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Dedications

In fond memory of my maternal great grandmother and my maternal grandmother

To my mother Mrs. G. Suseela and to my father Mr. T. E. Sarangan

In fond memory of my Tittu and Kutty

Preface

By Trotter approximations, see Trotter, 1958, we mean roughly speaking, the continuous dependence of a semigroup $\{S(t) : t \geq 0\}$ on its infinitesimal generator A, and vice versa, that is, the continuous dependence of A on $\{S(t) : t \geq 0\}$. In the proof of the main result from Trotter, 1958, it is not clear if the limit of the resolvents $R(\lambda, A_n)$, $n \in \mathbb{N}$ of A_n is itself a resolvent of some operator A. This was pointed out and also corrected by Kato, 1959. See Pazy [1]. Hence, such approximations are also called Trotter-Kato approximations. The objective of this research monograph is to present a systematic study on Trotter-Kato approximations of stochastic differential equations in infinite dimensions and applications.

A study on stochastic differential equations (SDEs) in infinite dimensions was initiated in the 1960s by Curtain [1] and Curtain and Falb [1, 2], followed in the 1970s by Métivier and Pistone [1], Chojnowska-Michalik [1], Ichikawa [1, 2], Haussman [1] and Zabczyk [1], among others, using the semigroup theoretic approach; and Pardoux [1, 2] using the variational approach of Lions [1] from the deterministic case. Note, however, that a strong foundation of SDEs in infinite dimensions in the semilinear case was first laid by the pioneering work of A. Ichikawa in 1982 in Ichikawa [3]. All these aforementioned attempts in infinite dimensions were generalizations of the celebrated work on stochastic ordinary differential equations introduced by K. Itô in the 1940's in Itô [1] and independently by Gikhman [1] in a different form. Today, the theory of SDEs in the sense of Itô, in infinite dimensions, is a well-established area of research; see, for instance, the excellent monographs by Curtain and Pritchard [1], Itô [2], Métivier [2], Belopolskaya and Dalecky [1], Rozovskii [1], Ahmed [1], Da Prato and Zabczyk [1], Kallianpur and Xiong [1] and Gawarecki and Mandrekar [1]. Throughout this book, we shall use mainly the semigroup theoretic approach as it is our interest to study Trotter-Kato approximations of mild solutions of SDEs in infinite dimensions.

To the best of our knowledge, Kannan and Bharucha-Reid, 1985, were the first to introduce and study Trotter-Kato approximations of linear stochastic integrodifferential evolution equations of Volterra-Itô type in Hilbert spaces. They considered the existence and uniqueness of mild solutions of such a class of equations and also proved the convergence of mild solutions of the Trotter-Kato approximating equations to the mild solution of the original stochastic equation in mean-square. Further, it was shown that the corresponding induced probability measures P_n converge weakly to P. They also obtained error estimates by introducing a version of the

Trotter-Kato approximation, namely a zeroth order approximation, that is, approximating a stochastic integrodifferential equation by a deterministic evolution equation, among others. Since then, such results were obtained for many classes of stochastic differential equations.

The book begins in Chapter 1 with a brief introduction and considering motivating problems like heat equations, an electric circuit, an interacting particle system, lumped control systems and the option and stock price dynamics, to study, in the following chapters, the abstract SDEs in infinite dimensions like stochastic evolution equations, including such equations with a delay, McKean-Vlasov stochastic evolution equations, neutral stochastic partial differential equations and stochastic evolution equations with Poisson jumps, including the Lévy martingales. The book also deals with abstract stochastic equations such as stochastic integrodifferential equations, uncertain stochastic evolution equations and stochastic evolution equations in unconditional martingale difference (UMD) Banach spaces.

In Chapter 2, to make the book as self-contained as possible and reader-friendly, all the necessary mathematical background that will be needed later on will be provided. As the book studies SDEs using mainly the semigroup theory, it is first intended to provide this theory including the fundamental Hille-Yosida theorem. Then, the Trotter-Kato theorem, which is the main topic of this monograph, is presented in detail. Next, some basics from probability and analysis in Banach spaces are considered like those of the concepts of probability and random variables, Wiener process, Poisson process and Lévy process, linear monotone operators, accretive operators and UMD Banach spaces, among others, and also state many fundamental theorems for an interested reader. With this preparation, we are ready to deal with stochastic calculus in infinite dimensions, namely the concepts of Itô stochastic integral with respect to Q-Wiener and cylindrical Wiener processes, stochastic integral with respect to a compensated Poisson random measure, stochastic integral with respect to square integrable Lévy martingales, stochastic integral in UMD Banach spaces and Itô's formula in various settings. In many parts of this book, the theory of stochastic convolution integrals is vital. This therefore motivates the consideration of all the necessary results from this theory without proofs. Moreover, we state some results on the convergence of stochastic convolutions, the stochastic Fubini theorem and the Burkholder type inequality in many forms. This chapter together with the appendices dealing with Pettis measurability theorem, convergence of analytic semigroups and operators on Hilbert spaces, more precisely, notions of trace class operators, nuclear and Hilbert-Schmidt operators, R-boundedness and γ-boundedness, etc. should provide a sound mathematical background. Since there are many excellent references on this background material such as Ahmed [1], Arendt et al [1], Barbu [1], Bharucha-Reid [1], Bichteler [1], Brzeźniak et al [1], Da Prato and Zabczyk [1, 2], Dunford and Schwartz [1], Gawarecki and Mandrekar [1], Hille and Philips [1], Hytönen et al [1], Ichikawa [3], Joshi and Bose [1], Kallenberg [1], Kato [2, 4], Knoche [1], Kunita [1], Liu [1], Marinelli, Di Persio and Ziglio [1], Métivier [1], Métivier and Pellaumail [1], van Neerven, Veraar and Weis [1, 2], Pazy [1], Peszat and Zabczyk [1], Prévôt and

Röckner [1], Protter [1], Stephan [1] and Yosida [2], to mention only a few, the objective here is to keep this chapter quite brief. Finally, the chapter includes some further topics on the Trotter-Kato theory like a Trotter-Kato type theorem and the Trotter-Kato theorem for nonlinear evolutions. We shall also touch upon the celebrated Trotter-Kato formula, see Trotter, 1959, and Kato, 1978, and its applications to resolve an optimal investment problem in incomplete markets in finance using the celebrated Feynman-Kac formula. It is interesting to point out here that we shall provide a proof of the latter formula using the Trotter-Kato formula in Appendix E.

Chapter 3 addresses the main results on Trotter-Kato approximations of many classes of SDEs in Hilbert spaces. The chapter begins by motivating this study from the pioneering work of Ichikawa, 1982, on semilinear stochastic evolution equations. Then, the Trotter-Kato approximations of semilinear stochastic evolution equations are introduced and their existence and uniqueness of mild solutions are established. It is also shown that the mild solutions of such approximating equations converge to the mild solution of the original equation in the mean-square sense. These results are then generalized to semilinear stochastic evolution equations with a delay. Next, a special form of a stochastic evolution equation is considered that is related to the so-called McKean-Vlasov measure-valued evolution equation. Trotter-Kato approximations are then introduced for McKean-Vlasov stochastic evolution equations. It is shown, as before, that the Trotter-Kato approximating equations have unique mild solutions and that these solutions converge to the mild solution of the original equation in mean-square. These results are subsequently generalized to McKean-Vlasov stochastic evolution equations with a multiplicative diffusion. Moreover, the chapter considers Trotter-Kato approximations of neutral stochastic partial differential equations, linear and semilinear stochastic integrodifferential equations, uncertain semilinear stochastic evolution equations from the control theory and stochastic evolution equations driven by Lévy martingales and Poisson random measures.

In Chapter 4, Trotter-Kato approximations of mild solutions of semilinear stochastic evolution equations are considered in the UMD Banach spaces using Lipschitz and local Lipschitz nonlinearities. In both these cases, to put it in simple terms, the Trotter-Kato approximating equations are introduced and are shown to have unique mild solutions. It is then proved that the solutions of these approximating equations converge to the mild solution of the original stochastic equation in mean.

In Chapter 5, we consider some applications of Trotter-Kato approximations to stochastic stability problems, in some sense. In other words, some interesting applications and consequences of the Trotter-Kato approximation, and its version, so-called a zeroth order approximation, of mild solutions of SDEs are studied. Using the latter, we shall begin by providing an estimate of the error in approximating a semilinear stochastic evolution equation by a deterministic evolution equation. As an application, we shall investigate a classical limit theorem on the dependence of the semilinear stochastic evolution equation on a parameter. It is interesting to mention here that the probability measures induced by the mild solutions of the Trotter-Kato approximating equations converge weakly to the probability measure induced by the mild solution of the original stochastic equation. This program is carried out later on for other classes of SDEs, namely stochastic evolution equations with a delay, McKean-Vlasov stochastic evolution equations, neutral stochastic partial differential equations and stochastic integrodifferential equations.

In the last Chapter 6, we study some applications of Trotter-Kato approximations to stochastic optimal control problems. More precisely, we begin with some interesting applications of Trotter-Kato approximations to inverse and optimal output feedback control problems for semilinear infinite dimensional systems with uncertain semigroup generators. In other words, we present several typical and nontypical control problems and their solutions. The nontypical problems are related to the control of the evolution of measures. We prove the existence of optimal feedback control laws for these systems in the presence of uncertainty of the principal operator. We consider both deterministic and stochastic systems. Some interesting applications to mass transfer problem, evasion problem and Hausdorff dimension problem are also considered. Lastly, we present the necessary conditions of optimality for the uncertain stochastic feedback control problem. The chapter also includes a brief discussion on the system identification and optimization. The book concludes with an interesting application of the Trotter-Kato formula to an optimal investment problem in incomplete markets.

I have tried to keep the work of various authors drawn from all over the mathematics literature as original as possible. I thank very much all of them whose works have been included in the book with due citations they deserve in the bibliographical notes and remarks and elsewhere. To the best of my knowledge, I believe that I have covered in this monograph all the work that I have known. I apologize to those authors in case I have failed to include their work. This is not deliberate.

Mexico City, Mexico T. E. Govindan
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Contents

Notations and Abbreviations

Abbreviations

a.e.	Almost everywhere
P-a.s.	Probability almost surely or with probability 1
i.i.d.	Independently and identically distributed
w.l.g.	Without loss of generality
l.s.c.	Lower semicontinuous
u.s.c.	Upper semicontinuous
HJB	Hamilton-Jacobi-Bellman
SDE	Stochastic differential equation
SEE	Stochastic evolution equation
SPDE	Stochastic partial differential equation
UMD	Unconditional martingale difference

Notations

$\square$	Signals end of proof
$:=$	Equality by definition
$I_B(x)$	Indicator function of a set B
$\mathbb{N}$	Set of natural numbers
R^n	n-dimensional Euclidean space with the usual norm, $n \in \mathbb{N}$
R	Real line, i.e., $R = (-\infty, \infty)$
R^+	Nonnegative real line, i.e., $R^+ = [0, \infty)$
$\bar{R}$	Extended real line, i.e., $\bar{R} = R \cup \{-\infty, \infty\}$
C	Complex plane
$\mathbb{K}$	Scalar field (R or C)
$\mathrm{Re}\,\lambda$	Real part of λ
$\mathrm{Im}\,\lambda$	Imaginary part of λ
$(X, \|\cdot\|_X)$	Banach space with its norm
$(X^*, \|\cdot\|_{X^*})$	Dual of a Banach space with its norm
${}_{X^*}\langle x^*, x\rangle_X$	Duality pairing between X^* and X
$\mathcal{B}(X)$	Borel σ-algebra of subsets of X

$M(X)$	Space of probability measures on $\mathcal{B}(X)$ carrying the usual topology of weak convergence
$BC(Z)$	Space of bounded continuous functions on Z with the topology of sup norm where Z is a normal topological space
$D(A)$	Domain of an operator A
$\rho(A)$	Resolvent set of an operator A
$R(\lambda, A)$	Resolvent of an operator A
$\mathrm{tr}\, Q$	Trace of an operator Q
$L(Y, X)$	Space of all bounded linear operators from Y into X
$L(X)$	$L(X, X)$
$L_1(Y, X)$	Space of all nuclear operators from Y into X
$L_2(Y, X)$	Space of all Hilbert-Schmidt operators from Y into X
$\Vert \cdot \Vert_{L_2}$	Hilbert-Schmidt norm
$(\Omega, \mathcal{F}, P)$	Probability space
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$	Filtered probability space
$L^p(\Omega, \mathcal{F}, P; X)$	Banach space of all functions from Ω to X which are p-integrable with respect to (w.r.t) P, $1 \le p < \infty$
$L^p(\Omega, \mathcal{F}, P)$	$L^p(\Omega, \mathcal{F}, P; R)$, $1 \le p < \infty$
$L^p([0, T], X)$	Banach space of all X-valued Borel measurable functions on $[0, T]$ which are p-integrable, $1 \le p < \infty$
$L^p[0, T]$	$L^p([0, T], R)$, $1 \le p < \infty$
$C([0, T], X)$	Banach space of X-valued continuous functions on $[0, T]$ with the usual sup norm
$B_b([0, T], X)$	Banach space of bounded and strongly Borel measurable functions on $[0, T]$ with values in X endowed with the supremum norm
$\{S(t) : t \ge 0\}$	C_0-semigroup
$\{\beta(t), t \ge 0\}$	Real-valued Brownian motion or Wiener process
$\{w(t), t \ge 0\}$	Q-Wiener process or cylindrical Wiener process
$\mathcal{L}(x)$	Probability law of x
$E(x)$	Expectation of x
$E(x\|\mathcal{A})$	Conditional expectation of x given $\mathcal{A}$
$Q^{1/2}$	Square root of $Q \in L(X)$
T^*	Adjoint operator of $T \in L(Y, X)$
T^{-1}	(Pseudo) Inverse of $T \in L(Y, X)$
$N(m, Q)$	Gaussian law with mean m and covariance operator Q
$T(\omega)$	Random operator
$N_w^2\left(0, T; L_2^0\right)$	Or simply $N_w^2(0, T)$ is a Hilbert space of all L_2^0-predictable processes Φ such that $\Vert\Phi\Vert_T < \infty$
$\to$	Strong convergence in $L(X)$ or in X
$\rightharpoonup$	Weak convergence
$M_T^2(X)$	Space of all X-valued continuous, square integrable martingales

$\mathscr{M}^2(Y)$	Space of all c*à*dl*à*g square integrable martingales in a Hilbert space Y w.r.t. $\{\mathfrak{F}_t\}$
$\langle\langle x(\cdot)\rangle\rangle$	The process of quadratic variation of x
$\int_0^t \Phi(s)dw(s)$	Itô stochastic integral w.r.t. $w(t)$
$\tilde{N}(t,A)$	Compensated Poisson random measure
$\int_0^t \int_Z \Phi(s,z)\tilde{N}(ds,dz)$	Stochastic integral w.r.t. a compensated Poisson measure $\tilde{N}(dt,du)$
$\int_0^t \Phi(s)dM(s)$	Stochastic integral w.r.t. a martingale $M \in \mathscr{M}^2(Y)$
$Q_1 \lesssim_A Q_2$	To express that there exists a constant c, depending only on A, such that $Q_1 \leq cQ_2$
$Q_1 \eqsim_A Q_2$	To express that $Q_1 \lesssim_A Q_2$ and $Q_2 \lesssim_A Q_1$
$a \lesssim b$	If there exists a constant $c > 0$ such that $a \leq cb$
$a \lesssim_{p_1,p_2,...,p_n} b$	If the constant c in $a \leq cb$ depends on the parameters $p_1, p_2, ..., p_n$

Chapter 1

Introduction and Motivating Examples

Stochastic differential equations, in the sense of Itô, arise quite naturally as mathematical models in many areas of science, engineering and finance. Problems such as existence and uniqueness of mild solutions; stability and optimal control; continuous dependence on initial values; Yosida and Trotter-Kato approximations, among others, of mild solutions of stochastic differential equations in infinite-dimensional spaces have been investigated by many authors, see Ahmed [1, 2], Bharucha-Reid [1], Curtain and Pritchard [1], Da Prato [1], Da Prato and Zabczyk [1, 3, 4], Gawarecki and Mandrekar [1], Govindan [13], Itô [2], Kallianpur and Xiong [1], Kotelenez [1], Liu [1], Liu and Röckner [1], Lototsky and Rozovsky [1], Mandrekar and Rüdiger [1], McKibben [2], Métivier [2], Peszat and Zabczyk [1] and Prévôt and Röckner [1], to mention only a few, and the references cited therein.

In this introductory chapter, we motivate the study of some of the abstract stochastic differential equations in infinite dimensions considered in this book by modeling real-life problems quite briefly such as a heat equation, an electric circuit, an interacting particle system, a lumped control system, the stock and option price dynamics and an optimal investment problem in incomplete markets. Rigorous formulations of some concrete problems and theoretical examples are taken up later on in the forthcoming chapters.

1.1 A Heat Equation

Let us begin with the following heat equation with a stochastic perturbation of the form

$$\begin{aligned} dx(z,t) &= \frac{\partial^2}{\partial z^2}x(z,t)dt + \sigma x(z,t)d\beta(t), \quad t>0, \qquad (1.1)\\ x(0,t) &= x(1,t) = 0, \quad x(z,0) = x_0(z), \end{aligned}$$

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where σ is a real number and $\beta(t)$ is a real standard Wiener process or a Brownian motion.

We shall also consider the semilinear stochastic heat equation

$$\begin{aligned} dx(z,t) &= \left[\frac{\partial^2}{\partial z^2}x(z,t) - \frac{x(z,t)}{1+|x(z,t)|}\right]dt \\ &\quad + \frac{\sigma x(z,t)}{1+|x(z,t)|}d\beta(t), \quad t>0, \qquad (1.2) \\ x_z(0,t) &= x_z(1,t)=0, \quad x(z,0)=x_0(z), \end{aligned}$$

where $|\cdot|$ is the absolute value on $R=(-\infty,\infty)$. We refer to Ichikawa [1, 3] for more details.

1.1.1 Stochastic Evolution Equations

The equation (1.2) can be formulated in the abstract setting as follows:

Let X and Y be real Hilbert spaces. Take $X = L^2(0,1)$ and $Y = R$. Define $A = d^2/dz^2$ with $D(A) = \{x \in X | x, x'$ absolutely continuous, $x', x'' \in X, x'(0) = x'(1) = 0\}$, $f : X \to X$ and $g : X \to L(Y,X)$ (space of all bounded linear operators from Y into X), where

$$f(x) = -\frac{g(x)}{\sigma} = -\frac{x}{1+||x||_X}, \quad x \in X. \qquad (1.3)$$

With this notation, equation (1.2) can be expressed as a semilinear stochastic evolution equation in X as

$$\begin{aligned} dx(t) &= [Ax(t)+f(x(t))]dt + g(x(t))dw(t), \quad t>0, \qquad (1.4) \\ x(0) &= x_0, \end{aligned}$$

where $w(t)$ is a Y-valued Q-Wiener process.

The semilinear stochastic equations of the form (1.4) will be discussed in detail in Section 3.1 and later on in Sections 3.6 and 6.1. More general semilinear stochastic equations with time-varying coefficients will be studied in Sections 3.1.1, 5.1.1, 5.2.1 and 5.3.1; and in Section 5.1.2 with a finite-dimensional noise.

1.2 An Electric Circuit

Let us consider an electric circuit connected in series in which there are two resistances, a capacitance and an inductance. Suppose that the current, $x(t)$ amperes, at time t flows through the circuit. We shall use volts for the voltage, ohms for the resistance R, henry for the inductance L, farads for the capacitance c, coloumbs for the charge on the capacitance, and seconds for the time as the units of measurement. Under this system of units, it is known that the voltage drop across the inductance is $\mathsf{L}dx(t)/dt$, that across the resistances R and R_1 is $(\mathsf{R}+\mathsf{R}_1)x(t)$ and that across the capacitance is q/c, where q is the charge on the capacitance. It is also known that $x(t) = dq/dt$. According to Kirchhoff's law, the sum of the voltage drops around the loop must be equal to the applied voltage

$$\mathsf{L}\frac{dx(t)}{dt} + (\mathsf{R}+\mathsf{R}_1)x(t) + \frac{q}{c} = 0. \tag{1.5}$$

Differentiating equation (1.5) with respect to t, we have

$$\mathsf{L}\frac{d^2x(t)}{dt^2} + (\mathsf{R}+\mathsf{R}_1)\frac{dx(t)}{dt} + \frac{1}{c}x(t) = 0. \tag{1.6}$$

The voltage across R_1 is applied to a nonlinear amplifier A_1. A special phase-shifting network P is provided to the output. This introduces a constant time lag between the input and the output P. Thus, the voltage drop across R in series with the output P is given by

$$e(t) = qg(\dot{x}(t-r)),$$

where q is the gain of the amplifier to R measured through the network. In view of this, equation (1.6) becomes

$$\mathsf{L}\frac{d^2x(t)}{dt^2} + \mathsf{R}\dot{x}(t) + qg(\dot{x}(t-r)) + \frac{1}{c}x(t) = 0.$$

Lastly, a second device is introduced in the circuit to help stabilize the fluctuations in the current. If $\dot{x}(t) = y(t)$, the controlled system is then given by

$$\begin{aligned}\dot{x}(t) &= y(t) + u_1(t)\\ \dot{y}(t) &= -\frac{\mathsf{R}}{\mathsf{L}}y(t) - \frac{q}{\mathsf{L}}g(y(t-r)) - \frac{1}{c\mathsf{L}}x(t) + u_2(t).\end{aligned} \tag{1.7}$$

The system (1.7) can be expressed in the matrix form as

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t) + G(\mathbf{X}(t-r)) + B\mathbf{U}, \tag{1.8}$$

where

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1/c\mathsf{L} & -\mathsf{R}/\mathsf{L} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$G(\mathbf{X}(t-r)) = \begin{pmatrix} 0 \\ -qg(y(t-r))/\mathsf{L} \end{pmatrix}.$$

Note that the controlled vector $\mathbf{U}$ is created and introduced by the stabilizer.

1.2.1 Stochastic Evolution Equations with a Delay

Let X and Y be real Hilbert spaces. Motivated by this electric circuit and also by the stochastic partial differential equations with a delay in general, see Taniguchi [1], let us consider the following stochastic evolution equation with a delay in X of the form

$$\begin{aligned} dx(t) &= [Ax(t)+f(x_t)]dt+g(x_t)dw(t), \quad t>0, \qquad (1.9)\\ x(t) &= \varphi(t), \quad t\in[-r,0], \quad 0\le r<\infty, \end{aligned}$$

where $x_t(s)=x(t+s), -r\le s\le 0$ is the finite history of x at t, $A:D(A)\to X$ (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t):t\ge 0\}$, $f:C\to X$, where $C=C([-r,0],X)$ (space of continuous functions from $[-r,0]$ to X), $g:C\to L(Y,X)$ and $w(t)$ is a Y-valued Q-Wiener process. It is assumed that the past process $\{\varphi(t),-r\le t\le 0\}$ is known.

Such stochastic evolution equations with a constant delay will be considered in Section 3.2. See also Sections 5.1.3 and 5.2.2.

1.3 An Interacting Particle System

Let us consider a chemical interacting particle system in which each particle moves in a space governed by the dynamics of the following system of N coupled semilinear stochastic evolutions equations

$$\begin{aligned} dx_k(t) &= [Ax_k(t)+f(x_k(t), \ _N(t))]dt+\sqrt{\mathbb{Q}}dw_k(t), \quad t>0, \qquad (1.10)\\ x_k(0) &= x_0, \quad k=1,2,\dots,N, \end{aligned}$$

where $\ _N(t)$ is the empirical measure given by

$$\ _N(t)=\frac{1}{N}\sum_{k=1}^{N}\delta x_k(t)$$

of the N particles $x_1(t)$, $x_2(t)$,...,$x_N(t)$ at time t. According to McKean-Vlasov theory, see McKean [1], Dawson and Gärtner [1] and Gärtner [1], among others, under suitable conditions, the empirical measure-valued process $\ _N$ converges in probability to a deterministic measure-valued function   as N goes to infinity. It may be noted that   is the probability distribution of a solution process of the abstract stochastic equation (1.11) that follows. See Kurtz and Xiong [1].

1.3.1 McKean-Vlasov Stochastic Evolution Equations

Motivated by this interacting particle system, consider the McKean-Vlasov stochastic evolution equation in a real Hilbert space X of the form

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t),\ \ (t))]dt + \sqrt{\mathbb{Q}}dw(t), \quad t > 0, \\ (t) &= \text{probability distribution of x(t)}, \\ x(0) &= x_0, \end{aligned} \tag{1.11}$$

where $A : D(A) \subset X \to X$ (possibly unbounded) is the infinitesimal generator of a strongly continuous semigroup $\{S(t) : t \geq 0\}$ of bounded linear operators on X, f is an appropriate X-valued function defined on $X \times M_{\gamma^2}(X)$, where $M_{\gamma^2}(X)$ denotes a proper subset of probability measures on X, $\mathbb{Q}$ is a positive, symmetric, bounded operator on X, $w(t)$ is an X-valued cylindrical Wiener process and x_0 is an X-valued random variable that is assumed known. We refer to Sections 3.3.1, 5.1.4 and 5.2.3 for further details.

More general McKean-Vlasov type stochastic systems with a multiplicative diffusion shall be considered in Section 3.3.2 and later on in Sections 5.1.5 and 5.2.4.

There are many situations that are modeled by equation (1.11). This phenomenon is common in biological and physical sciences. We mention examples like dynamics of charge density waves, chemical reactions, mean-field dynamics of soft spins, power flow in mobile communication networks, population biology, etc. Another interesting example is Kushner's equation arising in the study of nonlinear filtering. Given the history of the observation, the conditional probability law is governed by an equation of the McKean-Vlasov type.

1.4 A Lumped Control System

Let R^m be an m-dimensional Euclidean space. One way of stabilizing lumped control systems is to use a proportional-integral-differential (PID) feedback control. Let us consider now a linear distributed system of the form

$$\frac{dx(t)}{dt} = Ax(t) + f(x(t)) + Bu(t), \quad t > 0, \tag{1.12}$$

where $x(t) \in X$ denotes the state, $u(t) \in R^m$ is the control, $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \geq 0\}$ on X, and $B : R^m \to X$.

A PID control considered below is the feedback control $u(t)$ and it is given by

$$u(t) = K_0 x(t) - \frac{d}{dt} \int_0^t K_1(t-s)x(s)ds, \tag{1.13}$$

where $K_0 : X \to R^m$ is a bounded linear operator and $K_1 : [0, \infty) \to L(X, R^m)$ is a strongly continuous operator-valued map. The closed system corresponding to the PID control (1.13) is of the form

$$\frac{d}{dt}\left[x(t) + B\int_0^t K_1(t-s)x(s)ds\right] = (A + BK_0)x(t) + f(x(t)), \quad t > 0,$$

where it is known that $A+BK_0$ is the infinitesimal generator of an analytic semigroup.

1.4.1 Neutral Stochastic Partial Differential Equations

Motivated by this lumped control system, consider a neutral stochastic partial differential equation in a real Hilbert space X of the form

$$\begin{aligned} d[x(t)+f(t,x(t))] &= [Ax(t)+a(t,x(t))]dt \\ &\quad +b(t,x(t))dw(t), \quad t>0, \\ x(0) &= x_0, \end{aligned} \tag{1.14}$$

where $-A : D(-A) \subset X \to X$ (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$ on X, $a : R^+ \times X \to X$, where $R^+ = [0,\infty)$, $b : R^+ \times X \to L(Y,X)$ and $f : R^+ \times X \to D((-A)^{\alpha})$, $0 < \alpha \leq 1$, $w(t)$ is a Y-valued Q-Wiener process, and assume that the initial condition x_0 is known. We refer to Section 3.4 below for more details.

This class of equations will be studied in Sections 5.2.5 and 5.3.2.

1.5 A Hyperbolic Equation

Let us consider the following hyperbolic type deterministic integral equation

$$\begin{aligned} u_{tt}(t,z) &= \Delta u(t,z) + \int_0^t b(t-s)\Delta u(s,z)ds + f(t,z), \quad t>0, \\ u(t,0) &= u(t,\pi) = 0, \end{aligned} \tag{1.15}$$

where $\Delta = \partial^2/\partial z^2$, or the equivalent system

$$u_t = v, \quad v_t = \Delta u + \int_0^t b(t-s)\Delta u(s,\cdot)ds + f(t,\cdot).$$

Equation (1.15) can be expressed as

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + F(t), \quad t>0, \tag{1.16}$$

where

$$x = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

and

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ b(t)\Delta & 0 \end{pmatrix}.$$

1.5.1 Stochastic Integrodifferential Equations

Let us consider a stochastic version of the Volterra integrodifferential equation (1.16) of the form

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t-s)x(s)d\beta(s) + f(t), \quad t > 0, \qquad (1.17)\\ x(0) &= x_0, \end{aligned}$$

where A is a linear operator (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$ on a real Hilbert space X with domain $D(A)$, f belongs to a function space $\mathscr{A}$ on X-valued functions, $B(t)$ is a (not necessarily bounded) convolution kernel type linear operator on the domain $D(A)$ (for each $t \geq 0$) such that $B(\cdot)x \in \mathscr{A}$ for each $x \in D(A)$, x_0 is an X-valued known random variable and $\beta(\cdot)$ is a Hilbert-Schmidt operator-valued Brownian motion. For details, see Section 3.5.1 below and Kannan and Bharucha-Reid [1]. See also Sections 5.1.6 and 5.3.3 for a study on such equations.

We shall also be considering a semilinear stochastic integrodifferential equation of the form

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t,s)f(s,x(s))ds \\ &\quad + \int_0^t C(t,s)g(s,x(s))dw(s) + F(t,x(t)), \quad t > 0, \qquad (1.18)\\ x(0) &= x_0, \end{aligned}$$

where A is a linear operator (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$ on a real Hilbert space X; $B(t,s)_{0\leq s\leq t\leq T}$ and $C(t,s)_{0\leq s\leq t\leq T}$ $(0 < T < \infty)$ are linear operators mapping X into X, $F : [0,\infty) \times X \to X$, $f : [0,\infty) \times X \to X$ and $g : [0,\infty) \times X \to L(Y,X)$, $w(t)$ is a Y-valued Q-Wiener process and x_0 is a known random variable. For details, see Section 3.5.2 below.

See also Sections 5.1.7 and 5.3.4 for a study on such class of equations.

Integrodifferential equations arise quite naturally, for instance, in mechanics, electromagnetic theory, heat flow, nuclear reactor dynamics, and population dynamics, we refer to Kannan and Bharucha-Reid [1] and the references therein for details. A dynamic system with memory may lead to integrodifferential equations.

1.6 The Stock Price and Option Price Dynamics

This classical problem was introduced by Merton [3]. The total change in the stock price is composed of two types of changes. First, the normal vibrations in price, for instance, occur due to temporary imbalance between supply and demand, changes in capitalization rates, or other new information that causes marginal changes in the stock's value. It is known that this component is modeled by a standard geometric Brownian motion with continuous sample paths. Second, the abnormal vibrations in price occur due to the arrival of important new information about the stock which has a marginal effect on price. It is quite reasonable to expect that there will be active times in the stock when such information arrives and quiet times when it does not although these times are random. Naturally, important information arrives only at discrete points in time. This component is modeled by a jump process.

According to the general efficient market hypothesis of Fama [1] and Samuelson [1], the dynamics of the unanticipated part of the stock price motions should be a martingale. Since the dynamics are assumed to be a continuous-time process, the natural choice for the continuous component of the stock price change is a Wiener process, while for the jump component is a Poisson-driven process.

Given that some important information on the stock arrives, i.e., a Poisson event occurs, if $S(t)$ is the stock price at time t and $\mathbf{Y}$ is the random variable to determine the impact of this information, neglecting the continuous part, the stock price at time $t+h$, $S(t+h)$, will be the random variable $S(t+h) = S(t)\mathbf{Y}$, given that one such arrival occurs between t and $t+h$. We assume that $\mathbf{Y}$ has a probability measure with compact support, $\mathbf{Y} \geq 0$ and $\{\mathbf{Y}\}$ are *i.i.d.*.

As studied in Merton [2], the stock price returns are a mixture of both types and can be written as a stochastic differential equation of the form

$$\frac{dS(t)}{S(t)} = (\alpha - \gamma k)dt + \sigma d\beta(t) + dN(t), \quad t > 0, \tag{1.19}$$

where α is the instantaneous expected return on the stock, σ^2 is the instantaneous variance of the return, conditional that the Poisson event does not occur, $\beta(t)$ is a standard Wiener process and $N(t)$ is the Poisson process. We assume that $N(t)$ and $\beta(t)$ are independent. Here, γ is the mean number of arrivals per unit time and $k = E(\mathbf{Y} - 1)$, where $\mathbf{Y} - 1$ is the random variable percentage change in the stock price if the Poisson event occurs.

It is interesting to observe that the $\sigma d\beta(t)$ part describes the instantaneous part of the unanticipated return due to the normal price vibrations while the $dN(t)$ part describes the abnormal price vibrations. If $\gamma = 0$, the return dynamics would be identical to those considered in Black and Scholes [1] and Merton [4]. Equation (1.19) may be rewritten as

$$\frac{dS(t)}{S(t)} = (\alpha - \gamma k)dt + \sigma d\beta(t),$$

if the Poisson event does not occur, and

$$\frac{dS(t)}{S(t)} = (\alpha - \gamma k)dt + \sigma d\beta(t) + (\mathbf{Y} - 1),$$

if the Poisson event occurs.

After having established the stock price dynamics, let us now consider the dynamics of the option price. Suppose that the option price, W, can be written as a twice continuously differentiable function of the stock price and time; namely

$W(t) = F(S,t)$. Given that the stock price is modeled by equation (1.19), then the option return dynamics can be written similarly as

$$\frac{dW(t)}{W(t)} = (\alpha_W - \gamma k_W)dt + \sigma_W d\beta(t) + dN_W(t), \tag{1.20}$$

where α_W is the instantaneous expected return on the option, σ_W^2 is the instantaneous variance of the return, conditional that the Poisson event does not occur and $N_W(t)$ is a Poisson process with parameter γ. We assume that $N_W(t)$ and $\beta(t)$ are independent and $k_W \equiv E(\mathbf{Y}_W - 1)$, where $\mathbf{Y}_W - 1$ is the random variable percentage change in the option price if the Poisson event occurs.

1.6.1 Stochastic Evolution Equations with Poisson jumps

Motivated by this stock price and option price dynamics, consider the class of stochastic differential equations with Poisson jumps in a Hilbert space X of the form

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t))]dt + g(x(t))dw(t) \\ &\quad + \int_Z L(x(t),u)\tilde{N}(dt,du), \quad t > 0, \\ x(0) &= x_0, \end{aligned} \tag{1.21}$$

where $\tilde{N}$ is a compensated Poisson random measure associated with a counting Poisson random measure N; A, generally unbounded, is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$ on X; $f : X \to X$, $g : X \to L(Y,X)$ and $L : X \times Y \to X$ are some measurable functions; $w(t)$ is a Y-valued Q-Wiener process; and x_0 is a known random variable. We assume that $\tilde{N}(dt,du)$ is independent of $w(t)$.

Stochastic equations of the type (1.21) shall be considered in Section 3.7.2.

1.7 An Optimal Investment Problem in Incomplete Markets

In this section, we present a simple extension of the stochastic optimization problem of Merton [1, 2] when the market is incomplete.

A financial market consists of two assets, a riskless and a risky one. The riskless asset is a bond earning a constant interest rate, l, while the risky one is a stock whose price, $\{S(t), t \geq 0\}$, satisfies the stochastic differential equation

$$\begin{aligned} dS(t) &= (Y(t))S(t)dt + \sigma(Y(t))S(t)dw_1(t), \quad t > 0, \\ S(0) &= S_0 > 0. \end{aligned} \tag{1.22}$$

The stochastic factor, $Y(t)$, satisfies

$$\begin{aligned} dY(t) &= b(Y(t))dt + a(Y(t))\Big(\rho dw_1(t) \\ &\quad + \sqrt{1-\rho^2}dw_2(t)\Big), \quad t>0, \qquad (1.23) \\ Y(0) &= Y_0 \in R. \end{aligned}$$

Here, the process $w(t) = (w_1(t), w_2(t))$ is a standard Brownian motion with instantaneous correlation coefficient ρ.

Let us introduce the market price of the risk process $\{\lambda(t), t \geq 0\}$ as

$$\lambda(t) = \lambda(Y(t)) = \frac{(Y(t)) - l}{\sigma(Y(t))}. \qquad (1.24)$$

We now make the following assumptions on the model coefficients:

(i) The functions $: R \to R$ and $\sigma : R \to (0,\infty)$ are continuous.
(ii) The function $b : R \to R$ is continuously differentiable; and the functions $a : R \to (0,\infty)$ and $\lambda : R \to R$ are twice continuously differentiable.
(iii) The functions $a, 1/a, b, \lambda, a', b', \lambda', a''$ and λ'' are absolutely bounded.

Under these assumptions, the system of SDEs (1.22) and (1.23) has a unique strong solution, see Karatzas and Shreve [1].

An investor trades between the bond and the stock accounts in a finite (fixed) horizon $[0,T]$ by generating a random payoff at time T called the terminal time. The risk preferences of the investor at time T are modeled by a utility function, namely U_T.

It is assumed that $U_T : (0,\infty) \to R$ is strictly increasing, concave and twice continuously differentiable. Further, U_T satisfies

$$0 < \inf_{x>0}\left(-\frac{xU_T''(x)}{U_T'(x)}\right) \leq \sup_{x>0}\left(-\frac{xU_T''(x)}{U_T'(x)}\right) < \infty, \qquad (1.25)$$

$$0 < \inf_{x>0}\left(x^\gamma U_T'(x)\right) \leq \sup_{x>0}\left(x^\gamma U_T'(x)\right) < \infty, \qquad (1.26)$$

for some $\gamma > 0$ and $e^{(1+\gamma)z}U_T''(e^z)$ is a uniformly continuous function of $z \in R$.

Given an initial endowment $x > 0$ at time $t \in [0,T)$, the discounted allocations of the investor in the bond and the stock accounts at time $s \in [t,T]$ are denoted by π_s^0 and π_s, respectively. Then, the total discounted investment at time s, denoted by $X_s^{\pi,x,t}$, satisfies $X_s^{\pi,x,t} = \pi_s^0 + \pi_s$. We shall refer to $X_s^{\pi,x,t}$ as the discounted wealth. Given $\pi = \{\pi_s, s \in [0,T]\}$, the process $\pi^0 = \{\pi_s^0, s \in [0,T]\}$ is determined uniquely by the self-financing condition. Therefore, we shall identify a trading strategy, or a policy, with the process π. It can be shown that the process $\{X_s^{\pi,x,t}, s \in [t,T]\}$ satisfies

$$\begin{aligned} dX_s^{\pi,x,t} &= \sigma(Y(s))\pi_s\big(\lambda(Y(s))ds + dw^1(s)\big), \quad s \in [t,T], \qquad (1.27) \\ X_t^{\pi,x,t} &= x, \end{aligned}$$

for any policy π that belongs to the set of admissible policies defined as follows.

Let us define $\mathcal{A}$ to be the set of admissible policies that consists of all locally square-integrable $\mathcal{F}$-progressively measurable stochastic processes $\pi = \{\pi_s, s \in [0,T]\}$ such that for any initial condition $(x,t) \in (0,\infty) \times [0,T]$, the corresponding discounted wealth process $\{X_s^{\pi,x,t}, s \in [t,T]\}$ given by (1.27) is strictly positive. In addition, if $\gamma \geq 1$, it is required that

$$E\int_t^T (X_s^{\pi,x,t})^{-p}(1+\pi_s^2)ds < \infty, \quad \text{for all}\, p \geq 0. \qquad (1.28)$$

The objective of the investor is to maximize the expected utility of terminal wealth given today's information and overall the admissible strategies. We define the value funtion process $J(x,t)$ for each $(x,t) \in (0,\infty) \times [0,T]$ as

$$J(x,t) = \text{esssup}_{\pi \in \mathcal{A}} E\left(U_T\left(X_T^{\pi,x,t}\right)\big|\mathcal{F}_t\right). \qquad (1.29)$$

1.7.1 Hamilton-Jacobi-Bellman Equations

In the Markovian setting considered here, the value function process is typically associated with the Hamilton-Jacobi-Bellman (HJB) equation. Precisely, $J(x,t)$ is expected to have a functional representation of the form

$$J(x,t) = U(x,Y(t),t), \qquad (1.30)$$

where $U : \mathbb{D} \to R$ is a deterministic function that is defined on the domain

$$\mathbb{D} = (0,\infty) \times R \times [0,T]. \qquad (1.31)$$

If such a function U exists, it is called a value function of the optimal investment problem and it is expected to satisfy the HJB equation given by

$$\begin{aligned} U_t \quad &+ \quad \max_\pi\left(\frac{1}{2}\sigma^2(y)\pi^2 U_{xx} + \pi(\sigma(y)\lambda(y)U_x + \rho\sigma(y)a(y)U_{xy})\right) \\ &+\frac{1}{2}a^2(y)U_{yy} + b(y)U_y = 0, \qquad (1.32) \end{aligned}$$

with the terminal condition $U(x,y,T) = U_T(x)$. We refer to Zariphopoulou [1] and the references therein for details.

Note that the correlation coefficient ρ controls the incompleteness of the market, that is, when $|\rho| = 1$, the market is complete.

We shall consider the equation (1.32) in Section 6.3.

Chapter 2

Mathematical Machinery

This chapter introduces the necessary mathematical background from the semigroup theory, particularly the Trotter-Kato approximations, analysis and probability in Banach spaces, Itô stochastic calculus, stochastic convolution integrals and further topics on the Trotter-Kato theory, among others. We shall also include statements of some fundamental results that may be of independent interest.

2.1 Semigroup Theory

Let $(X, ||\cdot||_X)$ be a Banach space.

Definition 2.1 A one parameter family $\{S(t) : 0 \leq t < \infty\}$ of bounded linear operators mapping X into X is said to be a semigroup of bounded linear operators on X if

(i) $S(0) = I$ (I is the identity operator on X) and
(ii) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators, $\{S(t) : t \geq 0\}$, is defined to be uniformly continuous if

$$\lim_{t \downarrow 0} ||S(t) - I|| = 0.$$

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{ exists}\}, \tag{2.1}$$

where $D(A)$ is the domain of A, and

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt}|_{t=0} \quad \text{for } x \in D(A), \tag{2.2}$$

is said to be the infinitesimal generator of the semigroup $\{S(t) : t \geq 0\}$.

T. E. Govindan, *Trotter-Kato Approximations of Stochastic Differential Equations in Infinite Dimensions and Applications*, https://doi.org/10.1007/978-3-031-42791-6_2

Theorem 2.1 A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.

Proof See Pazy [1, Theorem 1.2, p. 2] or Ahmed [1, Theorem 1.2.3]. $\square$

Definition 2.2 A semigroup $\{S(t) : t \geq 0\}$ of bounded linear operators on X is said to be a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \downarrow 0} S(t)x = x \quad \text{for every} \quad x \in X. \tag{2.3}$$

A strongly continuous semigroup of bounded linear operators on X is called a C_0-semigroup. A C_0-semigroup $\{S(t) : t > 0\}$ is said to be compact if it is a compact operator.

Theorem 2.2 Let $\{S(t) : t \geq 0\}$ be a C_0-semigroup. Then, there exist constants $\alpha \geq 0$ and $M \geq 1$ such that

$$||S(t)|| \leq Me^{\alpha t} \quad \text{for} \quad 0 \leq t < \infty. \tag{2.4}$$

Proof See Pazy [1, Theorem 2.2, p. 4] or Ahmed [1, Theorem 1.3.1]. $\square$

If $\alpha = 0$, $\{S(t) : t \geq 0\}$ is defined to be uniformly bounded and if moreover $M = 1$ it is called a C_0-semigroup of contractions. If $M = 1$, $\{S(t) : t \geq 0\}$ is called a pseudo-contraction semigroup. A semigroup $\{S(t) : t \geq 0\}$ is said to be of negative type, or is exponentially stable if $||S(t)|| \leq Me^{-\alpha t}, t \geq 0$ for some constants $M > 0$ and $\alpha > 0$.

Corollary 2.1 If $\{S(t) : t \geq 0\}$ is a C_0-semigroup then $t \to S(t)x$, for every $x \in X$, is a continuous function from R^+ into X.

Proof See Ahmed [1, Corollary 1.3.2]. $\square$

Theorem 2.3 Let A be an infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$. Then,

(i) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)xds = S(t)x.$$

(ii) For $x \in X$,

$$\int_0^t S(t)xdx \in D(A) \quad \text{and} \quad A\left(\int_0^t S(t)xdx\right) = S(t)x - x.$$