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Paola Cantù
Georg Schiemer *Editors*

Logic, Epistemology, and Scientific Theories - From Peano to the Vienna Circle

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Editorial Address
Wiener Kreis Gesellschaft
Universitätscampus, Hof 1, Eingang 1.2
Spitalgasse 2-4, A–1090 Wien, Austria
Tel.: +431/4277 46504 (international) or 01/4277 46504 (national)
Email: vcs@univie.ac.at
Homepage: https://vcs.univie.ac.at/

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Paola Cantù • Georg Schiemer
Editors

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Editors
Paola Cantù
Centre Gilles Gaston Granger UMR 7304
Aix-Marseille Université/CNRS
Aix-en-Provence, France

Georg Schiemer
Department of Philosophy and Institute Vienna Circle
University of Vienna
Wien, Austria

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Part I
Logic, Epistemology, and Scientific Theories: From Peano to the Vienna Circle

Chapter 1
Introduction: Symbolic Logic and Scientific Philosophy

Paola Cantù and Georg Schiemer

Abstract The turn of the last century was a key transitional period for the development of symbolic logic and scientific philosophy. The Peano school, the editorial board of the *Revue de Métaphysique et de Morale,* and the members of the Vienna Circle are generally mentioned as champions of this transformation of the role of logic in mathematics and in the sciences. The articles contained in this volume aim to contribute to a richer historical and philosophical understanding of these groups and research areas in Italy, France and Austria. Specifically, the contributions focus on the following topics: a detailed investigation of the relation between structuralism and modern mathematics; different notions of definition and interpretation at the turn of last century; a closer understanding of the relation between the Vienna Circle, the Peano School and French philosophy in the first half of the twentieth century.

Keywords Symbolic logic · Scientific philosophy · Peano school · Vienna circle · Revue de Métaphysique et de Morale

1.1 The Peano School, the *Révue de Métaphysique et de Morale* and the Vienna Circle

The turn of the last century, i.e., from the nineteenth to the twentieth century, was a key transitional period for the development of symbolic logic and "scientific philosophy". The Peano school (with its members G. Peano, G. Vailati, A. Padoa, C. Burali-Forti, M. Pieri, and G. Vacca) and the Vienna Circle (H. Hahn, K. Menger,

P. Cantù
Centre Gilles Gaston Granger UMR 7304, Aix-Marseille Université/CNRS, Aix-en-Provence, France
e-mail: paola.cantu@univ-amu.fr

G. Schiemer (✉)
Department of Philosophy and Institute Vienna Circle, University of Vienna, Wien, Austria
e-mail: georg.schiemer@univie.ac.at

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R. Carnap, and K. Gödel, among others) are generally mentioned as champions of this transformation of the role of logic in mathematics as well as in the sciences. The change, often reconstructed as a key element in the history of analytic philosophy, was neither uniform nor consensual. Moreover, the philosophical conceptions associated with these research groups are generally presented as forms of (empirical) logicism, without taking into account relevant differences between the members of the groups, including the types of foundational problems they were interested in, the influence of classical traditions of thought (Leibniz, Kant, British and German empiricism), their historical interactions at international conferences and in the edition of journals, and finally, the peculiar collaborative and interdisciplinary dimensions of the two groups.

The editors of this volume co-directed an international scientific project to better understand how the modern conception of logic developed by these groups emerged from interactions with classical axiomatics and the Kantian, Leibnizian, and empiricist philosophical traditions as well as an epistemological consequence of collaborative and interdisciplinary undertakings. The project was entitled "The effect of interdisciplinary collaboration on early twentieth-century epistemologies. A comparison between the Peano school, the Vienna Circle, and the editorial board of the *Revue de Métaphysique et de Morale*: proto-structuralism and proto-pluralist logicism" (INTEREPISTEME) and funded by the French Scientific Center for Research.[1]

The reasons for the misrepresentation of the Peano group's and the Vienna Circle's epistemologies (and of their inner variants) as a coherent logicist understanding of mathematics are various. First, there was an overestimation in the literature of the three major 'isms' in the foundations of mathematics, an overestimation that was partly a result of the activities of the Vienna Circle itself, as they devoted the 1930 conference on *The Epistemology of the Exact Sciences* to logicism, formalism, and intuitionism. Second, there were tendencies to assimilate several different positions to a unique, clearly stated point of view, as in the case of Peano's *Formulario*,[2] Russell's remarks on Peano, and Neurath's efforts to present a unitary scientific perspective of the Vienna Circle in the redaction of the 1929 manifest. Third, these research groups have so far mainly been investigated in isolation, without a systematic analysis of their reciprocal connections and their interactions with classical philosophy.

The research project aimed to contribute to a richer historical and philosophical understanding of the three groups and research areas in Italy, France and Austria, as can be seen from the articles collected in this volume (see the next section for a

[1] International Emerging Action (IEA) PICS07887 INTEREPISTEME France-Austria (2018–2020) funded by CNRS and hosted by Centre Gilles Gaston Granger, Aix-Marseille Université, France. The articles published in this volume were first presented at several workshops funded by the project.

[2] Recent investigations have instead shown that Peano, Padoa, Burali-Forti, Vailati and Pieri had different viewpoints on the relation between mathematics and logic (see Luciano 2017 and Cantù 2022).

detailed overview). The project also aimed to contribute to the following topics: a detailed investigation of the relation between structuralism and geometry and the different notions of definition and interpretation at the turn of last century (see, e.g., Schiemer 2020; Giovannini and Schiemer 2021); a closer understanding of the relation between symbolic logic and previous traditions such as syllogistics (Cantù 2023) and of the relation between logic, history and didactics in the Peano School (Cantù and Luciano 2021).

A separate focus in the project concerned the philosophical contributions of the editorial group linked to the *Revue de Métaphysique et de Morale* which acted more as an aggregator and disseminator of new scientific ideas than as a standard research center. Documented are the contributions of Louis Couturat (Luciano 2012) and Maximilien Winter (see Alunni 2015). In the present volume, we include an analysis of the contributions by Léon Brunschvicg, as well as the first publication of an English translation of his 1909 article devoted to the philosophy of Henri Poincaré. Further insights into Brunschvicg's conception also emerge from a comparison with Émile Borel (see Mazliak and Sage, Chap. 3, this volume) which highlights a general resistance to a philosophy of mathematics focused on an overly abstract and dogmatic conception of set theory and too centered on a preliminarily given, logical classification of concepts.

Focusing on the originality of the epistemological and methodological approaches of these collaborative and interdisciplinary groups, the project took as a starting point the study of the origins and evolution of "scientific philosophy", a notion that was in fact polysemous, as it included different institutional projects and different philosophical traditions (e.g., Helmholtz, Brentano, Tannery, the Italian journal *Rivista di filosofia scientifica*, Russell, Husserl, neo-Kantism, American pragmatism, the Berlin and Vienna circles, Federigo Enriques, or Gaston Bachelard). The focus on the notion of scientific philosophy is not only important for understanding the relationship between the new conception of logic and the legacy of positivism. It is also relevant for clarifying the origin of some issues that are still at the center of a lively contemporary debate, such as the autonomy of philosophy from science (generally defended in the form of an anti-naturalism) and the role of axiomatics in the conceptual analysis of science. Scientific philosophy was not born suddenly in the 1930s but is rather a response to methodological questions common to different philosophical traditions, emerging already in the second half of the nineteenth and the beginning of the twentieth century. Moreover, logical empiricism was not the only viable approach to scientific philosophy. The discussion that took place during the 1936 international conference (see Bourdeau et al. 2018) shows how different groups had diverging views at the time. For instance, for Enriques, symbolic logic was not even part of the method of philosophical analysis of science, whereas for Hahn and others, it played an essential role.

The study of the relationship between the views defended by individual authors and the ideas expressed in the philosophical manifesto of the Vienna Circle finds an analogue in the difference between the variety of contrasting views expressed in the Peano school and the relatively uniform picture presented in the *Formulario*. The present volume paves the way for analyzing the extent to which the dynamics of collective and interdisciplinary interaction of knowledge has modified or influenced individual dynamics of scientific research, providing concrete examples to answer a question that is often debated in an abstract way in social ontology: to what extent does a group's research differ from the sum of individual contributions?

To investigate the relation between logicism and structuralism, or at least some form of proto-structuralism in the philosophy of mathematics, two approaches proved to be particularly fruitful: on the one hand, the analysis of an element too often neglected in the study of modern axiomatics, namely different types of definitions; on the other hand, the analysis of structures and of the relation between their definitions and applications. The study of definitions involved comparing different forms of definitions used in the Peano school: implicit, explicit, proper, improper, direct, indirect, by abstraction, etc. (Cantù 2022), but also the aim to reach a better understanding of the relationship between implicit definitions and axioms in the works of Enriques and Schlick. The attention given to the analysis of definitions does not only derive from efforts to make the foundations of mathematics more rigorous or to reduce mathematical concepts to logical concepts, as in the standard formulations of logicism. It is also motivated by the metatheoretical question of the relation between axioms and theorems. Definitions, far from being exclusively abbreviated writings or logical truths, reflect quite different practices and objectives: if implicit definitions play the role of principles of an axiomatic system, explicit definitions are classified according to their logical form and the criteria they must satisfy in order to guarantee certain metatheoretical properties of an axiomatic system.

The analysis of the use of structures shows that they are conceived differently, depending on whether they derive from a logico-linguistic analysis, from a logico-arithmetic development of the notion of order, or from an attention to physical applications. For instance, Peano's, Schlick's, and Carnap's respective notions of structure not only have different origins, but also play quite different roles in the construction of an axiomatic system. The study of the relationship between definability, the construction of axiomatic systems and the structural analysis of mathematics has highlighted the importance of the unpublished reflections of one of the members of the Vienna Circle, namely Kurt Gödel. The transcription and edition of some of Gödel's unpublished notebooks (the MaxPhil, see Crocco et al. 2021), to which the INTEREPISTEME project has contributed with financial support, has provided several interesting clues for a better understanding of Gödel's philosophy of mathematics.

1.2 Overview of the Collection

The research articles in this volume investigate the historical development of and the interconnections between the different philosophical schools from various perspectives, including essays on Peano and Enriques, Borel and Brunschvicg, and the Vienna Circle. The contributions of Joan Bertran San-Millán, Laurent Mazliak and

Marc Sage and Frédéric Patras focus on different key contributions to the foundations of mathematics around the turn of the last century: they investigate deep interactions between empirism and the development of abstract mathematics, showing the role of deductivism, but also of probability and conventionalism. Bertran San-Millán's "*Peano's geometry: from empirical foundations to abstract development*" develops a critical discussion of Giuseppe Peano's foundational work on the axiomatic presentation of projective geometry. By focusing on the Peano's two central writings on the topic, namely *Principii di Geometria* (1889b) and "*Sui fondamenti della Geometria*" (1894), Bertran San-Millán investigates a critical tension between two poles in Peano's account: on the one hand, the view that the basic components of geometry must be founded on intuition, and, on the other hand, Peano's advocacy of the axiomatic method and an abstract understanding of the axioms. By studying his empiricist remarks and his conception of the notion of mathematical proof, Bertran San-Millán argues that these two poles can be understood as compatible stages of a single process of construction rather than conflicting options. Mazliak's and Sage's article "*Altered states. Borel and the probabilistic approach to reality*" focuses on Émile Borel's contributions to probability theory and what the authors call the "probabilistic shift" in his work around 1905. Specifically, they examine the transition from Borel's studies of the structure of real numbers and a certain rejection of Cantor's abstract vision in the foundations of set theory, to the study of the calculus of probabilities. Moreover, Mazliak and Sage give an informative discussion of Borel's views on the applicability and usefulness of probabilities in scientific methodology, in particular, in the field of statistical mechanics as well as in sociology.

Frédéric Patras translates a text by Brunschvicg on Poincaré that was first published in the *Revue de Métaphysique et de Morale* in 1909. Patras's introduction highlights the scientific role played by the RMM in France and sketches some previously unseen similarities between Brunschvicg's and Poincaré's philosophy, relating to the defense of an anti-positivist form of rationalism, the centrality of mathematics and criticism rather than logic in the scientific method, and the focus on reality and physics. Brunschvicg's article introduces Poincaré's philosophy with the aim of proving that scientific hypotheses, while conventions, are not arbitrary. To clear Poincaré of the charge of nominalism, Brunschvicg cites numerous passages on truth and on the relationship between convenience, logical simplicity and applicability to the external world, but also the analysis of the continuum, and the use of probability theory in the study of the kinetic theory of gases.

Several articles contained in the volume focus on the philosophy of the Vienna Circle as well as its relation to other philosophical traditions. Massimo Ferrari's article "*Leibniz and the Vienna Circle*" focuses on the hitherto neglected influence of Leibniz and Leibnizianism both on the origins and development of the Vienna Circle. As Ferrari argues, this background suggests a re-assessment of the roots of Logical Empiricism beyond the dominant narrative, which has mainly overlooked the role of Leibniz in shaping the scientific world conception. The article starts by focusing on the significance of Leibniz for the Austrian philosophical tradition, which Otto Neurath has emphasized in order to better understand the rise of

Viennese empiricism. Ferrari then turns to the debate about Leibniz's metaphysics and logic at the very beginnings of twentieth century, specifically by Giuseppe Peano and his school, Louis Couturat, and Bertrand Russell. This research has strongly motivated the anti-Kantianism of the Vienna Circle. Moreover, Ferrari argues that the ambitious project of the Encyclopedia endorsed by the late Vienna Circle can be considered, to some extent, in connection with Leibniz's dream of a *scientia generalis*, although carried out from the point of view both of Neurath's and Carnap's physicalism.

Julien Bernard compares two different ways of specifying the scientific status of philosophy: Schlick, like many of the philosophers closed to the Vienna Circle, claimed the rise of a "scientific philosophy", while Husserl wanted to make philosophy a "rigorous science". Arguing that these expressions hide conceptions of science and of its relationship to philosophy that are in sharp opposition, the paper analyzes the polemic focusing on Schlick and Husserl but also on Weyl. After presenting the context of the polemic from the Weyl-Schlick correspondence, and highlighting the opposed role assigned by Husserl and Schlick to intuition and lived experience (*Erlebnis*) in the constitution of a scientific philosophy, Bernard also shows how Weyl, in constrast with Schlick's demands, retains a role for the synthetic a priori within the foundations of science, thereby accounting for the historicity of science.

The contributions by Pierre Wagner and Francesca Biagioli closely connect to the articles on the intellectual context of the Vienna Circle mentioned above. Both articles focus on different contributions to the method of implicit definitions in mathematics and in scientific knowledge more generally. In her article, "*Federigo Enriques and the philosophical background to the discussion of implicit definitions*", Biagioli aims to draw further insights on implicit definitions and the development of this notion from its first occurrence in German language in Enriques's "Principles of Geometry" (1907) to Schlick's *General Theory of Knowledge* (1918). Biagioli argues that Enriques offers one way to counter some of the classical objections against the early twentieth-century conceptualization of implicit definitions. Specifically, Enriques did not conflate the distinct notions that had been identified as implicit definitions in the recent history of mathematics, but he tried to offer an account of the process leading to structural definitions. The paper points out, furthermore, that Enriques's account differs significantly from Schlick's. The scientific interpretations of implicit definitions in Schlick's theory of knowledge depend on the coordination of the terms of abstract mathematical structures with physical realities. By contrast, Enriques addressed the problem of bridging the gap between abstract and concrete terms by identifying patterns within mathematics that provide a clarification of conceptual relations, and so also serve the purposes of applied mathematics. In his article, "*Schlick and Carnap on definitions*", Wagner develops a critical comparison of Carnap's and Schlick's respective accounts of the notion of definition. In the 1920s, both philosophers made an important use of definitions in their main publications: Schlick, in his *Allgemeine Erkenntnislehre* (1918) and Carnap in *Der Logische Aufbau der Welt* (1928). Wagner's paper provides an analysis of the kinds of definitions which are distinguished in these books and a few other papers and then proposes a systematic comparison of Schlick's and Carnap's diverging conceptions of definitions in the 1920s, relating them, in both cases, to their respective philosophical projects in the *Allgemeine Erkenntnislehre* and in the *Aufbau*.

The contributions by Paola Cantù and Frédéric Patras as well as by Gabriella Crocco also investigate different aspects of Carnap's philosophy of mathematics. Cantù and Patras in the article "*Russell and Carnap or Bourbaki? Two ways towards Structures*" focus on early contributions to mathematical structuralism. Specifically, they analyze a central difference between a logical notion of structure that can be traced back to the writings of Bertrand Russell and Rudolf Carnap, and a mathematical notion of structure, exemplified in the works by Bourbaki. As they argue, this coexistence gives rise to a fundamental ambiguity that affects contemporary structuralism. Philosophically, in one case the attention is rather centered on a foundational and reductionist perspective, as featured by the Whitehead-Russell *Principia* and the Carnapian project of the *Aufbau*: the scientific construction of the world around the idea of structure. In the other, the focus is on epistemological and dynamical issues, as exemplified by two key issues in Bourbaki's treatise: understanding the architecture of mathematics, offering a tool-kit to mathematicians. Crocco's article "*Carnap and Gödel, again*", re-addresses the analysis of Carnap's conception of logic and mathematics in Gödel's famous drafts of 'Is mathematics Syntax of Language?'. She critically responds to a recent defense of Gödel's arguments against Carnap's position developed in work by Greg Lavers, pointing out three important differences between her own understanding of Gödel's argument and Lavers's interpretation of it. These differences concern the appreciation of (a) Gödel's strategy of using, in any critical examination of his opponents, only arguments that can be accepted by them; (b) Gödel's analysis of Carnap's position in the 1950s; (c) Gödel's understanding of Carnap's philosophical project. Crocco argues that, contrary to Lavers's opinion, Gödel takes seriously the details of Carnap's original conception and does not overlook the novelty of its solutions in the 1930s and 1950s.

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Chapter 2
Peano's Geometry: From Empirical Foundations to Abstract Development

Joan Bertran San-Millán

Abstract In *Principii di Geometria* (1889b) and 'Sui fondamenti della Geometria' (1894) Peano offers axiomatic presentations of projective geometry. There seems to be a tension in Peano's construction of geometry in these two works: on the one hand, Peano insists that the basic components of geometry must be founded on intuition, and, on the other, he advocates the axiomatic method and an abstract understanding of the axioms. By studying Peano's empiricist remarks and his conception of the notion of mathematical proof, and by discussing his critique of Segre's foundation of hyperspace geometry, I will argue that the tension can be dissolved if these two seemingly contradictory positions are understood as compatible stages of a single process of construction rather than conflicting options.

Keywords Peano · Geometry · Axiomatic method · Empiricism · Deductivism

2.1 Introduction

During the last decades of the nineteenth century, foundational studies became a major field in geometrical research. In Italy, the publication of Fano's translation into Italian (1889)—made at Segre's request—of Klein's *Vergleichende Betrachtungen über neuere geometrische Forschungen* (1872) (commonly known as the Erlangen program) bolstered foundational investigations. Pasch's *Vorlesungen über neuere Geometrie* (1882) is also a key point of reference in this regard.

The growing importance of foundational studies ran parallel to the central role algebraic and projective geometry acquired in the last half of the century. The analytic development of projective geometry pioneered by geometers such as Plücker and Cremona made a pronounced impact on Italian scholarship.[1]

[1] See (Plücker 1831, 1868) and (Cremona 1873).

J. Bertran San-Millán (✉)
Centro de Filosofia das Ciências da Universidade de Lisboa (CFCUL), Lisbon, Portugal
e-mail: jbmillan@ciencias.ulisboa.pt

P. Cantù, G. Schiemer (eds.), *Logic, Epistemology, and Scientific Theories - From Peano to the Vienna Circle*, Vienna Circle Institute Yearbook 29,
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Grassmann's groundbreaking *Ausdehnungslehre* (1844, 1862) attracted attention in Italy from the late 1850s.[2] Klein's two papers on non-Euclidean geometry (1871, 1873) also played an important role. Furthermore, the effort at providing coordinates for projective geometry exclusively on a geometrical basis led by von Staudt was followed by De Paolis in 'Sui fondamenti della geometria proiettiva' (1880).[3]

All in all, the intense developments to which geometry was subjected in the second half of the nineteenth century became the fertile ground from which the Italian school of algebraic geometry and Peano's school could blossom, gaining international renown.[4]

Peano's work on geometry can be divided into two main areas: the development of a geometrical calculus, and the axiomatization of elementary and projective geometry from a synthetic point of view. In this paper, I will focus on this second aspect.[5] Specifically, I will investigate Peano's axiomatizations in *I Principii di Geometria logicamente esposti* (1889b) (hereinafter, *Principii di Geometria*) and 'Sui fondamenti della Geometria' (1894).

There seems to be a tension in Peano's construction of geometry in these two works. On the one hand, Peano insists that the basic geometrical concepts and propositions must have an empirical foundation. On the other hand, geometry starts from axioms, which cannot be attached to a single interpretation. In fact, Peano highlights the abstract character of the terms occurring in such axioms and argues that the demonstration of theorems from these axioms must proceed exclusively by logical means.[6]

By studying Peano's axiomatization of geometry, I will argue that the tension can be dissolved if these two seemingly contradictory positions are understood as compatible aspects of a single process of construction, rather than competing options. Specifically, I will explain that each stance corresponds to a specific phase in the construction of geometry. I will describe these two phases, and characterize their relationship by referring to a dispute between Peano and Segre. Accordingly, I will first claim that for Peano, the construction of geometry must rely on a pre-mathematical phase determined by the selection of a minimal set of axioms and fundamental concepts, which have to be verifiable by direct observation.

[2] On the Italian reception of Grassmann *Ausdehnungslehre*, see (Bottazzini 1985, 27–34).

[3] Von Staudt's *Geometrie der Lage* (1847) was translated into Italian by Pieri (1889), again at the request of Segre.

[4] For a panoramic view of nineteenth-century geometry, see (Gray 2007). On the connection between the development of projective geometry and modern logic, see (Eder 2021). On the development of projective geometry in Italy, see (Avellone et al. 2002).

[5] On Peano's geometrical calculus, see (Bottazzini 1985) and (Borga et al. 1985, 177–198). On the relationship between Peano's geometrical calculus and the axiomatization of geometry, see (Gandon 2006) and (Rizza 2009).

[6] Although some historical studies emphasize the abstract aspect in Peano's construction of geometry (see (Kennedy 1972)), others have observed the tension between empiricism and an abstract approach (see (Bottazzini 2001, 288–290), (Avellone et al. 2002, 378–386), (Gandon 2006, 253)). (Rizza 2009) also aims at dissolving this apparent tension.

Second, I will argue that the formulation of the axioms entails a selection, rearrangement and systematization of content given intuitively. I will claim that, although there is a close connection between the content of the axioms and the nature of the fundamental notions of geometry, the former do not completely determine the latter. In Peano's construction of geometry, there is a second phase, properly mathematical, where rather than being attached to a single system of objects as their sole interpretation, the axioms are understood as abstract postulates.

A study of Peano's criticism of Segre's treatment of hyperspace geometry will allow me to substantiate Peano's abstract understanding of the axioms. On the one hand, Peano's opposition to a purely abstract construction of geometry is motivated by its lack of empirical foundation, and hence relies on his requirements regarding the first pre-mathematical phase. On the other hand, Peano's abstract conception of postulates, in the second phase, can be better understood by alluding to two related notions of purity of method. Peano's advocacy of synthetic geometry, and thus of the independence of this discipline from metric considerations, is closely connected with his conception of the relation between the means to prove theorems and their content. In Peano's view, the content of geometrical laws is not determined by their informal wording, but rather by the deductive relations they establish with the axioms. This indicates that, in the properly mathematical phase, the specific meaning conveyed by these laws becomes irrelevant. From this stance, I will argue that Peano's abstract axiomatic approach can be framed within deductivism. In fact, deductivism squares in a natural way with Peano's notions of purity and his understanding of mathematical proofs regimented by logical means.

This paper is organized into three parts. In Sect. 2.2 will characterize Peano's understanding of the basic concepts of geometry and the requirement that they be empirically founded. In Sect. 2.3, I will explore Peano's critique of Segre's hyperspace geometry in order to contrast the former's empiricist stance with the latter's purely abstract approach. I will also describe Peano's conception of the content of geometrical propositions, and give his view on the nature of Desargues's theorem. This conception of content will inform, in the Sect. 2.4, Peano's views on the process of demonstration of geometrical propositions. From this standpoint, I will offer an explanation of Peano's abstract understanding of the axioms.

2.2 Empirical Foundation of Geometry

Peano's conception of the construction of a mathematical theory relies on a distinction between undefined and derived notions, and between unproven propositions, namely axioms or postulates, and theorems. In 'Sui fondamenti della Geometria' the undefined notions, the most basic concepts of geometry, are called '*primitive* notions' (1894, 116).[7] Peano states that the primitive notions must be "very simple ideas, common to all men" and "reduced to a minimum number" (1894, 116).

Both in *Principii di Geometria* (1889b, 77) and in 'Sui fondamenti della Geometria' (1894, 119), Peano states that the concepts of point and straight segment are the primitive notions of geometry. Specifically, the class of points **1** or p (as it is represented in *Principii di Geometria* and 'Sui fondamenti della Geometria', respectively), and the segment formation operation between two points (ab is the class of points that lie between a and b and is taken as a segment) are the fundamental concepts of Peano's construction of elementary geometry.[8]

The primitive notions of geometry are not defined, but Peano is very clear about the need to provide a secure grounding for them. Peano's claim that the primitive notions are known to any geometer (1894, 116) can be linked to his idea that they are intuitive (1891a, 67). In fact, Peano states that they must be acquired from experience (1894, 119) and that their properties are "experimentally true" (1889b, 56).

Besides the requirement that the primitive notions be acquired from experience, Peano also considers some methodological principles that are involved in the selection of these concepts. Attributing simplicity to the primitive notions is coherent with the idea that any other geometrical concept has to be defined in terms of them. In addition, precision and the reduction of the number primitive notions to the smallest possible are some of the most explicit methodological principles in Peano's presentations of logic, geometry or arithmetic (see, for instance, (1889a, 21)/(1973, 102), (1889b, 78) and (1895, 191–192)/(Dudman 1971, 28–30)).

Relying on an undisputed intuitive basis, simplicity, minimality, and precision guide Peano's selection of primitive notions. In 'Sui fondamenti della Geometria', he rules out the possibility of assuming the notion of space as primitive (1894, 117). In Peano's view, the notion of space is not, strictly speaking, necessary, and such an assumption moreover requires us to add further primitive notions that constitute space's common attributes, namely homogeneity, infinitude, divisibility, immobility, etc., which goes against the criterion of simplicity. Besides, the notion of line, surface and solid are not precise enough for a systematization of the intuitive basis of geometry, and thus are too indeterminate to be considered primitive (1894, 117–118). Instead, Peano proposes using the notions of straight line, plane and specific solid figures, since they can be defined in terms of classes of points and segments.

[7] Unless a reference to an English translation is included after a slash, all quotations from the sources are translated by the author. Page numbers refer to the most recent edition of the source or translation listed in the Bibliography.

[8] Although, strictly speaking, the binary segment formation operation is a primitive notion, Peano often refers to it as a ternary relation of incidence between a point and a segment, and represents it as '$c \epsilon ab$' (see (Peano 1889b, 61)). In fact, in (Peano 1894, 119), Peano makes it explicit that instead of reading '$c \varepsilon ab$' as 'c is a point of the segment ab', he prefers to read it as 'c lies between a and b'. Note however that 'ϵ' (or 'ε' in (1894)) is Peano's membership relation symbol and 'ab' is an individual term that refers to the result of applying the segment formation function to a and b. See (Marchisotto 2011).

In 'Sui fondamenti della Geometria', Peano takes pride in having constructed projective geometry with two primitive concepts, that is, one less than those of Pasch's presentation:

> Pasch, in his important book *Vorlesungen über neuere Geometrie* (Leipzig, 1882), developed Projective Geometry [*Geometria di Posizione*] assuming only three primitive concepts, namely the *point*, the *rectilinear segment* and the finite portion of a *plane*. But the third of these concepts can be reduced to the previous ones by assuming as the definition of the plane, or a part of it, one of its well-known generations [*generazioni*]. Therefore, having admitted the two concepts, point and rectilinear segment, we can define all the other entities, and develop the whole Projective Geometry [*Geometria di Posizione*]. (Peano 1894, 119)

Peano pays much attention to definitions in his construction of geometry, and to the fact that any derived notion can be nominally defined by means of primitive notions using logical symbolism. The formal resources provided by the language of his mathematical logic are instrumental in the formulation of precise and rigorous definitions. However, Peano does not develop a systematic account of the indefinability of the primitive notions. Such an account would prove to be an important issue in Peano's close mathematical environment: in 'Essai d'une théorie algébraique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque' (1901), Padoa informally characterizes the indefinability—in his terms, irreducibility—of a system of primitive notions with respect to a set of postulates.[9]

Despite the methodological principles that guide the establishment of a set of basic concepts, Peano acknowledges that there is some degree of arbitrariness in his selection. In the context of a specific theory, as long as the primitive notions make it possible to define all derived notions, there is no need to rely on a specific choice. According to Peano, if by means of a and b we can define c, and by means of a and c we can define b, then it is just a matter of preference to decide whether a and b, or a and c are the primitive notions (1889b, 78). Nevertheless, this arbitrariness has its limits. First, as Peano puts it in *Principii di Geometria*, "the signs **1** and $a'b$[10] (point and ray) could have been assumed instead of the signs **1** and ab (point and segment);

[9] I am indebted to an anonymous referee for bringing Padoa's account of the irreducibility of primitive notions into my attention.

[10] The ray function $'$ determines the class of points that lie beyond a point b relative to a point a. In 'Sui fondamenti della Geometria' (1894, 120), Peano defines $a'b$ as follows:

$$a, b, c \,\varepsilon\, \mathrm{p} \,.\, \supset : c \,\varepsilon\, a'b \,.=.\, b \,\varepsilon\, ac.$$

Note that, using $a'b$, ab could be defined:

$$a, b, c \,\varepsilon\, \mathrm{p} \,.\, \supset : c \,\varepsilon\, ab \,.=.\, b \,\varepsilon\, a'c.$$

See also (Peano, 1889b, §2, 61, Prop. 1).

this would not have been possible assuming the point and the straight line as undefined concepts" (1889b, 78).[11]

Second, Peano's remarks on arbitrariness are framed in a single theory—specifically, elementary geometry. Assuming the intuitive basis from which geometry is constructed, Peano does not seem to consider the possibility of building different geometries which might have conflicting sets of primitives. Late nineteenth-century empiricism in geometry is nuanced with respect to the role of intuition in the basic components of geometrical theories. In this regard, Peano's account diverges from Klein's. In a lecture delivered in September 2, 1893, Klein distinguishes between naïve intuition, which is inexact, and refined intuition, which comes as the result of an axiomatization (1911, 41–42)/(Ewald 1996, II, 959). In Klein's view, the inexactitude of spatial naïve intuition can be organized and systematized in different ways, and can actually form the foundation of different and equally justified geometries (1890, 572).[12] Peano does not draw such a distinction on intuition, and he does not suggest that the intuitive content from which the primitive notions of geometry are extracted is inexact. After all, as he states in 'Sui fondamenti della Geometria', the primitive notions are known by anyone who is familiar with geometry, and must already have terms that refer to them (1894, 116). The concepts of point and straight segment constitute, with the axioms, the basis of Peano's construction of elementary geometry. The same intuitive foundation remains for any specific theory derived from elementary geometry, including projective geometry.[13]

Assuming that the primitive notions cannot be defined, Peano refuses to even offer descriptions or elucidations about their nature. In *Principii di Geometria*, he affirms that concerning the primitive notions, "only [their] properties will be stated"

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[11] In 'Sui fondamenti della Geometria' (1894, 126), the concept of straight line (in Italian, *retta*) is defined as follows:

$$a, b \,\varepsilon\, \mathrm{p} \,.\, a \mathrel{-\!\!=} b \,.\, \supset \,.\, \mathrm{retta}(a, b) = b'a \cup \iota a \cup ab \cup \iota b \cup a'b,$$

where ιa is the class of objects that are equal to a (i.e., the singleton of a).

Following (Moore 1902, 144), Marchisotto (2011, 163) suggests that not only simplicity is behind Peano's choice of the segment formation operation as a primitive notion; the notion of segment is more fundamental than the concept of line with respect to a set of postulates based on spatial intuition. In their view, the fundamentality of the notion of segment also played a role in Peano's choice.

[12] I am indebted to an anonymous referee for suggesting me to consider Klein's account of intuition.

[13] In (1889b) and (1894), Peano's goal is to put forward a synthetic construction of geometry, one that does not rely on any non-geometrical notion. This could be seen as a specific way of systematising the kind of intuition that is relevant in geometry. However, Peano adopts an alternative way of systematising intuitive content in his work on the geometrical calculus (see, for instance, (1888) and (1898)). The geometrical calculus establishes a linear algebra and ultimately rests on the notion of number. On Peano's two ways of organising spatial intuitions, see (Rizza 2009, 357). On the relationship between Peano's geometric calculus and his synthetic axiomatization of elementary geometry, see (Gandon 2006).

(1889b, 78). These properties are expressed in the axioms. As Peano puts it in 'Sui fondamenti della Geometria':

> [I]t will be necessary to determine the properties of the undefined entity *p* [point], and of the relation *c ε ab* [*c* lies between *a* and *b*], by means of axioms or postulates. The most elementary observation shows us a long series of properties of these entities; we just have to collect these common notions [*cognizioni*], order them, and enunciate as postulates only those that cannot be deduced from simpler ones. (Peano, 1894, 119)

Peano's remarks that the primitive notions of geometry are acquired from experience, and that the axioms are the result of a systematization of the properties of the fundamental concepts, stand at the core of his construction of geometry. The combination of his specific choice of primitive notions and axioms constitute an analysis of the intuitions of space. This intuitive basis is selected, rearranged and regimented following, as we have seen, methodological criteria. The adoption of the axiomatic method plays a crucial role in this analysis, as it makes possible to systematically collect the most elementary properties of the notions of point and straight segment and build geometry in such a way that the deductive dependencies between axioms and theorems are made explicit.

Although Peano states that the axioms of geometry express the simplest properties of the primitive notions, they cannot be considered explicit definitions of these concepts. As stated above, the primitive notions are left undefined and geometry has to be constructed from axioms. Accordingly, although there is a close connection between the content of the axioms and the nature of the notions of point and straight segment, the former do not completely determine the latter. As Peano states, the axioms articulate a selection of the properties of the primitive notions, and as we will see in Sect. 2.4, there are multiple systems which can share the structural features stated in the axioms.[14] This specific relationship between the primitive notions and the axioms paves the way for an abstract understanding of the latter. I will consider such an understanding in Sect. 2.4.[15]

That said, Peano is not interested in constructing geometry as an abstract theory. The axioms must be founded on direct observation. Such a connection between the axioms and intuitive content is what makes them truly geometrical. In Peano's words:

> [A]nyone is allowed to allow those hypotheses that they want, and develop the logical consequences contained in those hypotheses. But for this work to deserve the name of Geometry, those hypotheses or postulates must express the result of the simplest and most elementary observations of physical figures. (Peano 1894, 141)

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[14] On a structuralist understanding of Peano's axiomatization of geometry, see (Bertran San-Millán 2022).

[15] Rizza (2009) suggests a similar idea. In his words:

> [T]he need to systematically organize spatial intuition around certain fundamental concepts can give rise to the concept of a formal structure as a type of organization of a given intuitive content. The choice of fundamental concepts and the articulation of geometry on their basis is carried out through the axiomatic method. (Rizza 2009, 366)

As the result of an analysis of spatial intuition, the axioms of geometry articulate the basic properties of the three-dimensional space. Of the 16 axioms of elementary geometry that are formulated in *Principii di Geometria*, axioms XV and XVI bear witness to Peano's empiricist stance:[16]

$$\text{(XV)} \qquad p \,\epsilon\, \mathbf{3} \,.\, \supset \therefore a \,\epsilon\, \mathbf{1} \,.\, a \,-\!\epsilon\, p : -\!\!=_a \Lambda.$$

$$\text{(XVI)} \quad p \,\epsilon\, \mathbf{3} \,.\, a \,\epsilon\, \mathbf{1} \,.\, a \,-\!\epsilon\, p \,.\, b \,\epsilon\, a'p \,.\, x \,\epsilon\, \mathbf{1} : \supset : x \,\epsilon\, p \,.\, \cup \,.\, ax \cap p \,-\!\!= \Lambda \,.\, \cup \,.\, bx \cap p \,-\!\!= \Lambda.$$

According to Peano, Axiom XV can be read as "Given a plane, there are points that are not contained in it", and Axiom XVI, "Given a plane, and two points from opposite sides of the plane, either each point of space lies on the given plane, or one of the segments that connect it to the given points meets the plane" (1889b, 89). Peano concludes that Axiom XVI states that the space is three-dimensional. Although, as we will see in the next section, Peano considers the possibility of a higher-dimensional space, he does not include any axiom in his construction of elementary geometry that postulates the existence of high-dimensional spaces. In fact, as we will see in Sect. 2.3.3, Axiom XVI would have to be dropped in an axiom system of a four-dimensional space. Had Peano understood his axiom system as a purely abstract structure, this limitation would not be justified.[17]

Furthermore, in 'Sui fondamenti della Geometria', Peano considers the proposition "Two straight lines lying in the same plane always have a point in common" as a possible axiom of projective geometry. He rejects such a possibility because this proposition is "not verified by observation, and it is indeed in contradiction with Euclid's theorems" (1894, 141). As Peano states, "projective Geometry originates from the postulates of elementary Geometry and, by means of appropriate *definitions*, it introduces new entities, called ideal points (both in Euclidean and non-Euclidean geometry)" (1894, 149). He explicitly claims that by means of these new entities all the axioms of elementary geometry are satisfied. All in all, for Peano projective geometry is derived from elementary geometry through definitions, and thus all the axioms of the former must be confirmed by direct observation.[18]

[16] Note that **3** is the class of classes of points that form a plane; Λ, depending on the context, is the empty set (Axiom XVI) or a propositional constant that means the absurd (Axiom XV); and a formula that contains an equality symbol with a letter attached to it as a subscript is the universal quantification of a biconditional.

[17] On the idea that Peano axiomatizes the properties of a three-dimensional space, see (Rizza 2009, p 362–364).

[18] Similarly, Pasch reflects on the addition of an axiom of continuity, but then rejects such a possibility on the grounds that it is inconsistent with his empiricist stance (1882, 125–127). I am indebted to an anonymous referee for suggesting me to consider Pasch's reflection on the axiom of continuity.

In *Principii di Geometria* (1889b, 84–85), Peano analyses the content of three of Pasch's axioms from *Vorlesungen über neuere Geometrie* (1882) and establishes correspondences between his axioms of linear geometry and Pasch's. In 'Sui fondamenti della Geometria' (1894, 120) Peano again acknowledges that his axioms of linear geometry essentially correspond to Pasch's.[19] Besides the postulates of linear geometry, Peano also shares with Pasch the requirement of an empiricist foundation of geometry.[20] Pasch's empiricism is idiosyncratic, but commonalities with Peano's account can nonetheless be found. In *Vorlesungen über neuere Geometrie* (1882, 3), Pasch claims that geometry is a natural science. He also offers a characterization of the basic concepts that echoes Peano's:

> The *basic concepts* [*Grundbegriffe*] *are not defined*; no explanation is able to replace that means which alone eases the understanding of those simple concepts that cannot be traced back to others, namely the reference to suitable physical objects [*geeignete Naturobjecte*]. (Pasch, 1882, 16)

As we will see in the next section, Peano also shares the reservations expressed by Genocchi—with whom Peano collaborated as assistant during the first years of the 1880s—concerning a purely abstract foundation of geometry.

2.3 Peano's Critique of Segre's Geometry of Hyperspaces

Although there is textual evidence concerning Peano's position on the foundations of geometry, his views can be better understood if they are juxtaposed with alternative conceptions of the basis of this mathematical theory. Peano's empiricism can thus be put into an explanatory context, especially on those occasions when he criticizes a purely abstract foundation of geometry. In fact, Peano's criticism is instrumental in understanding the role of an empirical foundation as a guiding principle in the axiomatization of geometry rather than an ad-hoc imposition. Moreover, he makes an effort to explain his views on the abstract character of geometrical proofs when he detects that certain mathematical reasonings lack rigour. On those occasions, Peano substantiates the claim that, in addition to this empirical foundation, there is a stage in the construction of geometry where it can be understood as an abstract discipline. The study of Peano's polemical exchange with Segre will serve as a transition between my accounts of the former's empiricism and the abstract nature of mathematical proofs.

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[19] See (Gandon 2006, 284–287) for a comparison between Pasch's (1882) axioms of projective geometry and Peano's (1889b) axioms of elementary geometry. See also (Borga et al. 1985, 206–211).

[20] On Pasch's empiricism and, in general, on his philosophy of mathematics, see (Schlimm 2010). On Pasch's influence on Peano's axiomatization of geometry, see (Borga et al. 1985, 52–54). Gandon (2006) offers an alternative account of Peano's empiricism and its relationship with Pasch's *Vorlesungen über neuere Geometrie*.

2.3.1 *Segre's Hyperspace Geometry*

As one of the driving forces behind the Italian school of algebraic geometry, Segre was highly influential in the popularization of Klein's *Erlangen program* in Italy.[21] He also made important contributions to hyperspace projective geometry and algebraic geometry. For the purposes of this paper, I will focus on Segre's work on the foundations of hyperspace geometry, which was heavily influenced by the works of Clebsch, Veronese and D'Ovidio.[22] Segre did not follow the axiomatic method and his foundational work on geometries of n-dimensions was constructed upon an abstract notion of point.

In 'Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni' Segre introduces the notion of point as follows:

> Let us consider any linear space of $n - 1$ dimensions. We will call point each of its elements, whatever their nature (which is of no importance to us). (Segre 1883, 39)

A point is presented just as an n-sequence of real numbers and Segre rejects any reflection upon its nature. In fact, Segre dismisses intuitions of space and, as a consequence, all linear spaces of a given number of dimensions are identified:

> All linear spaces with the same number of dimensions, whatever their elements are, can be regarded as identical to each other, since, as we have already noted, in studying them the nature of those elements is not considered, but only the property of linearity and the number of dimensions of the space formed by the elements themselves. (Segre 1883, 46)

Although Segre's characterization of a linear space (1883, 38) does not meet contemporary standards of rigour (nor, in reality, even Peano's),[23] its abstract character is fundamental to the incorporation of algebraic tools into geometry and the characterization of the relationships between linear spaces of different dimensions. It is at the essence of Segre's notion of linear space that, as he puts it in in 'Su alcuni indirizzi nelle investigazioni geometriche', "every space is contained in a higher one; and in the latter we may seek for forms which will simplify the study of given forms in the former" (1891a, 63)/(1904, 465).[24]

Segre published a long paper addressed to students, 'Su alcuni indirizzi nelle investigazioni geometriche' (1891a), in the first volume of *Rivista di matematica*. Despite the introductory and general character of the paper, it triggered an unusual response from Peano, who was the editor and one of the founders of the journal.

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[21] On Segre's leadership of the Italian school, see (Conte and Giacardi 2016) and (Luciano and Roero 2016).

[22] On Segre's contributions to the foundations of geometry, see (Brigaglia 2016).

[23] On a comparison between Segre's and Peano's definitions of a linear space, see (Avellone et al. 2002, 375–377).

[24] It is worth mentioning that other prominent members of the Italian school of algebraic geometry did not share Segre's point of view and argued for empiricism. Veronese, whose work on hyperspace geometry influenced Segre, advocated for using empirically-grounded basic concepts (1891, 611–612). On Veronese's work on the foundations of geometry, see (Cantù 1999). See also (Avellone et al. 2002, 380–385).

Peano placed his 'Osservazioni del Direttore sull'articolo precedente' (1891a) immediately after Segre's paper in the same volume of *Rivista di matematica*. Segre's reply (1891b) was also published, and this in turn prompted Peano's final reaction (1891b).[25]

The dispute sparked by Peano is mainly concerned with mathematical rigour and the use, in geometrical works, of principles lacking solid demonstration. However, Peano also criticizes Segre's construction of hyperspace geometry, and this will be the focus of my discussion in this section. In particular, I will consider the two main aspects of Peano's critique: on the one hand, the lack of empirical character of basic propositions and primitive notions of a foundational work on geometry; and, on the other, the unjustified analogical use of $n + 1$-dimensional geometry to obtain results of n-dimensional geometry.

2.3.2 The Abstract Foundation of Hyperspace Geometry

In 'Osservazioni del Direttore sull'articolo precedente', Peano insists on some ideas that he had suggested in *Principii di Geometria* and would develop in 'Sui fondamenti della Geometria'. Specifically, in his first reaction to (Segre 1891a), Peano puts forward his claim concerning the empirical character of the axioms of geometry. He suggests that geometry cannot be built upon "hypotheses contrary to experience, or [...] hypotheses which cannot be verified by experience" (1891a, 67).

Peano then elaborates on this view and suggests that there is a pre-mathematical phase in which the axioms are selected and formulated:

> Each author can assume those experimental laws that they please, and can make those hypotheses that they like best. The good choice of these hypotheses is very important in the theory to be developed; but this choice is made by way of induction, and does not belong to mathematics. Having made the choice of the starting point, it is up to mathematics (which, in our opinion, is a perfected logic) to deduce the consequences; and these must be absolutely rigorous. Whoever states consequences that are not contained in the premises might make poetry, but not mathematics. (1891a, 67)

These remarks complement the picture laid out in the previous section concerning the establishment of the axioms of geometry. For Peano, the foundation of geometry begins with a stage where the primitive notions are selected. The properties of these primitive notions are obtained by direct observation, and they are rearranged and systematized in a list of axioms. The axioms can be understood as experimental because they state the basic properties of the primitive notions, which are obtained from experience. Therefore, in Peano's view, the result of direct observation is not imposed upon a set of abstract axioms at a later stage; it is inherent in these axioms that they select, rearrange and regiment intuitive content. Once this pre- mathematical

[25] On the polemic between Peano and Segre, see (Manara and Spoglianti 1977), (Borga et al. 1985, 242–244), (Bottazzini 2001, 553–555), (Avellone et al. 2002, 372–385).

analysis has produced a specific list of axioms, it is followed by mathematics proper, which consists in the definition of derived notions and the demonstration of theorems. In the next section I will evaluate Peano's claim that mathematics is "a perfected logic".

With these assertions alone, Peano is ready to discredit Segre's foundations of hyperspace geometry, viewing them as not genuinely geometrical. If a point is characterized just as an n-sequence of numbers, the intuitive character attached to this concept is completely lost. Moreover, the primitive notions of geometry are no longer independent of the notion of number and thus the boundaries between geometry and analysis—which relies on the concept of number—become blurred.[26] In Peano's words:

> If any group of n variables is called a point [...], then it is well known that any discussion on the postulates of Geometry ceases; the theories that are deduced develop the consequences of the principles of arithmetic, and not of those of geometry; every result thus obtained is independent of any geometric postulate. (Peano 1891b, 157)

Peano advocates for an autonomous foundation of geometry, one which does not rely on non-geometrical notions. This is coherent with his synthetic approach in the construction of geometry, and implicitly encapsulates an idea of *purity of method*.[27] For Peano, Segre's foundation of hyperspace geometry is not pure and, moreover, lacks an account that connects the basic concepts with our intuitions of space.[28]

2.3.3 An Axiomatic Construction of Hyperspace Geometry

Let us now turn to the second aspect of Peano's critique of Segre's construction of hyperspace geometry. In 'Su alcuni indirizzi nelle investigazioni geometriche', Segre suggests three possible foundations of hyperspace geometry, which in turn correspond to three possible ways of defining points in an n-dimensional linear space (1891a, 59–61)/(1904, 460–463). The first is the one already considered, and takes points to be "any system of values of n variables (the coordinates of the point)" (1891a, 59)/(1904, 460). The second follows Plücker and characterizes points as "geometric forms of ordinary space, such as groups of points, curves, surfaces"

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[26] As Rizza (2009, 357) suggests, Peano does not rule out n-dimensional linear spaces, because they are used in ordinary mathematics; he does not accept them *in geometry*, since their existence is not supported by our intuitions of space.

[27] On the notion of purity of method, see (Arana 2008), (Detlefsen 2008), (Detlefsen and Arana 2011).

[28] Peano's empiricism and his critique of Segre's abstract foundation of n-dimensional geometries can be connected with the views of Genocchi, Peano's predecessor at the chair of infinitesimal calculus in Turin. In (1891, 614–615, fn. 2), Veronese reports Genocchi's dismissive and harsh judgement of hyperspace geometry, which can be found in (Genocchi 1877, 388–389). On Genocchi's views of hyperspace geometry and the polemic between Peano and Segre, see (Manara and Spoglianti 1977).

(1891a, 60)/(1904, 461). Finally, according to the third option, points in hyperspace are characterized as ordinary points, but "we omit the postulate concerning the three dimensions, and consequently modify some of those referring to the straight line and plane" (1891a, 60)/(1904, 462).

Concerning the first option, Segre already anticipates Peano's critique that it results in an algebra of linear transformations and it is thus no longer genuine geometry (1891a, 59)/(1904, 461). However, he makes it clear that this is not an issue for him, since, after all, "it is *mathematics* that is being made" (1891a, 59)/(1904, 461, fn. 2). In his response to Segre, Peano only considers Segre's third possible foundation, and it is on this matter that he levies his critiques.

Peano describes his proposal of an axiomatic construction of a four-dimensional geometry as follows:

> To move from the 3-dimensional space to the 4[-dimensional space], it is necessary to eliminate the 16th postulate, and then, without modifying those referring to the straight line and the plane, to admit the postulate, analogous to [postulates] 2, 7, 12, 15:
>
> A) *There are points outside ordinary space.*
>
> It follows as a consequence [. . .] that, in this way, *every proposition proved true using the 4-dimensional space ceases to hold in the 3-dimensional space*, since it is shown to be a consequence of postulates 1-15 and postulate A, and it is not shown to be a consequence of the postulates of elementary geometry alone. (Peano 1891a, 68)

In *Principii di Geometria*, 16 axioms establish the basis of elementary geometry.[29] Peano suggests axiomatizing the four-dimensional space by means of axioms I-XV and axiom A. In the aforequoted passage, he refers to Axioms II, VII, XII and XV:[30]

$$\text{(II)} \qquad a \in \mathbf{1} \,.\, \supset \therefore x \in \mathbf{1} \,.\, x -\!= a : -\!=_x \Lambda.$$

$$\text{(VII)} \qquad a, b \in \mathbf{1} \,.\, a -\!= b : \supset .\, a'b -\!= \Lambda.$$

$$\text{(XII)} \qquad r \in \mathbf{2} \,.\, \supset \therefore x \in \mathbf{1} \,.\, x -\!\in r : -\!=_x \Lambda.$$

$$\text{(XV)} \qquad p \in \mathbf{3} \,.\, \supset \therefore a \in \mathbf{1} \,.\, a -\!\in p : -\!=_a \Lambda.$$

Axiom II states that given any point a, there are points different from a. Axiom VII states that given two points a and b, if they are different, then the ray $a'b$ is non-empty (and thus there are points which lie in $a'b$). Axiom XII states that given a line r, there are points which do not lie on r. As indicated in the previous section, Axiom XV states that given a plane p, there are points which do not lie on p. These axioms are all existential and thus, as Peano states, analogous to the suggested axiom A; they postulate the existence of points that do not meet certain conditions.

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[29] In an Appendix, Peano also formulates a seventeenth axiom which postulates the continuity of the straight line (1889b, 90).

[30] Note that **2** is the class of classes of points that constitute straight lines, and recall that **3** is the class of planes, and Λ, depending on the context, is the empty set (axiom VII) or a propositional constant that means the absurd (axioms II, XII and XV). See Footnote 16.

Then, as a means of axiomatizing a four-dimensional space, Peano also proposes eliminating Axiom XVI:[31]

$$\text{(XVI)}\quad p \in \mathbf{3} \,.\, a \in \mathbf{1} \,.\, a -\!\in p \,.\, b \in a'p \,.\, x \in \mathbf{1} : \supset : \\ x \in p \,.\cup.\, ax \cap p -\!= \Lambda \,.\cup.\, bx \cap p -\!= \Lambda.$$

According to the construction put forward by Peano, any theorem that is demonstrated by means of the axiom system of a four-dimensional space cannot be considered a theorem of a three-dimensional space, since it has not been proved from axioms I-XVI. After all, if a theorem is deduced from axioms I-XV and A, then it cannot be considered a theorem of elementary geometry proper, since axiom A can play a role in its proof. Peano's argument attempts to block Segre's strategy, according to which results obtained in n + 1-dimensional linear spaces can be applied to n-dimensional spaces; the inclusion of axiom A in Peano's construction involves a a substantial use of n + 1-dimensional tools.[32]

Peano's conclusion is that Segre's analogical use of four-dimensional linear spaces to prove theorems of three-dimensional linear spaces is unjustified. In his words:

> Some writers, from the fact that many properties of plane figures are derived from properties of solid figures, deduce by analogy that properties of figures of ordinary space can be derived from considerations in 4-dimensional space. But the analogy is illusory. (Peano 1891a, 68)

A corollary of Peano's statement would be that, if the use of four-dimensional space in the proof of a three-dimensional theorem cannot be taken for granted, then the fact that "many properties of plane figures are derived from properties of solid figures" is also unjustified for similar reasons. Peano's conception of what constitutes a specific geometry, which can be connected with the notion of purity of method of proof, clarifies this issue.

2.3.4 Purity and Desargues's Theorem

In the first reply to Segre's (1891a) paper, Peano argues that linear, planar and solid geometry are constituted by specific axioms:

> [I]f by geometry of the straight line (1-dimensional) we mean that which develops the consequences of axioms 1-11; by plane geometry (2-dimensional) that which develops the

[31] Note that, in this context, Λ is the empty set. See Sect. 2.2 for an informal rendering of Axiom XVI.

[32] In my view, Peano's argument focuses on the fact that a theorem demonstrated in a four-dimensional space is *unjustified* in a three-dimensional space, and thus relies on its epistemological status rather than on its being true or false in a three-dimensional space. Bottazzini (2001, 303–304) reports an alternative interpretation of the aforequoted passage found in (Bozzi 2000, 104) and suggests that Peano might identify theory and interpretation.