more complex organizations of units. Such descriptions can help teachers and researchers organize instructional interactions with a range of elementary school students. Yet, the descriptions of number sequences of children do not follow a lockstep set of stages at the same ages—that is not what the developmental aspect of genetic epistemology means. So, the developmental aspect of genetic epistemology means that children tend to construct number sequences in a certain order, but there is great variation in how students at a particular age conceive of number.

And yet, the usefulness of genetic epistemology does not stop with description. The use and development of other Piagetian tools, such as accommodation and reflective abstraction (discussed in Chaps. 3 and 6), provide the means for explaining students' mathematical thinking and learning.[2]

Together, descriptions of mathematical thinking with explanations of how learning proceeded—or did not proceed—are key components of making models of students' mathematical thinking and learning (Steffe & Thompson, 2000; Ulrich et al., 2014). Consistent with Piaget's overall insight that knowledge need not be explained in terms of correspondence with reality, researchers using genetic epistemology recognize research as their process of knowledge construction. As a result, we must be careful to distinguish what we try to learn about student knowledge (our models of their knowing) from student knowledge itself. We cannot know whether these models, co-constructed with students, are what we would experience if we somehow became these students or otherwise fully adopted their ways of knowing. Students' ways of knowing are not directly accessible to us. Instead, the models are our ways of knowing that fit with our interactions with the students—they are what Steffe refers to as second-order knowledge (2010), or the mathematics of students.[3] Such models take extensive work for researchers to construct and refine (see Chap. 14). Robustly developed models can be regarded as legitimate mathematical ways of knowing—and thus, what gets considered to be mathematics gets expanded.

For example, when working to build fractions knowledge, students who are trying to draw 3/5 of a bar can learn to partition the bar into five equal parts, take out one part, and repeat the part to make three parts (Fig. 1.1). In other words, they can create 3/5 of a bar as 1/5 of the bar, another 1/5, and another 1/5. To observers, it might look like the student thinks of 3/5 as 3 times 1/5. However, that may not be the case. Third- through fifth-grade students taught Steffe and Olive (2010) that they may not think of 3/5 in this way. Rather, they may rely on part–whole meanings for the result, thinking of 3/5 as three parts out of five, despite the actions they took to make the 3/5. Because this way of thinking was a regularity in how students operated in a longitudinal teaching experiment (Steffe & Olive, 2010), Steffe and Olive formulated a scheme (see Chaps. 2 and 3 of this volume) to describe these students' way of thinking about fractions, the partitive fraction scheme (Steffe & Olive, 2010). This way of thinking about fractions is challenging to understand for those who conceive of fractions as multiples of unit fractions; it is hard to see that the students' meaning could be non-multiplicative when the actions look like what a person would do who thinks of 3/5 as 3 times 1/5. Indeed, a person with multiplicative meanings could engage in the same physical behavior for 3/5. However,

[2] We will use the term "students" rather than "children," since not all research has been with young children.

[3] First-order mathematical knowledge is the mathematical ways of knowing we have built to organize our own experiential worlds.

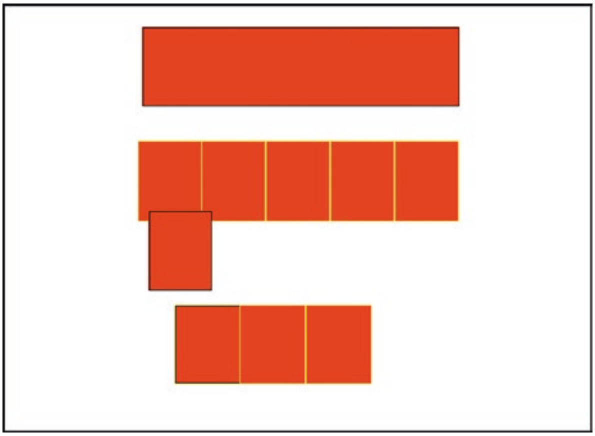

Fig. 1.1 3/5 as three of 1/5

differences quickly arise for students when fractions exceed the whole (e.g., Hackenberg, 2007; Norton & Wilkins, 2012; Steffe & Olive, 2010). Students who have constructed a partitive fraction scheme find drawing, for example, 7/5 of a bar, very mysterious. How can a person draw 7 parts out of 5?

This example shows both aspects of the usefulness of Piaget's views: The partitive fraction is an example of the use of scheme as a powerful tool for modeling student thinking, and this scheme is an expansion of ways of thinking with fractions that can be considered legitimately mathematical.

Second-order models of particular students can be very satisfying to make: When they are developed, they represent to the researcher an understanding of the ways of thinking of the students, and they can show why it makes sense that a student solved a problem or thought about a topic in a particular way. Thus, models can provide a researcher with a great sense of fit. And yet, the models are actually instruments of interaction (Steffe & Olive, 2010): They allow researchers to better interact with these particular students because the researchers can base problems and questions on the ways of thinking in the model. Doing so can facilitate communication about mathematical ideas with particular students. This aspect of models can also feel satisfying because it can engender a sense of connection between the researcher and students (see Chaps. 9 and 14).

And yet, if the models were only useful for the particular students with whom researchers were working in particular studies, that would be quite limiting as research. Fortunately, experience shows that is not at all the case. Another reason genetic epistemology is useful is that the models developed with a few students usually allow researchers to interact more broadly with other students who have similarities to the students from whom the models were made (see Chap. 14). So, as researchers build models for particular students, they are usually building models that are useful with a wide range of students.

Organization of the Book

We mention these models of students' mathematics because they portray how Piaget's genetic epistemology has contributed to mathematics education research. More importantly, Piaget provided a rich set of theoretical tools for pursuing this kind of research. The goal of this book is not to describe particular models of students' mathematics that have been developed but rather to describe the tools that mathematics educators use to construct such models. We have thus organized the main body of the book—Part 2, Chaps. 3, 4, 5, 6, 7, 8, 9, 10, and 11—around clusters of related constructs. To be precise, the first seven of those chapters describe constructs directly descended from Piaget's research, and the last two describe constructs developed later on but whose importance to mathematics education research warranted their inclusion in this volume.

The rest of the book—Parts 3 and 4, Chaps. 12, 13, 14, 15, 16, 17, 18, 19, 20, and 21—contains two parts corresponding to two different ways of building on the construct chapters in Part 2. The chapters in Part 3 each contain commentaries on the first part and on genetic epistemology more broadly. The chapters in Part 4 each summarize the research agenda of a younger mathematics education scholar who draws upon genetic epistemology in their work. These final chapters provide further examples of the utility and fecundity of this body of theory.

We could have adopted other organizational approaches such as a historical account of how ideas developed, by focusing on the various scholars who drew upon Piaget in their research, or by surveying key findings and contributions developed in this tradition. We adopted the current organization because we anticipated it would be most useful to scholars who want to learn about these tools to engage in mathematics education research. In other words, we organized the book looking forward to future research rather than trying to survey or summarize previous research. As a result, the contributions of many important mathematics educators who draw heavily upon Piaget's work may be underrepresented or omitted in these pages.

We have included Chap. 2 as an acknowledgment of the history of research and the intellectual heritage by which this body of theory has come to us. Dr. Les Steffe is one of the central scholars who draws upon Piaget's work to study children's mathematics and who trained many of the other authors to do the same. Chapter 2 presents Steffe's historical reflection on Piaget's influence on mathematics education.

This book was formulated to serve as a graduate textbook for those studying to become researchers in mathematics education. We sense that our field needs more such texts, especially regarding rich and complex bodies of theory such as genetic epistemology. We sincerely hope that this book provides a helpful starting point for those newer to these ideas and a productive resource for those more experienced. We have learned much from the chapters our excellent coauthors contributed, which makes us confident that what follows will be of value to the field. It is a joy to be continually engaged as learners: learners of mathematics, especially students' mathematics, and learners among the community of researchers trying to support quality mathematics education.

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Chapter 2
An Historical Reflection on Adapting Piaget's Work for Ongoing Mathematics Education Research

Leslie P. Steffe

Piaget and Modern Mathematics

Piaget was "rediscovered" (Ripple & Rockcastle, 1964) during the 1960s by mathematicians and mathematics educators whose goal was to reform mathematics curricula based on modern mathematics (e.g., Allendoerfer & Oakley, 1959; School Mathematics Study Group, 1965). Logical–mathematical structure served as the basic rationale for the new math programs that, in many cases, resembled collegiate mathematics. Although classical idealism, the doctrine that reality, or reality as we know it, is fundamentally mental, served operationally as the epistemological position of the reformers, empiricism and realism were still the more general positions in the United States as indicated by a return to behaviorism in the decade following the modernist programs. Problem solving, along with learning by discovery, was the major psychological emphases among the reformers (Pólya, 1945, 1981) for which Wertheimer's[1] (1945) work on productive thinking served as a basic rationale.

Piaget's genetic epistemology (Piaget, 1970) did not serve as an epistemological basis for the modern programs, nor was it explicitly emphasized at a conference held at Cornell University and the University of California to investigate implications of Piaget's work for mathematics education (Ripple & Rockcastle, 1964). The interest of the conference organizers was in exploring the implications of Piaget's

[1] Wertheimer was one of the three founders of Gestalt psychology along with Kurt Koffka and Wolfgang Köhler.

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stages of cognitive development as a rationale for the elementary programs because those programs were left without a psychological rationale (Ripple & Rockcastle, 1964). Piaget (1964) was invited to present four papers at the conference that he titled, "Development and learning," "The development of mental imagery," "Mother structures and the notion of number," and "Relations between the notions of time and speed in children." Although he made no reference to genetic epistemology in these papers, by presenting them, he did implicitly explain the concept of genetic epistemology that he presented at the Woodbridge Lectures at Columbia University in 1968 (Piaget, 1970).

> Genetic epistemology attempts to explain knowledge, and in particular scientific knowledge, on the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it is based. (p. 1)

Development vs. Learning

Although he could have oriented his papers as elaborations of genetic epistemology, his emphasis at the conferences was on explaining the cognitive development of number, space, and time as opposed to teaching such concepts and expecting them to be learned. He made a sharp distinction between development and learning in that development is a spontaneous process tied to the whole process of embryogenesis.

> Embryogenesis concerns the development of the body but it concerns as well the development of the mental functions. In the case of the development of knowledge in children, embryogenesis ends only in adulthood. … In other words, development is a process—which concerns the totality of the structures of knowledge. (Piaget, 1964, p. 8)

In contrast, he explained learning as presenting the opposite case.

> In general, learning is provoked by situations. … It is provoked in general as opposed to spontaneous. In addition, it is a limited process—limited to a single problem or to a single structure. (Piaget, 1964, p. 8)

Although I do not regard learning as such a limited process, the developers of the modern programs firmly believed that their programs could be learned and would either accelerate Piaget's account of the cognitive development of basic mathematical notions or essentially replace developmental processes.

> Most, if not all, of the major curricular reform projects have taken the logic (or structures) of the subject matter as a point of departure rather than psychological learning theory or studies of cognitive development. This point is made abundantly clear, for example, in Jeremy Kilpatrick's paper on the SMSG Program included in this report. (Ripple & Rockcastle, 1964, p. iii)

The curriculum developers considered Piaget to be an observer rather than a teacher, and the elasticity of the limits of children's minds was not considered as having been established.

> These reformers (and I speak now not only of SMSG) have been so successful in teaching relatively complex ideas to young children, and thus doing considerable violence to some old notions of readiness, that they have become highly optimistic about what mathematics can and should be taught in the early grades. (Kilpatrick, 1964, p. 129)

The lack of appreciation for genetic epistemology was addressed by Eleanor Duckworth, a former student of Piaget, who served as an intermediary between Piaget and the conference attendees. She addressed the teaching the "structure" of a subject matter.

> The pedagogical idea is that children should be taught the unifying themes of a subject matter area, after which they will be able to relate individual items to this general structure. (This seems to be what Bruner often means by 'teaching the structure' in the Process of Education). (Duckworth, 1964, p. 3)

Developmental vs. Mathematical Structure

The structural emphasis of the modern programs was not compatible with Piaget's emphasis on the structure of operational thought.

> An operation is an interiorized action. … Above all, an operation is never isolated. It is always linked to other operations and as a result it is always a part of a total structure. (Piaget, 1964. p. 7)

A major difficulty was that "structure" had very different meanings for Piaget and for the curriculum developers. Piaget's structures were second-order models, "the hypothetical models observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 1983, p. xvi). The mathematical structures of the modern programs were first-order models, or renditions of the mathematical knowledge of the curricular developers (cf. Steffe et al., 1983). This distinction between the mathematical thought of the child from the point of view of the adult and the adult's own mathematical knowledge that he or she would not attribute to the child has been and remains a major issue in the mathematics education of children. Even though it is assumed in genetic epistemology that the mathematical thinking of children as it evolves over time has something to do with mature mathematical thinking, it takes major decentering for an adult mathematical thinker to think as if he or she is a child (Thompson & Thompson, 1994). Further, in those cases where the adult does learn to think as if he or she is a child, developing models of how such an evolution might occur is quite exacting. Hermine Sinclair succinctly pointed out such difficulties at the level of child thought in attempts by mathematics educators to use Piaget's genetic epistemology at a conference held at Columbia University in 1970.

> At first sight it would seem that a psychological theory that is regarded by its author as a "by-product" of his epistemological research and is therefore principally directed toward the investigation of knowledge and its changes in the history of mankind, as well as in the growing child, is ideally suited to educational applications. (Sinclair, 1971, p. 1)

Sinclair used a metaphor to explain what she regarded as difficulties in trying to apply Piaget's stage theory of cognitive development in an attempt to provoke non-operational children to become operational.

> Piaget's tasks are like the core samples a geologist takes from a fertile area and from which he can infer the general structure of a fertile soil; but it is absurd to hope that transplanting these samples to a field of nonfertile soil will make the whole area fertile. (Sinclair, 1971, p. 1)

Preludes to IRON (Interdisciplinary Research on Number)

Piagetian Research

Professor Henry Van Engen introduced me to Piaget's work while I was a doctoral student at the University of Wisconsin, working as a research associate in the Research and Development Center for Cognitive Learning. Following Bridgeman (1927), mental operations were at the heart of Van Engen's meaning theory of arithmetic, where he defined the meaning of a symbol as an intention to act (Van Engen, 1949a, b). Piaget's emphasis on mental operations and operational thought was a major point of convergence and served as the basis of Van Engen's interest in Piaget (Van Engen, 1971). As a research associate, it was my wont to apply Piaget's theory of the development of number in the mathematics education of young children using scientific methods, which in part translated to investigating the importance of conservation of numerosity of first-grade children on arithmetical tasks using research design (Stanley & Campbell, 1963) and statistical methods (Steffe, 1966). I continued on with this program of research, which became known as "Piagetian Research" (Steffe & Kieren, 1994), for 7 years after joining the faculty of mathematics education at the University of Georgia, a time during which I directed ten doctoral students in applying Piaget's research in the mathematics education of children. Professor Charles Smock of the Psychology Department, who was a Piagetian, served on the committees of most of my doctoral students during that time, which was quite important because he was the one who eventually introduced me to Ernst von Glasersfeld.

A Change in Research Program

I was working as a realist and an empiricist in my attempts to *apply* Piaget's developmental theory in the mathematics education of children, and I was making only accretional rather than recursive progress. As a consequence of making only minimal progress, I abandoned my attempts to apply Piaget's research on number and quantity as well as my statistical method of application and taught a group of first graders with the help of two of my advanced doctoral students for an academic year in an attempt to let children teach me what was important in their numerical ways of operating (Steffe et al., 1976a, b). The importance of this work was two-fold. First, it was the impetus for the development and use of the constructivist teaching experiment as a legitimate scientific method of doing research (Steffe, 1983, 1991b; Steffe & Cobb, 1983; Steffe & Thompson, 2000; Steffe & Ulrich, 2013). Second, the finding that counting was the children's basic method of solving arithmetical tasks led to abandoning attempts to apply Piaget's idea that number is constructed as a synthesis of classification and seriation operations concurrently with the introduction of the arithmetical unit and to investigating children's construction of number sequences in the context of their spontaneous use of counting in solution of arithmetical tasks in teaching experiments (Steffe, 1994, 1996; Steffe et al., 1983; Steffe & Cobb, 1988).

A Fortunate Introduction

It was extremely fortunate, not only to me at the time but also to my doctoral students and to the field of mathematics education at large, that Professor Smock invited me to a seminar given by Ernst von Glasersfeld (Steffe, 2013). The seminar event arranged by Smock occurred around 1974, shortly after the demise of the modern mathematics movement and during the move back to behaviorism that occurred in the 1970s. At the time, the question concerning whether mathematics was invented or discovered held little sway with me even though I had read Piagetian basic books such as *The Child's Conception of Number* (Piaget & Szeminska, 1952), *The Child's Conception of Geometry* (Piaget et al., 1960), *The Child's Conception of Space* (Piaget & Inhelder, 1967), and *The Growth of Logical Thinking from Childhood to Adolescence* (Inhelder & Piaget, 1958). I understood that children developed mathematical knowledge, but what developed I regarded as a prelude to what was "out there" in some mathematical reality. My conception of mathematics was, and still is, widely shared by mathematics educators as well as mathematicians. According to Stolzenberg (1984), it is indisputable that the contemporary mathematician operates within a belief system whose core belief is that mathematics is discovered rather than created or invented by human beings.

My belief in the objective existence of mathematics was seriously questioned by a story Glasersfeld recounted in the seminar. The story, taken from Letvin et al. (1959), clarified that the only contact we have with what is "out there" is through our senses. When talking about a frog as a fly-catcher, he commented that:

> The system [the frog's visual system] as a whole makes the frog an efficient fly-catcher, because it is tuned for small dark "objects" that move about in an abrupt fly-like way. In the frog's natural habitat, as we, who observe the frog see it, every item that possesses the characteristics necessary to trigger the frog's detectors in the proper sequence is a fly or bug or other morsel of food for the frog. But if the frog is presented with a black bead, an air-gun pellet, or any other small dark moving item, it will snap it up as though it were a fly. In fact,