Studies in Universal Logic

Timothy J. Madigan
Jean-Yves Béziau
Editors

Universal Logic, Ethics, and Truth

Essays in Honor of John Corcoran (1937-2021)

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Studies in Universal Logic

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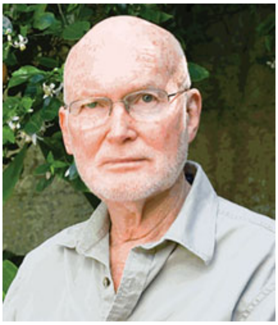
Preface



John Corcoran (March 20, 1937–January 8, 2021). (Photo Courtesy of Lynn Corcoran)

On January 13, 2021, JYB received the following email from the wife of John Corcoran, Lynn Corcoran:

Dear Jean-Yves

I write to you with sad news. John passed away on Friday, January 8.

As I was going through his recent emails, I saw your posting of World Logic Day tomorrow, January 14. I'm sure you are very busy at the moment, but if it would be possible to announce John's death, please do so.

Before he died, John had asked me to notify you and to tell you how much he appreciated your friendship, as well as your enthusiastic and imaginative support of logic. He greatly enjoyed his participation in the conference you organized in Istanbul. I accompanied him on that trip and it was a wonderful experience for both of us.

John became ill suddenly with a blood infection. Aggressive antibiotic treatment over the next two weeks was not successful. He entered hospice, where he received the care of extraordinary people.

Until the day he became ill, he was working on new papers and abstracts, supervising new translations of some of his papers into Arabic, Turkish, Spanish and German, and as always, mentoring young scholars via email. He lived a long and happy life, fully engaged in this world.

In keeping with John's long-standing wishes, there will be no funeral or memorial service.

Best,

Lynn

JYB then decided to organize a volume in honor of John Corcoran and invited his ex-student and long-time friend Tim Madigan to be co-editor of this volume.

We invited friends and colleagues to take part to this volume, and we are glad to have collected a rich collection of papers reflecting the many interests of John and the great variety of topics he has contributed to.

Rochester, NY, USA — Timothy J. Madigan
Rio de Janeiro, Brazil — Jean-Yves Béziau
July 14, 2023

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Introduction: John Corcoran as a Teacher, Mentor, and Friend

Timothy Madigan

> The caricature of logic as a meaningless game of symbol manipulation and the caricature of ethics as a rationalization of blind emotion must both be exposed. Logic and ethics are in fact inseparable and each is served by explicit recognition of its involvement with the other. ([1], p. 37)

When in 1987 I first took John Corcoran's *Introduction to Logic for Advanced Students* seminar as a graduate student in philosophy at the State University of New York at Buffalo, I entered it with considerable fear and trembling. Logic had never been a strong point of mine, and John's reputation on campus was that of a fierce taskmaster. Little did I realize at the time that I would not only love the course, but that it would have a profound impact upon my own thinking and teaching right up to the present day.

On the first day of class, John had us students look at six different standard textbooks on logic, written by some of the biggest names in the field. He asked us to examine how these books defined such standard terms as "validity," "soundness," "premises," and "syllogisms." Much to our astonishment, we found that there was little commonality and indeed much contradiction over the meanings of these basic terms. That was a lesson I have never forgotten, and I have used this technique myself when teaching courses in Logic and Critical Thinking. It demonstrated John's fierce commitment to clarity and precision, coupled with his recognition of human fallibility and the need for cooperation in learning.

From that point on in the course, instead of textbooks we relied on John's handwritten notes from our class discussions, which I have kept ever since and still refer to. Another aspect of John's teaching skills was his ability to connect with all of us by finding out what our own backgrounds were, and what areas of philosophy we were interested in. When he learned that I considered myself to be a pragmatist,

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rather than sneering at me as I had feared, he suggested that I write on Charles Peirce's concept of "abduction" and discuss it with the class. And when he learned that I was critical of organized – and indeed unorganized – religion, I could see that I had found a "soulmate." A few years later, I was grateful when he agreed to be a member of my dissertation committee, headed by his good friend Peter Hare, who had suggested that I write on the Victorian mathematician and logician W.K. Clifford and his 1877 essay "The Ethics of Belief." I much appreciated the helpful written and spoken comments John made regarding my work on Clifford's life and teachings, especially the claim made in "The Ethics of Belief" (known as "Clifford's Principle") that "It is wrong always, everywhere, and for anyone to believe anything on insufficient evidence." This was a statement that resonated with John, as can be seen in this quote from his 1989 essay "The Inseparability of Logic and Ethics":

> To a community of objective thinkers, any attempt to shield a proposition from the testing process reflects badly on those who believe it to be true. Shielding a proposition from testing is seen as shoddy, undignified, and ultimately absurd. A proposition not worth testing is not worth being taken seriously. ([1], pp. 38–39)

In many ways, John continued Clifford's advocacy of questioning and analyzing beliefs and belief-formation, for one's personal benefit but also for the good of the community. Like Peirce, John was a great believer in a Community of Inquirers, and shared Peirce's view (similar to Clifford's Principle) that the first rule of reasoning is "Do Not Block the Way of Inquiry."

Indeed, my fondest memory of John is sitting with him at his kitchen table, going over the proofs of "The Inseparability of Logic and Ethics" which appeared in *Free Inquiry* magazine in 1989 during the time I was an editor there. His careful usage of words, his eagerness – in fact his insistence – on strong criticism, his avoidance of ambiguity and vagueness, and his ability to write sentences that made sense in-and-of-themselves but also connected beautifully with the sentences immediately before and after provided me with a model for writing which I hope I have done justice to in the years since. He not only welcomed criticism, he demanded it. In later years, he would send out drafts of all his articles to his many friends, and nothing gave him greater joy than receiving back from them corrections, objections, and questions, all of which he took to heart. For John writing was always a collaborative effort.

It was a pleasure knowing and working with John for over 30 years, and I cherish the moments when I was able to attend the Buffalo Logic Colloquium events and join him and his many other former students, friends, and colleagues over beers and vigorous discussion afterwards. Whenever I begin to doubt that philosophy is a vital and life-affirming field, I need only think of John Corcoran to be reminded why I was attracted to this discipline and why I continue to toil in its fields.

I continued to stay in touch with John long after I graduated, for in his view anyone who had taken a course with him was a lifelong Logic partner. Since he had done a good deal of writing on the importance of George Boole to Logic, I let John know that I had visited the University of Cork in Ireland to participate in events commemorating the 200th anniversary of Boole's birth. Mentioning that I was a student of John Corcoran's was a perfect way to gain the respect among the

Booleans present, and John was delighted when I sent him a photo of myself next to a young man dressed as Boole. John only regretted that he was unable to attend the event in Cork himself, especially since – like me – he was proud of his Irish heritage.

In 2019, Jean-Yves Béziau, the co-editor of this volume, launched the First UNESCO World Logic Day, to foster and strengthen interactions between people having an interest in Logic. January 14 was chosen as the date, in part because it was the day on which Alfred Tarski was born in 1901. On learning of this, I arranged for my institution, Saint John Fisher College (now Saint John Fisher University), to be involved in this worthy endeavor. Given John's close connection with Tarski, I contacted him to see if he might be available to speak on my campus. But, being a wise man, after his retirement from the University of Buffalo in 2011, he and his wife Lynn had moved to Florida, and coming back to snowy Upstate New York in early January was not feasible. Nonetheless, John was there in spirit, as my colleague David White and I organized "An Open Discussion of John Corcoran's 'The Inseparability of Logic and Ethics' on Its 30th Anniversary" for our campus event. These words from the essay nicely express what World Logic Day is all about:

> The three facts that begin logic – that humans are neither omniscient nor infallible, that humans seek knowledge, and that improvement is possible – are three facts that serve to bring humans together. It is possible to cooperate in the goal, at once noble and practical, to overcome ignorance and fallibility as much as possible. Objectivity automatically involves cooperation and avoidance of deception, whether deception of others or by others, or even deception of and by oneself. It is said that the most destructive lies are those we tell to ourselves. ([1], p. 37)

I made it a point to wish John a Happy World Logic Day in 2019 and 2020, and received an immediate reply in return, for he was always punctual in responding to emails. I did the same in 2021 but was surprised when I didn't hear back from him. A few days later I learned the reason why – he had passed away on January 6 of that year. His widow Lynn let all his friends know that he had died doing what he loved, working on papers and translations of his essays into several languages, mentoring scholars across the globe, and enjoying the beautiful Florida weather. His death, like his life, was an inspiration to us all.

There is no better way to demonstrate John's core commitment to Logic than to reprint his "Farewell Letter to My Students" written upon his retirement in 2011, with his LogicLifetimeGuaranteeTM:

> "Dear Students,
>
> I am saying farewell after more than forty happy years of teaching logic at the University of Buffalo. But this is only a partial farewell. I will no longer be at UB to teach classroom courses or seminars, but nothing else will change. I will continue to be available for independent study. I will continue to write abstracts and articles with people who have taken courses or seminars with me. And I will continue to honor the LogicLifetimeGuaranteeTM you earned by taking one of my logic courses or seminars. As you might remember, according to the terms of the LogicLifetimeGuaranteeTM, I stand behind everything I teach. If you find anything to be unsatisfactory, I am committed to fixing it. If you forget anything, I will remind you. If you have questions, I will answer them – or ask more questions. And

if you need more detail on any topic we discussed, I will help you to broaden and deepen your knowledge. Stay in touch.

"I want to take this opportunity to say something about my intellectual development, and to leave you with some advice. In the four years that I was a graduate student I went to almost every philosophy colloquium. I met several famous philosophers. I asked each of them: 'What is your one piece of advice for a philosophy graduate student?' Only Paul Feyerabend said anything memorable. His advice was to find some fundamental problem that could serve as an anchor or focal point for a lifetime of philosophizing. Some time later I realized that I had already found such a problem: *What is proof?* This question gives rise to a series of epistemic, ontic, linguistic, logical, mathematical, and historical questions which still energize me.

"As I look back, I feel that for the first twenty-five or so years of my life I was being hindered by something. It felt like I was driving with my brakes on, carrying useless baggage, or slogging through a muddy swamp. Thinking that I was mysteriously and gratuitously granted belief in *the* truth was a terrible burden. What set me free was overcoming my need to be loyal to the beliefs I happened to have. *I had been afraid to doubt.* I remember discussing my fear of doubt with two of my high-school pals; but it wasn't until graduate school that I saw how destructive that fear was, and only then did I overcome it. I now realize the power of creative doubt. I now see that doubt is not to be feared and shunned, and that stubborn belief is the scary thing.

"It was only after working on the problem of proof that I discovered that doubt is often productive. A crucial property of proofs is their capacity to remove doubt – so if one lacks doubt, the detection of proof may be inhibited. And without the ability to doubt, some kinds of knowledge are difficult or even impossible. For instance, in order to find a proof of a given proposition (even one believed to be true), it is sometimes useful or even necessary to doubt it. Are the premises *really* known to be true? Does the chain of reasoning *really* show that the conclusion follows from the premises? In fact, the most direct method for verifying that an argumentation is a proof starts by doubting the conclusion. How can one doubt what one believes, or thinks one knows to be true? It seems paradoxical to say that people can doubt propositions they believe or even know. But mathematicians do this every day (as do non-mathematicians). In mathematics we often prove propositions that 'do not need proof'. Maybe the frequency of creative doubt in mathematics was one of the reasons Plato found mathematics so important in philosophical training.

"The experience of creating a doubt and the experience of having a doubt removed are both empowering, like the experience of grasping an ambiguity, detecting an implication, or perceiving a *non sequitur*. The experience produces self-knowledge, self-reliance, and self-confidence. It also overcomes the debilitating alienation generated by indoctrination, or by loyalty-motivated self-deception.

"Once I grasped the creative role of doubt and freed myself to employ it, instead of putting energy and emotion into protecting preconceptions that had been imposed on me, I was free to investigate anything and to follow any path. I became an autonomous member of the community of investigators, and thereby became collegial with people who had been ideological enemies.

"I discussed this theme with Alfred Tarski. He said that the motto of Jesus, 'The truth will set you free' was almost exactly backward: a better motto would be 'Be free to find truth.' One of my former students said that 'The truth sets you free' should be replaced with 'Doubt sets you free.'

"The courses I taught were mostly introductory, having no prerequisites and presupposing no previous knowledge. I tried to reconstruct the subject-matter from the ground up. I stressed the priority of self-education over authoritarian indoctrination, and I stressed the superiority of learning how to think over being told what to think. I tried to assist students to connect with the reality that logic is about so that they could become autonomous judges of the current state of logic. One of our class mottos was 'Ridicule the ridiculous'. I encouraged students to themselves become autonomous members of the community of investigators, and to discover and accept their own temperaments. Not every student is ready for intellectual freedom, and not every institution approves of it.

"Over the years I had been fortunate to have benefited from many great institutions and many dedicated students, but I treasure the University of Buffalo and its students above all others. After I settled in here at Buffalo, I had a feeling that I had arrived at my academic home: that this is my kind of institution, these are my kind of colleagues, these are my kind of students. There was confidence, dedication and competence, without conceit, affectation or pretension. I am grateful to all of the talented and energetic people that have made my years at UB so rich. I will miss the Buffalo Logic Colloquium and the fun at the dinners and parties afterward. I will miss seeing you.

"This above all: To thine own self be true.

"Warm regards,

John Corcoran "([2], p. 18).

As a proud recipient of John's LogicLifetimeGuaranteeTM, I can attest that he was a man of his word and lived up to every one of the assertions made in his Farewell Letter. This volume is a tribute to his work, and a demonstration of the positive impact he had – and continues to have – on his countless students, colleagues, and friends.

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Hugh MacColl and Christine Ladd-Franklin: 1877–1909

Francine F. Abeles

Abstract An outsider in the logic community, Hugh MacColl (1837–1909), achieved recognition of his work in logic belatedly in the nineteenth and early twentieth centuries. Together with George Boole, Augustus De Morgan, William Stanley Jevons, Charles S. Peirce, Ernst Schröder, John Venn, Christine Ladd-Franklin and others, MacColl considered logic as a calculus represented by the algebra of logic. In an article published in 1889 in *The American Journal of Psychology*, Ladd-Franklin wrote, “Nothing is stranger, in the recent history of Logic in England, than the non-recognition, which has befallen the writings of this author . . . .it seems incredible that English logicians should not have seen that the entire task accomplished by Boole has been accomplished by Maccoll [sic.] with far greater conciseness, simplicity and elegance.” Examining some of her work and the work of her contemporaries, I explore the possible reasons she had for holding this opinion.

Keywords Algebraic logic · Nineteenth century · Ladd-Franklin · MacColl

Mathematics Subject Classification 01A55 · 03-03 · 03B05

1 Introduction

In this paper, I discuss the possible reasons why Christine Ladd-Franklin {1847–1930) expressed the unusually laudatory opinion of Hugh MacColl’s (1837–1909) work in a paper she published on logic in 1889 where she wrote:

> Nothing is stranger, in the recent history of Logic in England, than the non-recognition, which has befallen the writings of this author . . . .it seems incredible that English logicians should not have seen that the entire task accomplished by Boole has been accomplished by

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> Maccoll [sic.] with far greater conciseness, simplicity and elegance; [21, p. 562; see also 2, p. 104]

What, exactly, was the task Boole accomplished? In his two books, the first from 1847 and the second from 1854, like Aristotle in his theory of syllogisms, Boole constructed a systematic set of principles that permitted valid inferences to be drawn [3, 4].

This paper has nine sections. Sects. 2 and 3 provide background information for Ladd-Franklin and MacColl. In Sect. 4, I analyze MacColl's first three papers titled "The Calculus of Equivalent Statements," published in *The Proceedings of the London Mathematical Society*, and his paper in *Mind* on the same topic [33, 41]. Section 5 describes similar work by Charles Sanders Peirce and MacColl. Section 6 deals with [21] Ladd-Franklin's paper in *The American Journal of Psychology*. Section 7 concerns the dispute that Jevons (1835–1882) and Venn (1834–1923) had with MacColl. A discussion of Ladd-Franklin's dissertation where she introduced her system of algebraic logic follows in Sect. 8. In the final section, I suggest possible reasons why she expressed her unusually complementary opinion of MacColl's work.

2 Christine Ladd's Early Publications

In 1877 and 1878, Christine Ladd authored six articles in *The Analyst*, a journal published by the Mathematics Department of Princeton University which after ten volumes through 1883 became *The Annals of Mathematics*. Of these six, two were solutions to problems posed by other authors. Problem No. 186 concerned three pairs of points in a plane that form three lines meeting in a point. Problem No. 236 dealt with homologous triangles. In a third piece, she posed a query asking which of two statements about a multiple point of order k is correct. One statement is by George Salmon (1819–1904) in Art. 73 of his *Higher Plane Curves*; the other statement is by Joseph A. Serret (1819–1885) in Art. 186 in his *Calcul Différentiel* [11–16, 54].

Between 1879 and 1881, while she was a graduate student at Johns Hopkins University, she published three papers in *The American Journal of Mathematics*, an important publication that James Joseph Sylvester (1814–1897) became editor of in 1879 [17–19].

Ladd-Franklin solved 82 problems and posed 53 problems for *The Educational Times*. She was the most prolific female contributor to *The Educational Times*, and her early solutions caught the eye of Sylvester who responded to a letter she wrote to him and subsequently invited her to study with him at Johns Hopkins University. She enrolled in 1878 and finished in mathematics and logic in 1882, having also taken courses in logic with Charles Sanders Peirce who became her thesis advisor and with the mathematician William Edward Story (1850–1930), who was the editor in charge of *The American Journal of Mathematics* from 1878 to 1882.

It is entirely possible that Ladd-Franklin first got to know and admire MacColl's strong mathematical abilities from his contributions to the *The Educational Times* in the 1870s, rather than, as she claimed, from his first three articles titled, "The Calculus of Equivalent Statements" published in *The Proceedings of the London Mathematical Society* in 1877 and 1878 which she could read easily because of her mathematical background, something most other logicians could not do. (Note: The eighth and last article in this series appeared in 1897.) In 1877, MacColl also published two short articles in *The Educational Times* titled, "Symbolical or Abbreviated Language, with an application to mathematical probability" and "Symbolical Language: No. 2" [July v. 29, pp. 91–92; November v. 29, p. 195].

Ladd-Franklin probably first came across MacColl's publications in *Mind*. Founded in 1876, it quickly became an important journal in philosophy where many eminent logicians published their work. Since logic was not a separate university department, it was included as a subject in philosophy departments. As a student at Hopkins, Ladd read issues of *Mind* regularly. And we know she read MacColl's long paper, "Symbolical Reasoning I" published there in 1880 because she refers to it in one of her early papers on logic, also published in *Mind*, "Some Proposed Reforms in Common Logic." Here, she mentions MacColl just once – in connection with his use of a single sign for a symmetrical and a nonsymmetrical relation [22, p. 82].

3 Hugh MacColl

From 1865 until his death, Hugh MacColl submitted questions and provided solutions to questions appearing in *The Educational Times and Journal of the College of Preceptors*, an established journal published in London from 1847 to 1918. In the second year of its publication, a mathematical department was added to serve as a place for both professionals and dilettantes to exhibit their mathematical skills. The section consisted of problems, solutions, and brief articles. MacColl was a prolific contributor to it. From 1865 to 1910, he posed or solved 144 problems of which 40 were on logic (https://educational-times.wcu.edu/ Also see [56, 57]).

Beginning in 1877, MacColl published several papers with the title, "The Calculus of Equivalent Statements" in the *Proceedings of the London Mathematical Society*, hereafter LMS. The next section of this paper is devoted to a detailed analysis of them.

4 MacColl's First Three London Mathematical Society Papers and His 1880 Paper in *Mind*

Each of these first three papers is organized as a set of definitions and a set of rules. The first paper has 11 definitions and 8 rules accompanied by many examples. He refers to his July article in the *The Educational Times* by title, "Symbolical Language," where he first introduced this analytical method to determine new limits of integration when the order of integration or the order of the variables in a multiple integral is changed. Also, he adds, to determine the limits of integration in applications of probability [26, 28].

In his second paper, he gives two more definitions and ten more rules. The first of these rules, Rule 12, is the definition: "The symbol A:B (which may be called an *implication*) asserts that the statement A implies B or that whenever A is true, B is also true, adding a note that the implication A:B and the equation A = AB are equivalent statements" [29, p. 177].

MacColl writes, "Rules 15, 17, 18 combined with the principle of contraposition ... will be found to include all the valid syllogisms, ... [29, p. 180]. Then, he analyzes the four types using his implication operator. Arranging the 19 ordinary syllogisms into the standard four figures, he then provides geometrical illustrations of his symbols. MacColl finds eight of these standard 19 to be redundant and declares that there are several more valid syllogism obtainable by admitting certain additional implications and nonimplications as factors in their premises. Despite his analysis, MacColl considers syllogistic reasoning too narrow.

Two more definitions and six additional rules are in the third paper, and on the first page is a set of 14 formulae for his calculus. Using his definition of an indeterminate statement (Def. 14) and his rule to reduce an indeterminate statement to its primitive form, Rule 21, he gives an example of reducing a statement to its primitive form. Using another definition (15) and three more rules (22–24), he gives a method that he calls, "Unit and Zero Substitution" followed by examples, adding in a Note that when the object is to eliminate x, rather than solve an implication with respect to x and x′, one should not omit any zero terms because the discovery of zero terms is the object of elimination [30, p. 21]. He solves two examples taken from Boole's *The Laws of Thought* and extends Boole's solution in the second of these two. And MacColl also suggests that his "calculus of statements and implications ... may be employed ... in investigating the causes of natural phenomena" [30, p. 26].

Then, he distinguishes his treatment of logic from that of Boole and of James Joseph Jevons by enumerating three main points. First, he uses a letter and every combination of letters *always* to denote a statement. Secondly, he denotes the symbol **:** as the statement following it is true provided that the preceding statement is true. Thirdly, he uses the accent symbol **’** to express denial and says it can be applied to a multinomial statement of any complexity.

Also, MacColl describes the origin of his logic method in his solution to Question No. 3440 accompanied by an introductory article, "Probability Notation" published in the *The Educational Times* for August 1871. The Question reads: A line is drawn at random across a window containing four equal rectangular panes; what are the respective chances that it crosses one, two, or three of the panes? He adds the following rather unusual statement:

> Shortly after this I gave up all mathematical investigations, and my thoughts did not again revert to the subject till two or three months before the appearance of my article on "Symbolical Language" in *The Educational Times* for July 1877. [30, p. 27]

In 1880, MacColl published a long paper in *Mind*, "Symbolical Language III" the first two papers on this topic, highly mathematical ones, appeared in *The Educational Times* in 1877:

> He begins by saying that mathematicians considered Boole's work an exciting development but not by logicians who saw their hitherto inviolate territory now for the first time invaded by a foreign power, and with weapons which they had but too much reason to dread. With these potent mysterious symbols mathematicians had already extended their dominion far and wide, whilst they, the successors of the illustrious Aristotle, had not added a single acre to these very restricted possessions and annex the sacred province of logic also to the over-grown empire of mathematics. [31–36, p. 46]

Remarking that Jevons fine 1864 book, *Pure Logic*, enabled logicians to reestablish themselves, MacColl suggests that now he could be a peacemaker between the two sciences and that they should unite "under some common appellation" [31–36, p. 47].

It is this paper that ignited the controversy between MacColl and Venn in the pages of *Nature*, discussed in Sect. 7 of this paper. In a letter to Bertrand Russell dated May 17, 1905, MacColl writes [37–39, 42]:

> When I found that my method (in his first LMS paper) could be applied to purely logical questions unconnected with the integral calculus or with probability, I sent a second and a third paper to the Mathematical Society . . . and also a paper to Mind (published January 1880). These involved me in a controversy with VENN & JEVONS of which I soon got tired, as I saw it would lead to no result.

MacColl suggests that the syllogistic method has for centuries prevented logic, the "noblest of the sciences," from joining the other sciences to explore fruitful regions in the common pursuit of truth. So, he has chosen to use the venerable Aristotelian syllogisms to illustrate his symbolical method. The constituents of his system are complete statements and he claims that *any* argument can be resolved into them:

> It is through and by means of the knowledge expressed by the *antecedent* that the reason reaches the knowledge expressed by the consequent. The latter becomes a means and a medium of progression in its turn, and so the reason moves onward from knowledge to knowledge. [31–36, p. 59]

What, actually, is MacColl saying? Firstly, he is making a *proposition*al interpretation of Aristotle's syllogistic system. Secondly, he is referring to *hypothetical* propositions not categorical propositions, as Aristotle did, and making a startling new claim for them, one that Peirce will make in a much clearer way 5 years later.

In 1885, Peirce stated that the hypothetical proposition deals not with the actual state of things but with what would occur if things were other than they are or may be. The result is that we now have a rule: "If A then B" such that later if we learn something which we don't currently know, i.e., A is true, then using this rule, we now know something new, i.e., that B is true [51, pp. 186–187].

Much earlier, Peirce had introduced the sign of inclusion, ⊂, into Boolean algebra to express "less than." His notation was the symbol, <. For statements A, B, A ⊂ B is true in just two instances: if A is false or if B is true. This relation between A and B is known as material implication [48, pp. 323–324].

MacColl divides "Logic" into two types, Pure and Applied, and divides "Reasoning" into two kinds, mental and symbolical. Symbols can be divided into two kinds, permanent and temporary. These comparisons, he states, comprise an analogy between algebra and the algebra of logic. Next, MacColl gives six definitions. The first of "factor," "compound statement," "multiple" (of each separate factor), and "product" (of all the factors); the second of a disjunctive statement; the third of his implication symbol; the fourth of his symbol for "equality"; the fifth of his symbol for "denial"; the sixth of his symbols "0" and "1." Then, he lists ten formulas that result from his definitions.

Referring to his second paper in *The Proceedings of The London Mathematical Society* series, he discusses the denial of implication and his symbol for it: the division operator. Then, he classifies all valid syllogisms into four sets of implications that are more general than the syllogisms, adding that he has given their proof in this second paper. He adds that a premise of a syllogism can be expressed as a simple implication or nonimplication, so that the entire syllogism is a complex implication. In his third paper in this series, he describes how his logical symbolism can be applied to discoveries in the physical sciences, concluding that his system is better than either that of Boole or of Jevons.

It is well known that Hugh MacColl interrupted his logical investigations around 1884 for about 13 years, a period which he mainly devoted to literature. In a letter to Bertrand Russell (1872–1970), dated May 17, 1905, MacColl recalls this break:

> When, more than twenty-eight years ago, I discovered my Calculus of Limits [...] I regarded it at first as a purely mathematical system restricted to purely mathematical questions. [...] When I found that my method could be applied to purely logical questions [...] I sent a fourth paper (in 1884) to the Math. Soc., on the "Limits of Multiple Integrals," which was also accepted. [1, p. 56]

Discussing MacColl's logical system, Irving Anellis writes:

> MacColl's conception of logical systems and their underlying structures were significantly different from Peirce's, namely, that whereas he considered implication to be fundamental to logical form and the basic syntactic elements of logic were propositions, Peirce considered illation understood as a generalized reflexive, transitive, asymmetrical relation, to be fundamental to the algebraic structure of logical systems and the basic syntactic elements to be the relata, i.e. the elements of the relations, interpretable as either terms, sets, classes, or propositions, depending upon the particular requirements of the system in its application. [2, p. 121]

5 Peirce and MacColl

MacColl lived in Boulogne-sur-Mer, France, where, it appears, Peirce visited him in mid May 1883. In a letter to Peirce dated May 16, 1883, MacColl wrote, "It will be a great pleasure indeed to me if you can stay a little while in Boulogne on your way to England. It is not often that I have the opportunity of making the personal acquaintance of my correspondents in logic and mathematics." There is no known response from Peirce. [Peirce Archives, Widener Library, Harvard University, Robin Catalog, MS. L261]

Peirce cited MacColl's work in logic for the first time in a long note on p. 24 in his 1880 paper "On the Algebra of Logic," referring to MacColl's use of the implication sign in his second paper in the sequence, "The Calculus of Equivalent Statements."

Peirce writes, "Mr. Hugh McColl (Calculus of Equivalent Statements, Second Paper, 1878a, 183) makes use of the sign of inclusion several times in the same proposition. He does not, however, give any of the formulae of this section" [49, note p. 25].

Peirce names four different algebraic methods of solving problems in the logic of non- relative terms, which have been proposed, namely, those of Boole, Jevons, Ernst Schröder (1841–1902), and McColl. Then, he claims he is adding a fifth method "which perhaps is simpler and certainly is more natural than any of the others" [49, p. 37]. Earlier in this paper, Peirce argues that A $\prec$ B or A implies B embraces both hypothetical and categorical propositions [49, p. 21].

Anellis asserts that Peirce's notion that there is no difference between categorical and hypothetical propositions came much earlier, referring to an entry in his Logic Notebook for November 14, 1865, citing MS 114 of Peirce's Logic Notebook of 1865–1909 [6, p. 337].

There, Peirce first declared that there is "no difference logically between hypotheticals and categoricals," thus virtually, if not yet actually, equating the copula of predication with implication. Anellis adds, This translation is, ..., strongly suggested in [Peirce 1870], and also, independently, ten years later, by Hugh MacColl in his 1880 paper on "Symbolical Reasoning," where he wrote in his Definition 3 that:

> The symbol :, which may be read "implies," asserts that the statement following it must be true, provided the statement preceding it be true.
>
> Thus, the expression *a : b* may be read "*a* implies *b*," or "If *a* is true, *b* must be true," or "Whenever *a* is true, *b* is also true." [2, pp. 102–103; 32, pp. 50–51]

In a second note, Peirce writes somewhat disparagingly:

> Mr. Hugh McColl, apparently having known nothing of logical algebra except from a jejune account of Boole's work in Bain's logic, published several papers on *a calculus of equivalent statements*, the basis of which is nothing but the Boolian algebra, with Jevon's addition and a sign of inclusion. Mr. McColl adds an exceedingly ingenious application of this algebra to the transformation of definite integrals. [49, p. 32]

In their discussion, Rahman and Redmond write:

> MacColl, ..., claimed the superiority of his methods over that of the "Boolian Logicians" for solving certain logical and mathematical problems, specifically those that involve questions of probability, and he indicated that his logic was developed explicitly in response to questions about probability. [53, p. 556 see also 53]

Edited by Peirce, in 1883, *Studies in Logic* was published containing papers by Peirce and his Johns Hopkins University students. In this book's "Preface," referring to Ladd's article, "On the Algebra of Logic," her doctoral dissertation under his supervision, Peirce wrote that contrary to Boole and himself, McColl [sic] successively and independently favored the addition sign to combine different terms into a single aggregate, thereby making logical addition a disjunction. He adds that to express *existence*, both he and MacColl use a sign for noninclusion. MacColl's is the negation sign,; Peirce's is a bar over his sign of inclusion, (illation), $-\!<$ [50, pp. iii, iv].

MacColl and Peirce disagreed principally in three ways: their notations for "implication," their notations for "the chance that a statement is true," and their different conceptions of probability. MacColl took every opportunity to distinguish his work from Peirce's. His short article, a single paragraph, actually, "A note on Prof. C. S. Peirce's probability notation of 1867" [35, p. 102] is a highly unusual request. Referring to his revised fourth paper subtitled, "On Probability Notation," in the series, "The Calculus of Equivalent Statements," he asked the Secretaries of the London Mathematical Society to support his contention about his and Peirce's notation which they did by writing:

> The Secretaries were directed by the Council to state that they had received a note from Mr. McColl which showed that the apparent coincidence of notation, in some few particulars, between himself and Prof. Peirce, was entirely accidental, and that Mr. McColl was not at that time acquainted with Prof. Peirce's paper. In fact, the revised fourth paper ("On Probability Notation," Proceedings of the London Mathematical Society, Vol. xi., No. 163) was communicated to the Society about nine months before the Author read Prof. Peirce's paper. [31–36, p. 102]

Peirce wrote an early paper on probability, "On an Improvement in Boole's Calculus of Logic" published in the *Proceedings of the American Academy of Arts and Sciences*, v. 7, 1867, pp. 250–261. On the very first page, Peirce gives his opinion of Boole's work:

> The principal use of Boole's calculus of logic lies in its application to problems concerning probability. It consists, essentially, in a system of signs to denote the logical relations of classes. [7, p. 3]

Peirce's notation is, "Let $\boldsymbol{b_a}$ denote the frequency of *b*'s among the *a*'s. Then considered as a class, if *a* and *b* are events, $\boldsymbol{b_a}$ denotes the fact that if *a* happens, *b* happens" [5, p. 583, ch. 8, note 109, 46].

In his fourth paper in the sequence, *The Calculus of Equivalent Statements*, subtitled, *Probability Notation*, MacColl defines his symbol, $\boldsymbol{x_a}$, to denote "the *chance* that the statement *x* is true *on the assumption that the statement* a *is true*" [31, p. 113].

Immediately, we see that their symbols, although much alike, have very different meanings corresponding to their fundamentally different ideas about the meaning of "probability."

Earlier, MacColl had submitted two short articles on probability notation to *The Educational Times*: the first, "Probability Notation," and the second, "Probability Notation No. 2." In the first paper, MacColl solves the problem of finding "the probability that any function $\phi(x, y, z, \ldots, u, v)$ will satisfy any condition M, when x, y, z, &c. are all taken at random between their respective limits" [24, p. 21].

In the second paper, MacColl solves Question 3385 which he then uses to solve another Question, no. 3440. In both of these, he uses another notation, p(r), which he defines as the probability of the occurrence of the rth event, . . . ,and p(: r) which denotes the probability of its nonoccurrence; so that p(r) + p(: r) = 1 [25, p. 29].

6 Ladd-Franklin's 1889 Paper

In this paper, Ladd-Franklin surveys the current state of logic from 1847 when Boole published his first logic book to the present time. She states that the problem of logic that Boole solved was that he gave an organized method to handle very many complex premises that in a subject or predicate provide a description of other terms. The process requires the elimination of certain terms; the ordinary syllogism "consists of elimination in the simplest possible case" [21, p. 543].

"The logic of the nonsymmetrical affirmative copula, 'all a is b,' was first worked out by Mr. MacColl." Citing his three London Mathematical Society papers, her remarkable quote follows. Further on, she provides a more detailed explanation of his accomplishment writing:

> But Mr. Maccoll has completely solved the problem of logic,- to throw the multiform propositions of real life into a single standard form of expression, to condense the information that interests us by the elimination of certain terms which we do not care for, and to state the information which is left in the form of any terms which we happen to wish to see described . . . .Mr Maccoll chooses for the expression of his particular propositions the simple denial of his universals, and he writes them very properly with the sign of negation attached to the affirmative copula; but he does not discuss their treatment in cases of any complexity. Neither does Mr. Peirce, who has worked out independently the Logic of the same copula. [21, p. 563]

What Ladd-Franklin asserts is that MacColl has created a complete logic system much better than Boole's scheme of symbolical reasoning which she finds "immensely complicated." Her reasoning is based on her observation that he has worked out the "logic of the nonsymmetrical affirmative copula, 'all a is b'" [21, pp. 548, 562]. In her table of "Four Different Statements of Fact," she gives the two nonsymmetrical universal statements: All of a is b; none but a is b, followed by the two nonsymmetrical particular statements: not all of a is b; some besides a is b. Next, she gives the two symmetrical universal statements: none of a is b; all but a is b, followed by the two symmetrical particular statements: some of a is b; not all but a is b [21, p. 583].

In this article, among many other motifs, she comments on the work and opinions of the British logicians, particularly Venn, Jevons, and John Keynes (1852–1949). She writes:

> When Mr. Venn said there is only one system of Logic, he seems to have had in mind only Jevons; and Jevon's work in Symbolic Logic does certainly not amount to a system, but merely to the absence of a system. Since then Mr. Keynes has published his treatment of the non-symmetrical copula, [. . .]. [21, p. 557]

The copula of Boole's system is the negative symmetrical, no *a* is *b*. She adds that Schröder gave this copula its final form in 1877, and his treatment should have superseded Boole's. The fact that it hasn't, she suggests, is because he doesn't have an English commentator [21, p. 559]. Then, she remarks that, without any mention of MacColl, Keynes has also written the entire logic of the nonsymmetrical affirmative copula, all a is b. She adds that Venn, in a footnote on p. 372, in his book, *Symbolic Logi*c, claims:

> After a careful study [of this scheme], aided by a long correspondence with the author, I am unable to find much more in it than the introduction of one more scheme of notation to express certain modifications and simplifications of a part of Boole's system. [21, pp. 562–563]

In her *Mind* article published a year later, she explains the role of the copula: "With the symmetrical copulas, subject and predicate can be freely interchanged; with the nonsymmetrical copulas, subject and predicate change places upon the condition that their quality is also interchanged" [22, p. 85].

7 Jevons's and Venn's Dispute with MacColl

It is well known that MacColl developed a logic system based on propositions, which he believed would improve and supersede the symbolic logic of Boole. Peirce and Schröder were sympathetic to MacColl's work. In Britain however, his work was criticized by William Stanley Jevons and John Venn.

Responding to Jevons's criticism that in the *Proceedings of the London Mathematical Society*, and in *Mind*, he rejects equations in favor of implications, MacColl published an article in *The London, Edinburgh and Dublin philosophical Magazine and Journal of Science* titled, "On the Diagrammatic and Mechanical Representation of Propositions and Reasoning" [34]. Jevons had sent MacColl a copy of his 1880 book, *Studies in Deductive Logic*. In the Preface, referring to MacColl's articles, Jevons describes the important differences between his work and MacColl's, particularly his preference for implications to equations, concluding that there is no advantage in them. He adds that "His proposals seem to me to tend toward throwing Formal Logic back into its ante-Boolean confusion" [8, p. xv].

MacColl partly agrees with Jevons, writing that he uses implication in logic only when it leads to solutions that are the simplest, shortest, and most elegant. To support his claim, MacColl adds that in his first and fourth papers in *Proceedings of the London Mathematical Society*, he used the equational form, but in his second and third papers there, he used the implicational form.

In a second paper, in the form of a Letter to the Editors of the journal, titled, *Implication and Equational Logic*, MacColl gives his opinion of Venn's diagrammatic method for solving problems in logic. Venn had sent him a copy of his paper published in the July issue of this journal. MacColl disputes Venn's contention that his diagrammatic method is superior to rival methods, adding that Venn does not appreciate his method; presumably, he is referring to MacColl's implication operator.

Venn published his book, *Symbolic Logic*, in 1881. There, Venn concluded that he was unable to find anything of importance in MacColl's work other than a new notation scheme to express simplifications and modifications of part of Boole's logic. Because in MacColl's system every letter and every combination of letters always denotes a statement, Venn disagrees with MacColl's conclusion that this requires "an essentially different treatment of the whole subject" [See Jevons's 1881 review of Venn's book 9, 10;58–60, p. 372 note 1].

MacColl and Venn continued their disagreement into 1881 in a series of articles titled "Symbolical Logic" published in *Nature*, an important multidisciplinary international scientific publication founded in 1869. Venn wrote two articles; MacColl wrote three [58, 59]. All were published in volumes 23 and 24 of the journal. (See also [8, 9, pp. 485–487]; for a fuller discussion, see [44].)

8 Ladd-Franklin's Paper in *Studies in Logic*

In her dissertation, Ladd-Franklin adds a note to her discussion of the validity of inferences. "Those syllogisms in which a particular conclusion is drawn from two universal premises become illogical when the universal proposition is taken as not implying the existence of its terms" [20, p. 39]. In support, she cites McColl's 1880 *Mind* article, "Symbolical Reasoning" and Peirce's, paper, "On the Algebra of Logic" that had appeared the same year.

She states that Algebras of Logic can be divided into two classes by how they express the "quantity" of propositions, either to the copula or to the subject. In the first set, two copulas are needed, one universal and one particular**.** In the second set, the algebras have one copula, =. MacColl's and Peirce's logic systems belong to the second set. In a note to her statement, she writes that according to Peirce, every algebra of logic has *two* copulas, one for propositions of nonexistence and the other for propositions of existence. Is there a disagreement between her classification and Peirce's? She states that *particular* propositions denote existence and *universal* propositions denote nonexistence, adding that the quantified copula, **:** or **–<**, is positive for universal propositions and negative for particular propositions.

By quantifying the copula, she pairs the *type* of proposition, particular, universal, with the *state* of the proposition, existence, nonexistence. So, there really is no disagreement between her classification and Peirce's. She requires one copula for her algebra of logic [20, pp. 23–25].

In a comparison, Ladd-Franklin states that MacColl uses *a*/*b* to mean/the statement that any object, *a*, implies the statement it is also *b*. Mr. Peirce's symbol for the same copula is a modification of ≤ , namely, **–<**. She adds that propositions using the copula **:** or **–<** are called inclusions.

In her dissertation, she had presented an algebra of logic that was a variant of the system devised by Boole, Peirce, and Schröder. Its fundamental relation is *exclusion* expressed by the symbol $\overline{\mathbf{V}}$, rather than the inclusion relation, **<** , in the classical algebra of Boole and his contemporaries. The expression *a* V *b* is defined as *a* is partly *b*. **V** and $\overline{\mathrm{V}}$ are symmetrical and intransitive. Inclusion, by contrast, is nonsymmetrical and transitive. Changing an inclusion into its equivalent exclusion requires changing the sign of the predicate. Using the copula –<, the quantity of the subject is universal; that of the predicate is indeterminate. But with $\overline{\mathrm{V}}$, the quantity of both subject and predicate is universal. She further states that the copulas V and $\overline{\mathrm{V}}$ are intransitive copulas, they are symmetrical, and *A* V *B* can be read forward or backward, the same for $A\ \overline{\mathrm{V}} B$. From this fact, there is no formal difference between the subject and predicate that the advantages of her algebra follow [20, pp. 23–27].

9 Conclusion

It seems then that there are four principal reasons for Ladd-Franklin's extraordinarily laudatory opinion of MacColl's work which she expressed in the quotation that is the subject of this paper. First, his work connects well with hers in her dissertation topic through their mutual focus on syllogisms. Secondly, that the character of a logic system is determined by the type of copula that represents the propositions is the basis for both of their calculi. She worked with the symmetrical affirmative copula; he worked with the nonsymmetrical affirmative copula. Thirdly, both thought that their systems reflected real, i.e., natural reasoning processes. Fourthly, she agreed with Schröder's opinion that MacColl's calculus was a preliminary stage of Peirce's algebra of logic. Schröder had accepted MacColl's priority in the formulation of a propositional logic [45, p. 20]. She saw and approved of the many connections between MacColl's and Peirce's work, implication especially, where she considered Peirce's notation to be better than MacColl's because "it expresses an unsymmetrical relation by an unsymmetrical symbol [20, pp. 24–25]. Finally, both she and Peirce approved of MacColl's work, while the mainstream British logicians Jevons and Venn were critical of him ([1, p. 68]; also see [44]). So, by acclaiming MacColl, she was asserting the superiority of both hers and Peirce's opinions, i.e., those of the American logicians, over those of the British logicians.

Oddly, she had no correspondence with MacColl who continued to search for recognition from the logic community. In 1900, he traveled to Paris and gave a long paper at The First International Congress for Philosophy titled "Symbolic Logic and Its Applications" [42, pp. 135–184]. In 1908, she gave a paper "Epistemology for the Logician" at the 1903 International Congress of Philosophy Meeting in Heidelberg [23, pp. 664–670].

Ladd-Franklin's initial interest in and admiration for MacColl centered on their mutual work in mathematical logic. But after 1889, her interest in logic shifted to its use in philosophical and psychological settings. MacColl never left the realm of pure mathematical logic.

Acknowledgments In my talk on Ladd-Franklin for the July 2021 conference of the joint session of the British Society for the History of Mathematics and the Canadian Society for History and Philosophy of Mathematics, I have included discussions of Ladd-Franklin's 1883 and 1889 papers but with entirely different emphases.

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