Athanase Papadopoulos *Editor*

Surveys in Geometry II

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Preface

This is the second volume of a collection of surveys on topics that are at the forefront of current research in geometry. They are intended for graduate students and researchers. Some of these surveys are based on lectures given by their authors to middle-advance and graduate students, and all of them can be used as bases for courses on geometry. Each chapter concentrates on a topic which I consider particularly interesting and which is worth highlighting. The topics include Riemann surfaces, metric geometry, Finsler geometry, Riemannian geometry, projective geometry, symplectic geometry, Teichmüller spaces and combinatorial group theory.

I would like to thank Elena Griniari for her kind support and care for this project, and the reviewers of the various chapters for their valuable anonymous work. My warm thanks go to all the authors, for a fruitful and friendly collaboration.

Nisyros (Dodecanese), Greece
September 2023

Athanase Papadopoulos

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Editor and Contributors

About the Editor

Athanase Papadopoulos (born 1957) is Directeur de Recherche at the French Centre National de la Recherche Scientifique. His main fields of interest are geometry and topology, the history and philosophy of mathematics, and mathematics and music. He has held visiting positions at the Institute for Advanced Study, Princeton (1984–1985 and 1993–1994), USC (1998–1999), CUNY (Ada Peluso Professor, 2014), Brown University (Distinguished Visiting Professor, 2017), Tsinghua University, Beijing (2018), Lamé Chair of the State University of Saint Petersburg (2019), and has had several month visits to the Max-Plank Institute for mathematics (Bonn), the Erwin Schrödinger Institute (Vienna), the Graduate Center of CUNY (New York), the Tata Institute (Bombay), Galatasaray University (Istanbul), the University of Florence (Italy), Fudan University (Shanghai), Gakushuin University (Tokyo) and Presidency University (Calcutta). He is the author of more than 220 published articles and 45 monographs and edited books.

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Chapter 1
Introduction

Athanase Papadopoulos

Abstract This chapter contains a description of the various subjects covered in the book.

Keywords Conformal geometry · Metric geometry · Teichmüller spaces · Surfaces immersed in 3-manifolds · Symplectic geometry · Grassmann spaces · Finite homogeneous metric spaces · Polytopes · The Gauss–Bonnet formula · Isoperimetry · Coxeter groups

AMS Codes 12D10, 14H55, 20B25, 26C10, 30F10, 30F20, 20F55, 30F30, 30F60, 32G15, 51F20, 51F15, 51M35, 53A35, 53D30, 53C70, 57K10

This is the second volume of a collection of multi-authored surveys in geometry. The project of publishing these volumes arose from the conviction that the mathematical community is in need of good surveys of topics that are at the heart of current research. Thus I have asked some colleagues and friends to write an expository article on a subject they have been working on, which I find particularly important or interesting. I hope the result will be of use to mathematicians, both beginners and experienced.

The topics surveyed in the present volume include the conformal and the metric geometry of surfaces, Teichmüller spaces, surfaces immersed with prescribed extrinsic curvature in 3-dimensional manifolds, symplectic geometry, the metric theory of Grassmann spaces, finite homogeneous metric spaces, projective metric spaces, regular and semi-regular polytopes, the Gauss–Bonnet formula and its higher-dimensional versions, isoperimetry in finitely generated groups, and Coxeter groups. Let me review now each chapter in some detail.

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Chapter 2, by Norbert A'Campo and me, is titled *Geometry on surfaces, a source for mathematical developments*. It is an overview of the theory of surfaces equipped with various structures: volume forms, almost complex structures, Riemannian metrics of constant curvature, quasiconformal structures, and others. Several topics discussed originate in the Riemann mapping theorem and its generalization, the uniformization theorem. The theory of Riemann surfaces is intertwined with topology and combinatorics. Higher-dimensional structures are also discussed. We also review several applications of a colored graph associated with a branched covering of the sphere, which we call a *net*, or a *Speiser net*. Nevanlinna, Teichmüller and others obtained criteria, using this graph, for the determination of the type of a simply connected surface, that is, to know whether the surface is conformally equivalent to the Euclidean plane or to the unit disc. The same graph appears in the theory of dessins d'enfants as well as in a realization theorem of Thurston concerning the characterization of some branched coverings of the sphere. Combinatorial characterizations of polynomials and rational maps among a class of branched covers of the sphere are discussed. We also survey the notion of rooted colored trees associated with slalom polynomials, and certain graphs that are used in the stratification of the space of monic polynomials. New models of the Riemann sphere and of hyperbolic 2- and 3-spaces appear at several places of the survey, some of them in an unexpected manner, using algebra (the ring of polynomials). By analogy, some constructions based on the field of complex numbers are extended to arbitrary fields.

Chapter 3, by Ken'ichi Ohshika, is titled *Teichmüller spaces and their various metrics*. This chapter is a survey of three different Finsler metrics on Teichmüller space: the Teichmüller metric, Thurston's asymmetric metric and the earthquake metric. In particular, the author presents some recent results he obtained with Y. Huang, H. Pan and A. Papadopoulos on the so-called earthquake metric, also introduced by Thurston. He reviews a duality established in that paper between the tangent space at an arbitrary point of Teichmüller space equipped with the earthquake norm and the cotangent space at the same point equipped with Thurston's co-norm. He also reports on a result, also obtained in that paper, saying that the earthquake metric is not complete, and he provides a description of its completion.

Chapter 4 by Marc Troyanov is titled *Double forms, curvature integrals and the Gauss-Bonnet formula*. We recall that the Gauss–Bonnet formula for surfaces is a major achievement of nineteenth century differential geometry, and is one of the very good examples of how topology is closely related to geometry. In its simplest form, the formula says that for a closed surface S equipped with a Riemannian metric, the integral of the Gaussian curvature is equal (up to a universal constant) to the Euler characteristic of the surface. There are much more evolved forms of the formula. Troyanov, in Chap. 4, recalls that the generalized Gauss–Bonnet formula is due to the effort of several mathematicians, namely, K. F. Gauss, P. Bonnet, J. Binet, and W. von Dyck, and that extensions of this formula to higher-dimensional Riemannian manifolds were obtained in the twentieth century by H. Hopf, W. Fenchel, C. B. Allendoerfer, A. Weil and S.S. Chern. The extended formula establishes relations between the Euler characteristic of a smooth, compact

Riemannian manifold with (possibly empty) boundary and a curvature integral over the manifold plus a boundary term that involves a combination of curvature and the second fundamental form over the boundary. In this chapter, the author revisits the higher-dimensional formula using the formalism of double forms, a tool introduced by G. de Rham and further developed by R. Kulkarni, J. Thorpe and A. Gray in the 1960s and 1970s. In particular, he surveys the history and the techniques around Chern's version of the formula, which uses É. Cartan's moving frame formalism. He explores the geometric nature of the boundary term and he provides examples and applications. This chapter is also an occasion for Troyanov to make a detailed account of the various tools that were used by various authors in the proof of the higher-dimensional Gauss–Bonnet formula, starting with H. Hopf's problem on the *Curvatura Intega*, including the theories of double forms with their geometric applications, the Pfaffian, the moving frame, the Gauss–Kronecker curvature of hypersurfaces, the Lipschitz–Killing curvature, and several other notions.

Chapter 5, by Graham Smith, is titled *Quaternions, Monge–Ampère structures and κ-surfaces*. This chapter combines Riemannian geometry, conformal geometry, symplectic geometry and quaternionic geometry (an analogue of complex geometry where the field of complex numbers is replaced by that of quaternions). The study is based on a work of F. Labourie. The latter developed, in his paper *Problèmes de Monge-Ampère, courbes pseudo-holomorphes et laminations*, published in 1997, a theory of surfaces immersed in three-manifolds with prescribed extrinsic curvature. This theory has found applications in hyperbolic geometry, general relativity, Teichmüller theory and other domains. Labourie's key insight is that if the second fundamental form of an immersed surface of prescribed extrinsic curvature is positive definite (the surface is then called infinitesimally strictly convex), then its Gauss lift is a pseudo-holomorphic curve for some suitable almost complex structure. This allows the application of Gromov's theory developed in his paper *Pseudo-holomorphic curves in symplectic manifolds* (1987), and in particular, his compactness results for families of immersed surfaces of prescribed extrinsic curvature in 3-dimensional Riemannian manifolds. Smith, in this chapter, builds on Labourie's work, of which he gives a quaternionic reformulation. This leads him to simpler proofs of Labourie's results, and at the same time, to generalisations to higher-dimensions. Two theorems of Labourie are in the background: a compactness result for sequences of quasicomplete pointed infinitesimally strictly convex immersed surfaces in a complete, oriented, 3-dimensional Riemannian manifold with prescribed curvature, and a description of the accumulation points of such a sequence. A key result of this chapter establishes a relation between the solutions of the 2-dimensional Monge–Ampère equation and pseudo-holomorphic curves.

Chapter 6, by Peter Kristel and Eric Schippers, is titled *Lagrangian Grassmannians of polarizations*. This chapter is an introduction to polarization theory in symplectic and orthogonal geometries. In this setting, one starts with a triple of structures on a real vector space, namely, an inner product, a symplectic form and a complex structure, the triple satisfying a compatibility condition that ensures that when we are provided with two out of these three structures, we can reconstruct the third one, if it exists (which is not always the case). The specification of such a compatible triple is equivalent to a decomposition of the complexified ambient vector space into the eigenspaces of the complex structure. A familiar example of such a triple of structures is that of a Kähler manifold, where the manifold is equipped with three compatible structures: a Riemannian metric, an integrable complex structure and a symplectic form. Kristel and Schippers adopt the natural point of view of fixing *one* of these structures, and studying the space of structures of a second type allowing the reconstruction of a compatible structure of the third type. For example, we can fix a symplectic manifold and study the space of integrable complex structures on the manifold such that this manifold admits a compatible Riemannian metric. In the case where either the symplectic form or the inner product is fixed, we get a Grassmannian of polarizations. Polarizations appear in several contexts of algebraic and complex geometry: in the study of moduli spaces, in the theory of metaplectic and spin representations, and in conformal field theory. In Chap. 6, Kristel and Schippers survey this circle of ideas, in which the underlying vector spaces are allowed to be infinite-dimensional, emphasizing the symmetry of the symplectic and orthogonal settings. The authors consider two particular situations: the Riemannian one, in which the polarization is an orthogonal decomposition, and the symplectic one, in which the polarization is a symplectic decomposition. The potential fields of applications of this theory of polarizations include representation theory, loop groups, complex geometry, moduli spaces, quantization, and conformal field theory.

The next three chapters are concerned with metric geometry.

Chapter 7, by Árpád Kurusa, is titled *Metric characterizations of projective-metric spaces*. The author starts by recalling Hilbert's Problem IV of the list of problems he proposed in 1900 at the Paris International Congress of Mathematicians. The problem asks for the construction of all the projective metrics on an open convex subset of projective space, that is, the metrics whose geodesics are the intersections of this open subset with the projective lines of the ambient space. Kurusa addresses the general question of characterizing such spaces under the effect of adding some additional conditions on the metric. On the same occasion, he surveys notions like Hilbert and Minkowski metrics, projective center, Ptolemaic metric, Erdös ratio, conics in metric spaces, general Finsler projective metrics, the Ceva and Menelaus properties, and other properties of triangles in a projective-metric setting. Notions such as plane and line perpendicularity, equidistance of lines, bisectors, medians, bounded curvature and others, that hold in a general metric space, are used. These properties were introduced in such a general setting by Herbert Busemann. The questions of characterizing Minkowski planes, of Hilbert geometries and of the three classical geometries, which were extensively investigated by Busemann, are addressed by Kurusa, who also mentions several open problems on this topic.

Chapter 8, by the same author, titled *Supplement to "Metric Characterization of projective-metric spaces"*, is a supplement to the previous chapter whose goal is to provide proofs of two theorems. The first one, due to B. B. Phadke, says that a projective-metric space is a Minkowski plane if all equidistants to geodesics are geodesics. The proof that the author provides is different from Phadke's original proof. The second theorem, due to the author, says that a projective-metric plane has the Ceva property (a generalization of the classical Ceva property) if and only if it is either a Minkowski plane or a model of hyperbolic or elliptic geometry. The proofs are long, which is the reason why these theorems are included in a separate chapter.

Chapter 9, by Boumediene Et-Taoui, is titled *Metric problems in projective and Grassmann spaces*. In this chapter, the author studies metric problems in real, complex and quaternionic spaces concerning equiangular lines and equi-isoclinic n-subspaces. Let us recall the setting. We take $\mathbb{F}$ to be the real, complex or quaternionic field. Let $p \geq 3$ and $r \geq 2$ be two integers. A collection of lines in $\mathbb{F}^r$ is said to be *equiangular* if any two lines in this collection make the same nonzero angle. A set of p n-subspaces in $\mathbb{F}^r$ is said to be *equi-isoclinic* if this set spans $\mathbb{F}^r$ and if any two lines in this set make the same non-zero angle. As the author notes, such structures appear, under various names, in fields such as discrete geometry, combinatorics, harmonic analysis, frame theory, coding theory and quantum information theory. The author addresses two natural questions, namely, (i) How many equiangular lines can be placed in $\mathbb{F}^r$? (ii) How many equiangular equi-isoclinic n-subspaces can be placed in $\mathbb{F}^r$? The chapter is a survey of the work done and the developments due to various authors in the last 70 years on these and related questions, interpreted in the setting of the metric geometry of projective and Grassmann spaces. This metric setting involves classical and fundamental works of Menger, Blumenthal, Lemmens, Seidel and others, as well as works of the author himself with co-authors.

Chapter 10, by Valeriĭ Berestovskiĭ and Yuriĭ Nikonorov, is titled *On the geometry of finite homogeneous subsets of Euclidean spaces*. The authors review recent results on finite homogeneous metric spaces, that is, spaces on which their isometry group acts transitively. They are especially interested in finite homogeneous metric subspaces of a Euclidean space that represent vertex sets (assumed to lie on a sphere) of compact convex polytopes whose isometry groups are transitive on the vertex set. In particular, the authors are led to the classification of regular and semiregular polytopes in Euclidean spaces, according to whether or not they satisfy the normal homogeneity property or the Clifford–Wolf homogeneity property on their vertex sets. The metric spaces that satisfy these two properties constitute a remarkable subclass of the class of homogeneous metric spaces. These properties are stronger than the usual homogeneity properties used for homogeneous metric spaces. The definitions of normal and Clifford–Wolf homogeneity involve a property satisfied by the isometry taking one point to the other, in the definition of homogeneity. The fact that such a study is closely related to the theory of convex polytopes in Euclidean spaces makes it natural to first check the presence of these properties for the vertex sets of regular and semiregular polytopes. Berestovskiĭ and Nikonorov are then led to the study of the m-point homogeneity property and to the notion of point homogeneity degree for finite metric spaces. They discuss several recent results, in particular, the classification of polyhedra with all edges of equal length and with 2-point homogeneous vertex sets, and they present results on the point homogeneity degree for some important classes of polytopes. While discussing these classification results, the authors explain in detail the main tools used for the study of the relevant objects, and they discuss prospects for future results, presenting several open problems.

Chapter 11, by Gue-Seon Lee and Ludovic Marquis, is titled *Discrete Coxeter groups*. It is an introduction to Coxeter groups, with a focus on how these groups can be used for the construction of discrete subgroups of Lie groups. Coxeter groups are geometrically defined groups. They were introduced in 1934 by H. S. M. Coxeter. They are generated by reflections in some spaces. They generalize the Euclidean reflection groups and the symmetry groups of regular polyhedra. They appear in several areas of mathematics, in particular in the theory of representations of discrete groups in Lie groups. The topics discussed in this chapter include, besides the general Coxeter groups, the theories of reflection groups in hyperbolic space, convex cocompact projective reflection groups, projective reflection groups, divisible and quasi-divisible domains in Hilbert geometry, and Anosov representations. The latter constitute a generalization of the class of discrete convex cocompact representations of hyperbolic groups into rank one Lie groups to the setting of representations of hyperbolic groups into semi-simple Lie groups. Anosov representations were introduced by Labourie in his paper *Anosov flows, surface groups and curves in projective space* (2006), in which he studies Hitchin representations.

Chapter 12, by Bruno Luiz Santos and Marc Troyanov, is titled *Isoperimetry in finitely generated groups*. The setting is that of infinite finitely generated groups equipped with word metrics associated with finite symmetric sets of generators. An isoperimetric inequality is an inequality between the size of an arbitrary finite set and the size of its boundary (for an appropriate definition of boundary). In this chapter, the authors revisit the work done in the 1980s–1990s by N. Varopoulos, T. Coulhon and L. Saloff-Coste on isoperimetric inequalities in finitely generated groups, adding new results and establishing relations with other topics. In particular, they obtain lower bounds for the isoperimetric quotient that appears in the isoperimetric inequality in terms of the $\mathcal{U}$-transform, which is a variant of the classical Legendre transform of a function, or its generalization, the Legendre–Fenchel transform, which is used in physics. The chapter also includes a review of some basic elements from geometric group theory (growth functions, amenability, the Cheeger constant, etc.) as well as some basic elements from the theory of $\mathcal{U}$-transform, including some computational techniques, and the relation between the $\mathcal{U}$-transform and the Legendre transform.

The first two chapters of this volume are based on lectures given by the authors at two thematic programs at Banaras Hindu University, in December 2019 and December 2022, which I organized with Bankteshwar Tiwari. The programs were funded by CIMPA (Centre International de Mathématiques Pures et Appliquées), IMU (International Mathematical Union), SERB (Science and Engineering Research Board of the Government of India), NBHM (National Board of Higher Mathematics of the Government of India), and SRICC (Sponsored Research and Industrial Consultancy Cell) and ISC-BHU (the Institute of Science of Banaras Hindu University).

Chapter 2
Geometry on Surfaces, a Source for Mathematical Developments

Norbert A'Campo and Athanase Papadopoulos

Abstract We present a variety of geometrical and combinatorial tools that are used in the study of geometric structures on surfaces: volume, contact, symplectic, complex and almost complex structures. We start with a series of local rigidity results for such structures. Higher-dimensional analogues are also discussed. Some constructions with Riemann surfaces lead, by analogy, to notions that hold for arbitrary fields, and not only the field of complex numbers. The Riemann sphere is also defined using surjective homomorphisms of real algebras from the ring of real univariate polynomials to (arbitrary) fields, in which the field with one element is interpreted as the point at infinity of the Gaussian plane of complex numbers. Several models of the hyperbolic plane and hyperbolic 3-space appear, defined in terms of complex structures on surfaces, and in particular also a rather elementary construction of the hyperbolic plane using real monic univariate polynomials of degree two without real roots. Several notions and problems connected with conformal structures in dimension 2 are discussed, including dessins d'enfants, the combinatorial characterization of polynomials and rational maps of the sphere, the type problem, uniformization, quasiconformal mappings, Thurston's characterization of Speiser nets, stratifications of spaces of monic polynomials, and others. Classical methods and new techniques complement each other.

Keywords Geometric structure · Conformal structure · Almost complex structure (J-field) · Riemann sphere · Uniformization · The type problem · Rigidity · Model for hyperbolic space · Cross ratio · Belyi's theorem · Riemann–Hurwitz formula · Chasles 3-point function · Branched covering · Type

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problem · Dessin d'enfants · Slalom polynomial · Slalom curve · Space of monic polynomials · Stratification · Fibered link · Divide · Speiser curve · Speiser graph · Line complex · Quasiconformal map · Almost analytic function · Net · Speiser net

AMS Classification 12D10, 26C10, 14H55, 30F10, 30F20, 30F30, 53A35, 53D30, 57K10

2.1 Introduction

Given a differentiable surface, i.e., a 2-dimensional differentiable manifold, one can enrich it with various kinds of geometric structures. Our first aim in the present survey is to give an introduction to the study of surfaces equipped with locally rigid and homogeneous geometric structures.

Formally, a geometric structure on a surface S is given by a section of some bundle associated with its tangent bundle TS. We shall deal with specific examples, mostly, volume forms, almost complex structures (equivalently, conformal structures, since we are dealing with surfaces) and Riemannian metrics of constant Gaussian curvature. We shall also consider quasiconformal structures on surfaces. Foliations with singularities, Morse functions, meromorphic functions and differentials on almost complex surfaces induce geometric structures that are locally rigid and homogeneous only in the complement of a discrete set of points on the surface. Laminations, measured foliations and quadratic differentials are examples of less homogeneous geometric structures. They play important roles in the theory of surfaces, as explained by Thurston, but we shall not consider them here.

A theorem of Riemann gives a complete classification of non-empty simply connected open subsets of $\mathbb{R}^2$ that are equipped with almost complex structures. Only two classes remain! This takes care at the same time of the topological classification of such surfaces without extra geometric structure: they are all homeomorphic. The classical proof of this topological fact invokes the Riemann Mapping Theorem, that is, it assumes the existence of an almost complex structure on the surface. Likewise, only two classes remain in the classification of non-empty open connected and simply connected subsets of $\mathbb{R}^2$ that are equipped with a Riemannian metric of constant curvature: the Euclidean and the Bolyai–Lobachevsky plane. The latter is also called the non-Euclidean or hyperbolic plane.

No classification theorem similar to that of simply connected open subsets of $\mathbb{R}^2$ holds in $\mathbb{R}^3$, even if one restricts to contractible subsets. See [113] for the historical example, now called "Whitehead manifold", which, by a result of Gabai [39], is a manifold of small category, i.e., it is covered by two charts, both of which being copies of $\mathbb{R}^3$ that moreover intersect along a third copy of $\mathbb{R}^3$.

We shall be particularly concerned with almost complex structures, i.e., conformal structures, on surfaces. The theory of such structures is intertwined with topology. This is not surprising: Riemann's first works on functions of one complex variable gave rise at the same time to fundamental notions of topology. He conceived the notion of "n-extended multiplicity" (*Mannigfaltigkeit*), an early version of n-manifold, he introduced basic notions like connectedness and degree of connectivity for surfaces, which led him to the discovery of Betti numbers in the general setting (see Andé Weil's article [112] on the history of the topic), he classified closed surfaces according to their genus, he introduced branched coverings, and he was the first to notice the topological properties of functions of one complex variable (one may think of the construction of a Riemann surface associated with a multi-valued meromorphic function). At about the same time, Cauchy, in his work on the theory of functions of one complex variable, introduced path integrals and the notion of homotopy of paths. We shall see below many such instances of topology meeting complex geometry.

In several passages of the present survey, we shall encounter graphs that are used in the study of Riemann surfaces. They will appear in the form of:

1. Speiser nets associated with branched coverings of the sphere: these are used in Thurston's realization theorem for branched coverings (Sect. 2.6.3), in the type problem (Sect. 2.7.2), in the theory of dessins d'enfants (Sect. 2.8.1) and in a cell-decomposition of the space of rational maps (Sect. 2.8.4);
2. rooted colored trees associated with slalom polynomials (Sect. 2.8.2);
3. pictures of monic polynomials used for the stratification of the space of slalom polynomials (Sect. 2.8.3).

At several places, we shall see how familiar constructions using the field of complex numbers can be generalized to other fields. Conversely, algebraic considerations will lead to several models of the Riemann sphere and of 2- and 3-dimensional hyperbolic spaces. Relations with the theory of knots and links will also appear.

Let us give now a more detailed outline of the next sections:

In Sect. 2.2 we present a few classical examples of rigidity and local rigidity results in the setting of geometric structures on n-dimensional manifolds. A theorem due to Jürgen Moser, whose proof is sometimes called "Moser's Trick", deals with the classification up to isotopy of volume forms on compact connected oriented n-dimensional manifolds. We show how this proof can be adapted to the symplectic and contact settings. A local rigidity result (which we call a Darboux local rigidity theorem) gives a canonical form for volume, symplectic and contact forms on non-empty connected and simply connected open subsets (S, ω) of $\mathbb{R}^n$.

In dimension two, almost complex structures are also locally rigid, and we present a Darboux-like theorem for them. The question of the existence and integrability of J-structures on higher-dimensional spheres arises naturally. We survey a result due to Adrian Kirchhoff which says that an n-dimensional sphere admits a J-structure if and only if the $(n + 1)$-dimensional sphere admits a parallelism, that is, a global field of frames. This deals with the question of the existence of J-fields on higher-dimensional spheres, which we also discuss in the same section: only S^6 carries such a structure.

Section 2.3 is concerned with the first example of Riemann surface, namely, the Riemann sphere. We give several models of this surface. Its realization as the projective space $\mathbb{P}^1(\mathbb{C})$ leads to constructions that are valid for any field k and not only for $\mathbb{C}$. In the same section, we review a realization of $\mathbb{P}^1(\mathbb{C})$ with its round metric as a quotient space of the group $\mathrm{SU}(2)$ of linear transformations of $\mathbb{C}^2$ of determinant 1 preserving the standard Hermitian product. The intermediate quotient $\mathrm{SU}(2)/\{\pm \mathrm{Id}\}$ is isometric to $P^3(\mathbb{R})$ and also to the space $T_{l=1}\mathbb{P}^1(\mathbb{C})$ of length 1 tangent vectors to S^2. In this description, oriented Möbius circles on $\mathbb{P}^1(\mathbb{C})$ (that is, the circles of the conformal geometry of $\mathbb{P}^1(\mathbb{C})$) lift naturally to oriented great circles on $S^3 = \mathrm{SU}(2)$. Also, closed immersed curves without self-tangencies lift to classical links (that is, links in the 3-sphere).

In Sect. 2.4, we present another model of the Riemann sphere, together with models of the hyperbolic plane and of hyperbolic 3-space. A model of the Riemann sphere is obtained using algebra, namely, fields and ring homomorphisms. In this model, the point at infinity of the complex plane is represented by $\mathbb{F}_1$, the field with one element. The notion of shadow number is introduced, as a geometrical way of viewing the cross ratio. The hyperbolic plane appears as a space of ideals equipped with a geometry naturally given by a family of lines. In this way, the hyperbolic plane has a very simple description which arises from algebra. The cross ratio is used to prove a necessary and sufficient condition for a generic configuration of planes in a real 4-dimensional vector space to be a configuration of complex planes. We then introduce the notions of compatible (or J-conformal) Riemannian metric and we prove the existence and uniqueness of such metrics on homogeneous Riemann surfaces with commutative stabilizers. We describe several models of spherical geometry (surfaces of constant curvature +1), and of 2- and 3-dimensional hyperbolic spaces in terms of the complex geometry of surfaces. We then study the notion of J-compatible Riemannian metrics. An existence result of such metrics is the occasion to characterize homogeneous Riemann surfaces up to bi-holomorphic equivalence.

In Sect. 2.5, we reduce generality by assuming that the surface S is an open connected and path-connected non-empty subset of the real plane $\mathbb{R}^2$. Fundamental results appear. For instance, the theorem saying that two open connected and path-connected non-empty subsets of the real plane $\mathbb{R}^2$ are diffeomorphic, a consequence of the Riemann Mapping Theorem. This theorem says that any nonempty open subset of the complex plane which is not the entire plane is biholomorphically equivalent to the unit disc. The Riemann Mapping Theorem generalized to any simply connected Riemann surface (and not restricted to open subsets of the plane) is the famous Uniformization Theorem. It leads to the type problem, which we consider in Sect. 2.7.

The next section, Sect. 2.6, is concerned with some aspects of branched coverings between surfaces. A classical combinatorial formula associated with such an object is the Riemann–Hurwitz formula. It leads to some natural problems which are still unsolved. A combinatorial object associated with a branched covering of the sphere is a Jordan curve that passes through all the critical values and which we call a *Speiser curve*. Its lift by the covering map is a graph we call a *net*, or *Speiser net*, an object that will be used several times in the rest of the survey. A theorem of

Thurston which we recall in this section gives a characterization of oriented graphs on the sphere that are Speiser graphs of some branched covering of the sphere by itself. Thurston proved this theorem as part of his project of understanding what he called the "shapes" of rational functions of the Riemann sphere. In the same section, we introduce a graph on a surface which is dual to the net, often known in the classical literature under the name *line complex*, which we use in an essential way in Sect. 2.7. We reserve the name line complex to another graph.

The type problem, reviewed in Sect. 2.7, is the problem of finding a method for deciding whether a simply connected Riemann surface, defined in some specific manner (e.g., as a branched covering of the Riemann sphere, or as a surface equipped with some Riemannian metric, or obtained by gluing polygons, etc.) is conformally equivalent to the Riemann sphere, or to the complex plane, or to the open unit disc. We review several methods of dealing with this problem, mentioning works of Ahlfors, Nevanlinna, Teichmüller, Lavrentieff and Milnor. Besides the combinatorial tools introduced in the previous sections (namely, nets and line complexes), the works on the type problem that we review use the notions of almost analytic function and quasiconformal mapping.

In the last section, Sect. 2.8, combinatorial tools are used for other approaches to Riemann surfaces, in particular, in the theory of dessins d'enfants, in applications to knots and links and in the theory of slalom polynomials. Two different stratifications of the space of monic polynomials are presented.

2.2 Rigidity of Geometric Structures

In this section, we give several examples of locally rigid structures on surfaces. A classical example of a non-locally rigid structure is a Riemannian metric on any manifold of dimension ≥ 2.

2.2.1 Volume, Symplectic and Contact Forms

Moser's theorem says that only the total volume of a smooth volume form on a connected compact manifold matters, namely, two volume forms of equal total volume are isotopic. More precisely:

Theorem 2.2.1 (Moser [77]) *Let M be a compact connected oriented manifold of dimension n equipped with two smooth volume forms ω_0 and ω_1 of equal total volume. Then there exists an isotopy ϕ_t, $t \in [0, 1]$, satisfying $\phi_t^*(t\omega_1 + (1-t)\omega_0) = \omega_0$. In particular, we have $\omega_0 = \phi_1^*\omega_1$.*

Proof Clearly $\omega_1 = f\omega_0$ for some positive function f, since for any $p \in M$ and for any oriented frame $X_1, \cdots, X_n$ at p we have $\omega_0(X_1, \cdots, X_n) > 0$ and $\omega_1(X_1, \cdots, X_n) > 0$. It follows that $t \mapsto \omega_t = t\omega_1 + (1-t)\omega_0$ is a path of volume forms that connects the form ω_0 to the form ω_1 and we have

$$\begin{aligned} \frac{d}{dt}\int_{[M]} \omega_t &= \int_{[M]} \frac{d}{dt}\omega_t \\ &= \int_{[M]} \omega_1 - \omega_0 \text{ (differentiating the formula for } t \mapsto w_t) \\ &= 0 \text{ (since the two forms have the same volume).} \end{aligned}$$

Thus, the de Rham cohomology class $[\omega_1 - \omega_0]$ vanishes on the connected manifold M, therefore there exists a smooth $(n-1)$-form α with $d\alpha = \omega_1 - \omega_0$. Hence, $\frac{d}{dt}\omega_t = d\alpha$.

In order to construct the required isotopy ϕ_t satisfying $(\phi_t)^*\omega_t = \omega_0$, we need a time-dependent vector field X_t whose flow ϕ_t^X induces the isotopy ϕ_t and such that the equality $(\phi_t^X)^*\omega_t = \omega_0$ holds. Differentiating, using the Cartan formula and the fact that $d\omega_t = 0$, yields

$$0 = \frac{d}{dt}(\phi_t^X)^*\omega_t = (\phi_t^X)^*(d(i_{X_t}\omega_t) + d\alpha) = (\phi_t^X)^*(d(i_{X_t}\omega_t + \alpha)).$$

The family of vector fields $X = (X_t)_{t\in[0,1]}$ defined by $i_{X_t}\omega_t = -\alpha$ satisfies the above equation. The equation $i_{X_t}\omega_t = -\alpha$ has, for a given $(n-1)$-form α, has a unique solution, since for each $t \in [0, 1]$, ω_t is a non-degenerate volume form. Therefore the family of forms $(\phi_t^X)^*\omega_t$ is constant, hence $(\phi_1^X)^*\omega_1 = \omega_0$ as required. □

The above result also holds for a symplectic form, that is, a closed nondegenerate differential 2-form, at the price of a stronger assumption. The proof works verbatim. Thus we get:

Theorem 2.2.2 (J. Moser) *Let M be a compact connected oriented manifold of dimension n equipped with two symplectic forms ω_0 and ω_1 of equal periods, i.e., with equal de Rham cohomology classes. Assume that the forms are connected by a smooth path ω_t of symplectic forms with constant periods, i.e., for all $t \in [0, 1]$, $[\omega_t] = [\omega_0]$ in $H^2_{\mathrm{dR}}(M)$. Then there exists an isotopy ϕ_t, $t \in [0, 1]$, with $\phi_t^*\omega_t = \omega_0$. In particular, $\omega_0 = \phi_1^*\omega_1$.* ■

The so-called "Moser trick" works as a "simplification by d" in the equation $di_{X_t}\omega_t = -d\alpha$ and it amounts to noticing that for a volume form ω and for an $(n-1)$-form β the equation $i_X\omega = \beta$ has a unique solution X.

From symplectic structures, we pass to contact forms and contact structures.

A *contact form* α on an n-dimensional manifold M is a pointwise non-vanishing differential 1-form such that at each point p in M the restriction of $(d\alpha)_p$ to the kernel of α_p is non-degenerate.

A *contact structure* on M is a distribution of hyperplanes in the tangent bundle TM given locally as a field of kernels of a contact form.

Moser's method also works, even more simply, without "trick" and without extra stronger assumption, for families of contact structures and it gives another proof of the Gray stability theorem for contact forms [41]:

Theorem 2.2.3 (Gray [41]) *Let $(\alpha)_{t\in[0,1]}$ be a smooth family of contact forms on a compact manifold M. Then there exists a t-dependent vector field X_t on M with flow ϕ_t and* $\mathrm{kernel}(\phi_t^*\alpha_t) = \mathrm{kernel}(\alpha_0)$. *In particular, there exists a family of positive functions $(f_t)_{t\in[0,1]}$ such that for all $t \in [0, 1]$, $\phi_t^*\alpha_t = f_t\alpha_0$.*

Proof A measure for the variation of the kernel $[\alpha_t]$ of α_t is the restriction $\dot{\alpha}_{t\,|[\alpha_t]}$ of $\dot{\alpha}_t = \frac{d}{dt}\alpha_t$ to the kernel $[\alpha_t]$. By the non-degeneration of the restriction $d\alpha_{|[\alpha_t]}$, there exists a unique t-dependent vector field X_t in the distribution (that is, the family of subspaces) $[\alpha_t]$ with $\dot{\alpha}_{t\,|[\alpha_t]} + i_{X_t} d\alpha_{t\,|[\alpha_t]} = 0$. Hence the kernels of $\phi_t^*\alpha_t$ do not vary since $\frac{d}{dt}\phi_t^*[\alpha_t] = \phi_t^*(\dot{\alpha}_{t\,|[\alpha_t]} + i_{X_t} d\alpha_{t\,|[\alpha_t]}) = 0$. □

For more applications, see [73]. The use of a proper exhaustion allows us to extend the above theorem to pairs of volume forms on connected non-compact manifolds of equal finite or infinite total volume.

Furthermore, the above proofs work also in a relative version: if the forms coincide on a closed subset A, then the time-dependent vector field X_t vanishes along the subset A and generates a flow that fixes the subset A. The Darboux type rigidity theorems for volume, symplectic and contact forms follow:

Theorem 2.2.4 (Local Darboux Rigidities) *Let ω be a volume or a symplectic form, and let α be a contact form on an n-, $2n$- or $(2n+1)$-manifold M respectively. Then at each point of M there exists a coordinate chart $(x_1, \cdots, x_n)$ or $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ or $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ respectively such that the volume form is expressed by $\omega = dx_1 \wedge \cdots \wedge dx_n$, the symplectic form by $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ and the contact form by $\alpha = dz - y_1 dx_1 - \cdots - y_n dx_n$.* ■

Remark The classical Darboux theorem holds in the setting of symplectic geometry, see [29]. This theorem says that any symplectic manifold of dimension $2n$ is locally isomorphic (in this setting, it is said to be symplectomorphic) to the linear symplectic space $\mathbb{C}^n$ equipped with its canonical symplectic form $\sum dx \wedge dy$. As a consequence, any two symplectic manifolds of the same dimension are locally symplectomorphic to each other.

2.2.2 Almost Complex Structures

An almost complex structure J on a differentiable surface S is an endomorphism of the tangent bundle of S satisfying $J^2 = -\mathrm{Id}$. More precisely, $J = \{J_p \mid p \in S\}$ is a smooth family of endomorphisms of tangent spaces $J_p : T_pS \to T_pS$ such that at each point $p \in S$, we have $J_p^2 = -\mathrm{Id}_{T_pS}$. The standard example is $(\mathbb{R}^2, J)$ where J is the constant family of endomorphisms given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This corresponds to the plane $\mathbb{C}$ equipped with multiplication by i.

The following proof is not based upon the above method.

Theorem 2.2.5 (Local J-Rigidity in Real Dimension 2) *Let J be an almost complex structure on a surface S. Then at each point $p \in S$ there exists a coordinate chart (x, y) such that $J(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$ holds.*

Proof *(Sketch)* First construct, using a partition of unity, an almost complex structure J_0 on the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ such that the structures J and J_0 are isomorphic when restricted to open neighborhoods U of p on S and U_0 of 0 on T. Let ω be a volume form on T and let g_{ω,J_0} be the associated Riemannian metric $g_{\omega,J_0}(u, v) = \omega(u, J_0(v))$. Let f be the real function on T satisfying $f(0) = 0$ and solving the partial differential equation

$$d(df \circ J_0) = -k_{g_{\omega,J_0}}\omega,$$

where $k_{g_{\omega,J_0}}$ is the Gaussian curvature of the metric g_{ω,J_0}. By the Gauss–Bonnet Theorem, $\int_T k_{g_{\omega,J_0}}\omega = 0$, therefore the equation admits a solution by Fourier theory. Now use the Gauss curvature formula:

$$k_{g_{e^{2f}\omega,J_0}}\omega = k_{g_{\omega,J_0}}\omega + d(df \circ J_0) = 0.$$

The metric $g_{e^{2f}\omega,J_0}$ has constant curvature 0, therefore (T, J_0) is bi-holomorphic to $\mathbb{C}/\Gamma$ for some lattice Γ (a 2-generator discrete subgroup), which shows the statement for a local chart at $0 \in (T, J_0)$, and hence also for a local chart at any $p \in (S, J)$. □

For a detailed proof of Theorem 2.2.5, see [8, p. 114–117]. This theorem shows that every almost complex structure on a differentiable surface S determines in a unique way a holomorphic structure in the usual sense (that is, a structure defined by an atlas of local charts with values in $\mathbb{C}$ and holomorphic local changes).

Exercise 2.2.1 Give a proof of Theorem 2.2.5 using Moser's trick.

Remark The first definition of an almost complex structure is due to Charles Ehresmann who addressed the question of the existence of a complex analytic structure on a topological (resp. differentiable) manifold of even dimension, from the point of view of the theory of fiber spaces; cf. Ehresmann's talk at the 1950 ICM [33]. Ehresmann mentions the fact that H. Hopf addressed the same question from a different point of view. He notes in the same paper that by a method proper to even-dimensional spheres he showed that the 4-dimensional sphere does not admit any almost complex structure, a result which was also obtained by Hopf using different methods. See also McLane's review of Ehresmann's work [84]. Ehresmann and MacLane also refer to the work of Wen-Tsün Wu [114, 115], who was a student of Ehresmann in Strasbourg.

Remark The Nijenhuis tensor is an obstruction to local integrability of J-fields in higher dimensions, where Theorem 2.2.5 does not hold in the general case, see [8, p. 124–125]. Real dimension 2 is very special!

2.2.3 Almost Complex Structures on n-Spheres

The existence and integrability of J-structures in dimension 2 is very special. We mentioned that the 4-sphere S^4 does not admit any J-field (Ehresmann and Hopf), but the 6-sphere does.

Clearly only spheres of even dimension can carry J-fields. Adrian Kirchhoff, in his PhD thesis (ETH Zürich 1947) [57] established a relationship between two non-obviously related structures on spheres S^{2n} and S^{2n+1} of different dimensions; we report on this now.

Recall that a *parallelism* on a smooth n-manifold is a global field of frames, that is, a field of n tangent vectors which form a basis of the tangent space at each point.

Theorem 2.2.6 (Kirchhoff [58]) *The sphere S^n, $n \geq 0$, admits a J-field if and only if the sphere S^{n+1} admits a parallelism.*

Proof The case $n = 0$ is special: the tangent space TS^0 is of dimension 0, therefore $J = \mathrm{Id}_{TS^0}$ is a J-field and S^1 admits a parallelism.

"Only if" part for $n > 0$: In $V = \mathbb{R}^{n+2}$ with the standard basis $e_0, e_1, \cdots, e_{n+1}$, let S^n be the unit sphere in the span $[e_1, \cdots, e_{n+1}]$. Let S^{n+1} be the unit sphere of V. Assume that J is a J-field on S^n. Let $L : S^{n+1} \to \mathrm{GL}(V), v \mapsto L_v$, be the continuous map satisfying $L_v(e_0) = v,\ v \in S^{n+1}$, defined as follows:

- First, for $v \in S^n$, seen as the equator of S^{n+1}, we set
  $L_v(v) = -e_0,\ L_v(e_0) = v,$
  $L_v(u) = v + J_v(u - v),\ u \in [v, e_0]^\perp.$
- For $v \in S^{n+1}$, we can write $v = \sin(t)e_0 + \cos(t)v',\ v' \in S^n,\ t \in]-\pi, \pi[$. We then set
  $L_v = \sin(t)\mathrm{Id}_V + \cos(t)L_{v'}.$

We have $L_v \circ L_v = -\mathrm{Id}_V$ for $v \in S^n$, hence $L_v = \sin(t)\mathrm{Id}_V + \cos(t)L_{v'} \in \mathrm{GL}(V)$ for $v \in S^{n+1}$, since the eigenvalues are $\sin(t) \pm \cos(t)i$. Observe that $T_{e_0}S^{n+1} = e_0 + [e_0]^\perp$ and $[e_0]^\perp = [e_1, \cdots, e_{n+1}]$. The differential $(DL_v)_{e_0} : T_{e_0}S^{n+1} \to V$ at e_0 of L_v maps the space $T_{e_0}S^{n+1}$ onto an affine space of dimension $n+1$ in $T_{L_v(e_0)}V$ that intersects transversely the ray $[v]$. Then, for $v \in S^{n+1}$, the images

$(DL_v)_{e_0}(e_1), \cdots, (DL_v)_{e_0}(e_{n+1}) \in T_v V$ define a frame in $T_v S^{n+1} = v + [v]^{\perp}$ by the projection parallel to $[v]$ onto $v + [v]^{\perp}$.

"If" part: Work backwards. □

Ehresmann in his ICM talk [33] mentions Kirchhoff's results [57].

In fact, by a celebrated result of Jeffrey Frank Adams [1], only the spheres S^1, S^3, S^7 admit a parallelism. This implies that S^4 does not admit any J-field and S^6 does. Adams' result was obtained several years after Kirchhoff's result.

The question of the existence of a complex structure on S^6 is still wide open. How the Nijenhuis integrability condition for a J-field on S^6 translates into a property of framings on S^7 is the subject of a recent paper [67].

We end this section on rigidity by a word on exotic spheres: Any two differentiable manifolds of the same dimension are locally diffeomorphic. But such manifolds may be homeomorphic without being diffeomorphic. The first examples of such a phenomenon are Milnor's exotic 7-spheres [75]. In later papers, Milnor constructed additional examples.

2.3 The First Compact Riemann Surface

A Riemann surface is a complex 1-dimensional real manifold, or a 2-dimensional manifold equipped with a complex 1-dimensional structure, that is, an atlas whose charts take values in the Gaussian plane $\mathbb{C}$, with holomorphic transition functions.

In this section, we shall deal with the simplest Riemann surface, the Riemann sphere.

2.3.1 The Riemann Sphere

The familiar round sphere in 3-space, together with its group of rigid motions, can be seen as a holomorphic object: its motions are angle-preserving. It is also a one-point compactification of the field of complex numbers. We shall see that this construction as a one-point compactification can be generalized to an arbitrary field.

First we ask the question:

Why do we need the Riemann sphere?

The statement: "Every sequence of complex numbers has a convergent subsequence" is very true, indeed true for bounded sequences. The statement is salvaged without this assumption if we introduce a wish object w with the property that every sequence of complex numbers, for which no subsequence converges to a complex number, converges to w. In this way, from the familiar Gaussian plane $\mathbb{C}$, we gain a new space, $\mathbb{C} \cup \{w\}$ in which the above statement improves from very true to true. This topological construction is the familiar one-point compactification of non compact but locally compact spaces.

Very true is also the statement: "The ratio $\frac{a}{b}$ is well defined as long as $(a, b) \neq (0, 0)$". Again the statement becomes true if we introduce by wish a new object $w \notin k$ with $\frac{a}{0} = w, a \neq 0$. This algebraic construction applies to any field k, and not only $\mathbb{C}$.

In both constructions the object w appears as a newcomer, an immigrant, with a special restricted status.

It is Riemann who gave an interpretation of the new set $X = \mathbb{C} \cup \{w\}$ together with a very rich structure Σ on it, for which the new element gains unrestricted status. In short, the automorphism group of (X, Σ) acts transitively on this space, that is, the space X is homogeneous.

The above topological construction also shows that the newcomer w is above any bound, so from now on we use the symbol ∞ for w.

Here is another construction of an infinity, valid for any field.

Let k be a field. An element $\lambda \in k$ can be interpreted as a linear map $a \in k \mapsto \lambda a \in k$. Its *graph* $G_\lambda \subset k \times k$ is the vector subspace $\{(a, \lambda a) \mid a \in k\}$ of dimension 1 in $k \times k$. So we get an embedding $\iota : \lambda \in k \mapsto \mathbb{P}^1(k)$ of the field k in the projective space $\mathbb{P}^1(k)$ of all 1-dimensional vector subspaces in $k \times k$. The vector subspace $G = \{(0, b) \mid b \in k\}$ is the only one which is not in the image of the embedding ι.

The element λ can be retrieved from G_λ as a slope: indeed, for any $(a, b) \in G_\lambda$, if $(a, b) \neq (0, 0)$ then $a \neq 0$ and $\lambda = \frac{b}{a}$.

So the missing vector subspace G corresponds to the forbidden fraction $\frac{1}{0} = \infty$ and can be called G_∞.

In the case where $k = \mathbb{R}$, this is the well-known embedding of $\mathbb{R}$ in the circle of *directions up to sign*. Extending $\iota : k \cup \{\infty\} \to \mathbb{P}^1(k)$ by $\iota(\infty) = G_\infty$ gives the interpretation of $k \cup \{\infty\}$ as the projective space $\mathbb{P}^1(k)$. Each linear automorphism A of the k-vector space k^2 induces a self-bijection G_A of $k \cup \{\infty\} = \mathbb{P}^1(k)$. If the matrix of A is the 2×2-matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, k)$, then $G_A(G_\lambda) = G_{\lambda'}$ with $\lambda' = \frac{a\lambda+b}{c\lambda+d}$. The transformation $G \in \mathbb{P}^1(k) \mapsto G_A(G) \in \mathbb{P}^1(k)$ or $\lambda \mapsto \frac{a\lambda+b}{c\lambda+d}$ is called a fractional linear or Möbius transformation. Note that in particular $G_A(G_\infty) = \frac{a}{c}$.

Given a general field k, an important structure on $\mathbb{P}^1(k)$ is provided by a 4-point function which we shall study in Sect. 2.4.3.

At this stage, we restrict to the case $k = \mathbb{C}$. The above construction of $\mathbb{P}^1(k)$ for an arbitrary field k gives the familiar construction of $\mathbb{P}^1(\mathbb{C}) = (\mathbb{C}^2 - \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of nonzero complex numbers.

The set $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$ carries many structures. First, there is the structure of a differentiable manifold given by the following atlas: We set $U_0 = \mathbb{P}^1(\mathbb{C}) \setminus \{G_\infty\}$ and $U_\infty = \mathbb{P}^1(\mathbb{C}) \setminus \{G_0\}$. Observe that $U_0 = \{G_\lambda \mid \lambda \in \mathbb{C}\}$ and that every $G \in U_\infty$ is of the type $G'_\sigma = \{(\sigma b, b) \mid b \in \mathbb{C}\}$ for $\sigma \in \mathbb{C}$.

Define maps $z_0 : U_0 \to \mathbb{C}$ by $z_0(G_\lambda) = \lambda$ and $z_\infty : U_\infty \to \mathbb{C}$ by $z_\infty(G'_\sigma) = \sigma$. Both maps are bijections. For $G \in U_0 \cap U_\infty$ the two maps are related; indeed, $z_0(G) z_\infty(G) = 1$. It follows that the system $((U_0, z_0), (U_\infty, z_\infty))$ is an atlas for a manifold structure with coordinates functions (z_0, z_∞). Its quality is hidden in the quality of the coordinate change. For $G \in U_0 \cap U_\infty$, from the above implicit relation it follows that $z_\infty(G) = 1/z_0(G)$, $z_0(G) = 1/z_\infty(G)$. This coordinate change is differentiable; therefore $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$ is a smooth manifold with charts in the Gaussian plane $\mathbb{C}$.

The smooth 2-dimensional real manifold $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$ is diffeomorphic to the unit sphere in the three-dimensional real vector space $\mathbb{R}^3$. More precisely, the coordinate change $\phi_{\infty,0} : \mathbb{C}^* = z_0(U_0) \to z_\infty(U_\infty) = \mathbb{C}^*$ is given in terms of the natural coordinate z on $\mathbb{C}^*$, by $\phi_{\infty,0}(z) = 1/z$. The smooth map $\phi_{\infty,0} : \mathbb{C}^* \to \mathbb{C}^*$ is moreover holomorphic, so the above atlas provides $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$ with the structure of a Riemann surface. This Riemann Surface is the *Riemann Sphere*.

Let U be an open subset of the Gaussian plane $\mathbb{C}$. Riemann defined a map $\phi : U \to \mathbb{C}$ to be holomorphic without using an expression that evaluates the map at given points. The idea is the following. The real tangent bundles TU and $T\mathbb{C}$ come with a field m_i of endomorphisms. (The notation m_i stands for "multiplication by i".) The value $m_{i,p}$ of the field m_i at the point p is the linear map $m_{i,p} : T_pU \to T_pU,\ u \mapsto iu$. To be holomorphic by Riemann's definition is given by the following property of the differential:

$$(D\phi)_p(m_{i,p}(u)) = m_{i,\phi(p)}((D\phi)_p(u)).$$

In words, this means that the differential $D\phi$ is $\mathbb{C}$-linear.

Riemann's characterization of holomorphic maps together with the local J-Rigidity Theorem 2.2.5 allows us to define a Riemann surface (S, J) as a real 2-dimensional differentiable manifold S equipped with a smooth field of endomorphisms $J : TS \to TS$ of its tangent bundle satisfying $J \circ J = -\mathrm{Id}_{TS}$.

The Riemann Sphere is the first example of a compact Riemann surface. The most familiar non-compact Riemann surface is the Gaussian plane $\mathbb{C}$. Another most important Riemann surface is the unit disc in $\mathbb{C}$. This is also the image of the southern hemisphere by the stereographic projection from the North pole onto a plane passing through the equator. This projection is holomorphic. The importance of the unit disc stems from the fact that it is equipped with the Poincaré metric, which makes it a model for the hyperbolic plane.

2.3.2 The Group SU(2) *and Its Action on the Riemann Sphere*

Now that we are familiar with the Riemann sphere, we study a group action on it.

Let $< u, v >_{\mathrm{Herm}}$ be the usual Hermitian product on $\mathbb{C}^2$. This is the complex bilinear form on $\mathbb{C}$ defined by $< u, v >_{\mathrm{Herm}} = u_1\bar{v_1} + u_2\bar{v_2}$. The Hermitian perpendicular $L^\perp$ to a complex vector subspace L is again a complex vector subspace.

The group of determinant 1 linear transformations of $\mathbb{C}^2$ that preserve $< u, v >_{\mathrm{Herm}}$ is the group SU(2) consisting of all matrices of the form $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$, $(a, b) \in \mathbb{C}^2$, $a\bar{a} + b\bar{b} = 1$. This group acts on the Riemann sphere by Möbius transformations, in fact, by rotations. The map $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \mapsto (a, b)$ defines a diffeomorphism $\mathrm{SU}(2) \to S^3$ and induces a Lie group structure on the sphere S^3.

The group SU(2) acts transitively by conformal automorphisms on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. The stabilizer of $L = \{(\lambda, 0) \mid \lambda \in \mathbb{C}\}$ in SU(2) is the group $\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)$, $a \in \mathbb{C}, a\bar{a} = 1$, which is isomorphic to the group of complex numbers of norm 1. The quotient construction induces a Riemannian metric on

$$\mathbb{P}^1(\mathbb{C}) = \mathrm{SU}(2)/\mathrm{Stab}_{\mathrm{SU}(2)}(L).$$

A *marked element* in $\mathbb{P}^1(\mathbb{C})$ is a pair (L, u) where $u = (a, b) \in \mathbb{C}^2$, $a\bar{a} + b\bar{b} = 1$ and $L = [u] = \{\lambda u \mid \lambda \in \mathbb{C}\}$. Note that this representation is redundant since u determines $L = [u]$.

The group SU(2) acts simply transitively on marked elements in $\mathbb{P}^1(\mathbb{C})$.

The involution $L \mapsto L^\perp$ extends to marked elements: map $(L, u) = (L, (a, b))$ first to $u^\perp = (\bar{b}, -\bar{a})$ and next to $(L^\perp, u^\perp)$ with $L^\perp = [u^\perp]$.

A marked element (L, u) determines a path in $\mathbb{P}^1(\mathbb{C})$ by $L_u : t \in [0, \pi] \mapsto L_u(t) = [\cos(t)u + \sin(t)u^\perp]$, which in fact is a simple closed curve. Its velocity at $t = 0$ is a length 1 tangent vector $V_u \in T_{[u]}(\mathbb{P}^1(\mathbb{C}))$. Observe that $V_u = V_{-u}$ and $V_{iu} = -V_u$. The path L_u lifts to $H_u^\perp : t \in [0, \pi] \mapsto H_u^\perp(t) = \cos(t)u + \sin(t)u^\perp \in S^3$, which is a geodesic from u to $-u$ perpendicular to the foliation on S^3 by the Hopf circles $H_v = \{v' \in S^3 \mid v' = \lambda v\}$, $v \in S^3$. Hopf circles H_v map to points, and geodesics $H_u^\perp$ map to simple closed geodesics in $\mathbb{P}^1(\mathbb{C})$.

The map $\pm u \in S^3/\{\pm \mathrm{Id}\} = \mathbb{P}^3(\mathbb{R}) \mapsto V_u \in T(\mathbb{P}^1(\mathbb{C}))$ induces a bijection onto the length 1 vectors to $\mathbb{P}^1(\mathbb{C})$. Observe that SU(2) acts almost simply transitively on length 1 tangent vectors to $\mathbb{P}^1(\mathbb{C})$. The quotient group $\mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm \mathrm{Id}\}$ acts simply transitively on length 1 tangent vectors.

2.4 All Three Planar Geometries and Hyperbolic 3-Space Simultaneously

2.4.1 A Stratification of the Riemann Sphere Arising from Algebra

Bernhard Riemann was aware of the (Riemann) sphere being the complex plane union a point at infinity. His point of view on complex analysis was very geometric. In this section, we wish to describe an incarnation of the Riemann sphere which arises from algebra. For more details on this model, see [8, Chap. 3, §3.1] and [9, Chap. 1, §8.3].

The starting object is the set Σ of surjective ring homomorphisms from the ring $\mathbb{R}[X]$ of polynomials in one unknown X with real coefficients to a field F. On the set Σ we introduce two equivalence relations. The first relation, $\sim$, declares $f\colon \mathbb{R}[X] \to F$ and $f'\colon \mathbb{R}[X] \to F'$ to be equivalent if there exists a field isomorphism $\phi\colon F \to F'$ with $f' = \phi \circ f$.

The relation $f \sim f'$ holds if and only if the ideals kernel(f), kernel(f') in $\mathbb{R}[X]$ are equal.

The second relation, $\sim_X$, requires $f \sim f'$ and moreover $f(X) = f'(X)$ holds in $\mathbb{R}[X]/\text{kernel}(f) = \mathbb{R}[X]/\text{kernel}(f')$.

Up to field isomorphism, there are only three fields, F, that are hit by a surjective ring homomorphism $f : \mathbb{R}[X] \to F$, namely, the fields $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{F}_1$ where $\mathbb{F}_1$ is the field with one element, that is, the field where $0 = 1$ holds. The field $\mathbb{F}_1$ corresponds to the ideal $\rho = \mathbb{R}[X]$ consisting of the whole ring, which is prime, maximal but not proper.

Exercise The two fields $\mathbb{R}$, $\mathbb{F}_1$ have only the identity as automorphism and the field $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ only two automorphisms as $\mathbb{R}$-algebra, but as many field automorphisms the power set of the real numbers.

In the following we will describe the quotient sets $\Sigma/\sim$, $\Sigma/\sim_X$ together with natural structures on these sets.

All ideals in $\mathbb{R}[X]$ are principal, that is, any such ideal is generated by a single element (it is obtained by multiplication of such an element by an arbitrary element of the ring). Kernels of $f \in \Sigma$ are prime ideals, that is, the quotient of $\mathbb{R}[X]$ by such an ideal is an integral domain (the product of any two nonzero elements is nonzero). Thus, we have three kinds of kernels of f, namely, $\rho = (1) = \mathbb{R}[X]$, $(X-a)$, $a \in \mathbb{R}$, and $((X-a)^2 + b^2)$, $a, b \in \mathbb{R}, b > 0$. Therefore the set $\Sigma/\sim$ is identified with $\mathbb{C}_+ \cup \mathbb{R} \cup \{\rho\}$. Here we use the notation $\mathbb{C}_\pm = \{a + bi \mid a, b \in \mathbb{R}, \pm b > o\}$ for the upper/lower half planes.

The kernel of the ring homomorphism f is not sufficient in order to describe its class in $\Sigma/\sim_X$ if kernel$(f) = ((X-a)^2 + b^2)$. One needs moreover to specify a root $a + bi \in \mathbb{C}_+$ or $a - bi \in \mathbb{C}_-$. Thus the set $\Sigma/\sim_X$ is a disjoint union of 4 strata $\Sigma/\sim_X = \mathbb{C}_+ \cup \mathbb{C}_- \cup \mathbb{R} \cup \{\rho\}$.

The fields $\mathbb{C}, \mathbb{R}, \mathbb{F}_1 = \{0\}$ are realized as sub-$\mathbb{R}$-algebras in $\mathbb{C}$, so an alternative description of the set $\Sigma/\sim_X$ is the set of $\mathbb{R}$-algebra homomorphism from $\mathbb{R}[X]$ to $\mathbb{C}$.

We shall see that the set $R = \Sigma/\sim_X$ and its strata carry a rich panoply of structures. The set $R = \mathbb{C}_+ \cup \mathbb{C}_- \cup \mathbb{R} \cup \{\rho\}$ is identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Structures, such as the Chasles three point function (defined below) on $\mathbb{R} \subset R$, or the hyperbolic geometry on $\mathbb{C}_+$, will appear naturally. Naturally means here that the construction that leads to the structure commutes with the $\mathbb{R}$-algebra automorphisms of $\mathbb{R}[X]$. For instance, it commutes with the substitutions that consist in translating X to $X - t$, $t \in \mathbb{R}$, or with stretching X to λX, $\lambda \in \mathbb{R}^*$. The ideal $(X - a)$ maps to the ideal $(X - a - t)$ by translation and to $(X - \frac{a}{\lambda})$ by stretching.

A first example is the Chasles 3-point function $\text{Ch}(A, B, C)$ on the stratum $\mathbb{R}$ consisting of the ideals $(X - a), a \in \mathbb{R}$, defined as follows: Given three distinct such points, $A = (X - a)$, $B = (X - b)$, $C = (X - c)$, define $\text{Ch}(A, B, C) = \frac{b-a}{c-a}$.

In words, $\text{Ch}(A, B, C)$ is the ratio of the monic generators of B and C evaluated at the zero of the monic generator of A.

The next example is the 4-point function cross ratio $\mathrm{cr}(A, B, C, D)$: for 4 distinct points $A = (X - a), B = (X - b), C = (X - c), D = (X - d)$, define $\mathrm{cr}(A, B, C, D) = \mathrm{Ch}(A, B, C)\mathrm{Ch}(D, C, B) = \frac{b-a}{c-a} \cdot \frac{c-d}{b-d}$. In words, this is Chasles evaluated at the first three points times Chasles evaluated at the last three points in the reverse order. It is truly a remarkable fact that the cross ratio function extends to a 4-point real function on $\mathbb{P}^1(\mathbb{R})$ and to a 4-point complex function on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\rho\}$ if one transfers the above wishful calculus with $w = \infty$ to ρ.

The multiplicative monoid $\mathbb{R}^*[X]$ of polynomials which do not vanish at 0 has also automorphisms that do not directly fit with the interpretation as polynomials with unknown X. In particular, they are not ring automorphisms, but monoid automorphisms. A main example is the *twisted palindromic symmetry*: perform on a polynomial $P(X)$ the substitution $X \to \frac{-1}{X}$, followed by stretching with factor $(-X)^{\mathrm{degree}(P)}$. (The palindromic symmetry is said to be twisted, because of the minus signs.) Then the ideal $(X - a)$ maps to the ideal $(-X(\frac{-1}{X} - a)) = (1 + aX) = (X + \frac{1}{a})$. The Chasles function restricted to $\mathbb{R}^*$ does not commute with the symmetry $a \mapsto \frac{-1}{a}$, but the cross ratio commutes. (This property is among the ones that make the cross ratio more natural than the Chasles 3-point function.) This symmetry, which is an involution, extends to a fixed point free involution $\sigma_{\mathbb{P}}$ of $\mathbb{R} \cup \{\rho\} = \mathbb{P}^1(\mathbb{R})$. Remarkably, the symmetry $\sigma_{\mathbb{P}}$ commutes with the cross ratio cr. In this sense, cr is more natural than Ch.

The above operations of real translation and stretching, i.e., substituting $X - t$ for X or λX for X with $t \in \mathbb{R},\ \lambda \in \mathbb{R}^*$, together with the twisted palindromic symmetry induce bijections of the set $\{(X - u)(X - \bar{u}) \mid u \in \mathbb{C}_+\}$ of monic polynomials of degree 2 without real roots. Composing these bijections generates a group G. It is a remarkable fact that this group is, as an abstract group, isomorphic to the group $\mathrm{PGL}(2, \mathbb{R})$. It is also a remarkable fact that the abstract group $\mathrm{PGL}(2, \mathbb{R})$ carries a unique structure of Lie group. So there is also a topology on G, which allows us to define the subgroup $G_0 \subset G$ as the connected component of the neutral element in the Lie group $G = \mathrm{PGL}(2, \mathbb{R})$. The group G_0 is isomorphic to the group $\mathrm{PSL}(2, \mathbb{R})$.

The fixed point free involution $\sigma_{\mathbb{P}}$ on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\rho\} = \partial \bar{C}_+$ extends to $\mathbb{C}_+$ by putting $\sigma_{\mathbb{P}}(u) = \frac{-1}{u}$ for an involution with i as unique fixed point.

The group G_0 acts transitively and faithfully on the above strata $\mathbb{C}_\pm$ and on $\mathbb{R} \cup \{\rho\}$. From this action one gets a topology on the strata and also, as we will explain, a geometry on $\mathbb{C}_+$. It is also remarkable that this geometry, in fact, the planar hyperbolic geometry, can also be explained in a more elementary way in term of the interpretation as ideals.

The action of G_0 on $\mathbb{C}_+$ is the so-called modular action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{C}_+$. Thinking of an element $g \in G_0$ as a real 2×2 matrix of determinant 1 up to sign, $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the action on $u \in \mathbb{C}_+$ is given by $(g, u) \mapsto \frac{au+b}{cu+d}$.

The modular action of G_0 on $\mathbb{C}_+$ extends to the projective action of $G = \mathrm{PGL}(2, \mathbb{R})$ on $\partial \bar{\mathbb{C}}_+ = \mathbb{P}^1(\mathbb{R})$ and also to the projective action of the complex group $\mathrm{PGL}(2, \mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$.

The hyperbolic geometry on $\mathbb{C}_+$ can also be defined in terms of ideals in the following rather elementary way.