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New Trends in Lyapunov Exponents

NTLE, Lisbon, Portugal, February 7–11, 2022

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Preface

The concept of characteristic exponent along an orbit of a dynamical system was introduced by Aleksandr Mikhailovich Lyapunov (1857–1918), who made fundamental contributions to the theory of differential equations and of dynamical systems. Known nowadays as Lyapunov exponent, this concept measures the sensitivity of an orbit to its initial condition. Roughly speaking, a negative Lyapunov exponent corresponds to stable orbit behavior, the kind characterized by Aleksandr Lyapunov, while positive Lyapunov exponents are associated with irregular or chaotic orbit behavior.

A systematic study of Lyapunov exponents followed the classical Multiplicative Ergodic Theorem of Valery Oseledets in 1965, where the concept is defined in the context of linear cocycles. A linear cocycle is a skew-product dynamical system acting on a vector bundle, which preserves the linear bundle structure and induces a measure preserving dynamical system on the base. Lyapunov exponents quantify the average exponential growth of the iterates of the cocycle along fiber-invariant subspaces, which are called Oseledets subspaces.

An important class of examples of linear cocycles are the ones associated with discrete, one-dimensional ergodic Schrödinger operators. Such an operator is the discretized version of a quantum Hamiltonian. Its potential is given by a time series, that is, the potential is obtained by evaluating an observable along the orbit of an ergodic transformation.

The iterates of a linear cocycle can be thought of as a multiplicative (noncommutative) stochastic process. A relevant and difficult problem is to understand the statistical properties of such processes, under appropriate assumptions.

Some of the main topics of study in the theory of Lyapunov exponents are concerned with their positivity and simplicity; dichotomy between uniform hyperbolicity and zero Lyapunov exponents; regularity properties such as continuity, modulus of continuity or even smoothness; the structure of their level sets; their behavior on non-typical sets; generalizations of Oseledets' theorem to other settings and applications to fields such as Mathematical Physics, Differential Equations and Geometry.

This monograph contains a collection of survey articles describing recent research trends in these and related topics. The articles are authored by participants of the workshop “New trends in Lyapunov exponents” that took place between February 7 and 11, 2022, at ISEG-ULisboa (Lisbon School of Economics & Management of the Universidade de Lisboa) in Lisbon, Portugal. The workshop was part of the scientific activities organized within the research project PTDC/MAT-PUR/29126/2017 funded by FCT (Fundação para a Ciência e Tecnologia), Portugal. It also received partial support from ISEG and the following research centers: CMAT, CEMAPRE and CMAFcIO. For many participants, this gathering marked their first in-person international event after two years of travel restrictions resulting from the Covid-19 pandemic.

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Contents

Lyapunov Exponents for Linear Homogeneous Differential Equations

Mário Bessa

Abstract We consider linear continuous-time cocycles $\Phi : \mathbb{R} \times M \to \mathrm{GL}(2, \mathbb{R})$ induced by second order linear homogeneous differential equations $\ddot{x} + \alpha(\varphi^t(\omega))\dot{x} + \beta(\varphi^t(\omega))x = 0$, where the coefficients α, β evolve along the orbit of a flow $\varphi^t : M \to M$ defined on a closed manifold M and $\omega \in M$. We are mainly interested in the Lyapunov exponents associated to most of the cocycles chosen when one allows variation of the parameters α and β. The topology used to compare perturbations turn to be crucial to the conclusions.

Keywords Differential equations · Linear cocycles · Linear differential systems · Multiplicative ergodic theorem · Lyapunov exponents

2010 Mathematics Subject Classification Primary: 34D08, 37H15; Secondary: 34A30, 37A20

1 Introduction

1.1 Linear Differential Systems

Let Φ_A^t be a matricial solution of the autonomous differential equation $\dot{U}(t) = A \cdot U(t)$ where A is a $n \times n$ matrix of the same order as $U(t)$. Given $v \in \mathbb{R}^n$, obtaining the asymptotic growth of the number

$$\frac{1}{t} \log \|\Phi_A^t \cdot v\| \tag{1}$$

is an exercise of finite dimensional spectral analysis. The Lyapunov spectrum is characterized by the Lyapunov exponents (the limit of (1) when $t \to \infty$) and

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its eigendirections. When a perturbation B of A is allowed, say by considering B uniformly near A, the perturbed system $\dot{U}(t) = B \cdot U(t)$ originates the solution Φ^t_B exhibiting a well-understood behaviour [36]. The problem starts to gain enormous complexity when we consider the non-autonomous case $\dot{U}(t) = A(t) \cdot U(t)$, where $A(t)$ depends on the parameter t. The calculation of Lyapunov exponents, as well as their stability, turns out to be a substantially more difficult issue. A classical way of looking at non-autonomous linear differential equations is to consider linear differential systems i.e. linear continuous-time cocycles.

1.2 Kinetic Cocycles

It has been known by *Liouville theory* (see e.g. [45]), established almost two centuries ago, that there are huge constraints when we try to apply analytical methods to integrate most functions. As we cannot always have explicit solutions a qualitative approach to understand the asymptotic behaviour of solutions of differential equations proved to be an efficient approach to deal with this difficulty. We intend to analyse the asymptotic behaviour of solutions of second order linear homogeneous differential equations of the form

$$\ddot{x}(t) + \alpha(\varphi^t(\omega))\dot{x}(t) + \beta(\varphi^t(\omega))x(t) = 0, \tag{2}$$

with coefficients α and β displaying a certain regularity (L^p or C^r with $r \geq 0$), varying along the orbits of a flow φ^t and admitting a small perturbation on the parameters α and β. This flow φ^t is usually consider to be aperiodic and so for each orbit we obtain a particular differential equation which results in dealing with infinitely many differential equations at the same time. We will try to describe the Lyapunov spectrum of a linear cocycle associated with (2) when some perturbation on its coefficients is made. The details about these type of cocycles will be presented later on Sect. 2.2 but the idea is very simple. We consider a flow $\varphi^t : M \to M$ preserving a measure defined in M and a linear variational equation $\dot{U}(\omega, t) = A(\varphi^t(\omega)) \cdot U(\omega, t)$ with generator

$$\begin{aligned} A\colon M &\longrightarrow \mathbb{R}^{2\times 2} \\ \omega &\longmapsto \begin{pmatrix} 0 & 1 \\ -\beta(\omega) & -\alpha(\omega) \end{pmatrix}. \end{aligned} \tag{3}$$

Hence, the infinitesimal generator A is of a particular type after all. The flow φ^t will label a certain differential equation where A captures its coefficients.

1.3 The Harmonic Oscillator

Equation (2) represents the simple damped harmonic oscillator free from external forces where α (frictional force) and β (frequency of the oscillator) are functions depending on $\omega \in M$ described by the flow $\varphi^t : M \to M$ for $t \in \mathbb{R}$. When the frictional force and the frequency of the oscillator are constant or, more generally, when α and β are first integrals with respect to φ^t, then (2) can be easily solved. When this is not the case, explicit solutions could be difficult to obtain. When $\alpha \neq 0$ (damped case) we have the solution $\Phi_A^t \in \mathrm{GL}(2, \mathbb{R})$ and when $\alpha = 0$ (the frictionless case) we have $\Phi_A^t \in \mathrm{SL}(2, \mathbb{R})$. Clearly, a perturbation theory for the frictionless case deserves some kind of care because perturbations will have to maintain $\alpha = 0$.

1.4 The Main Goal

Given (3) and fixing the *position* and *velocity* $(x(0), \dot{x}(0))$ we are interested in describing the Lyapunov spectrum of Φ_A^t when $t \to \infty$ of the pair $(x(t), \dot{x}(t))$. More particularly, we intend to address the following problem:

- Fixing a certain regularity of the parameters α and β (L^p, L^∞, C^0, C^1, ...) and
- providing the parameter space with a conforming topology we ask:
- For the 'majority' of parameters considered (dense/residual/open+dense) what kind of Lyapunov spectrum do we expect to have?

In Sect. 3 we discuss the case when parameters evolve on L^p, in Sect. 4 we consider parameters on C^0 and finally, in Sect. 5 we consider the parameters evolving on C^r. The only case where there is already results in literature is the L^p one. Hence, considering the C^0 and the C^r cases we will only address some open questions. Finally, in Sect. 6 we consider a particular model of a third order linear homogeneous differential equation and follow [18] to show how to remove zero Lyapunov exponents on a partial hyperbolic cocycle by a small C^0 perturbation.

2 Kinetic Linear Cocycles

2.1 Linear Cocycles

Let $(M, \mathcal{M}, \mu)$ be a probability space and let $\varphi : \mathbb{R} \times M \to M$ be a *measurable flow* in the sense that it is a measurable map and

(1) $\varphi^t : M \to M$ given by $\varphi^t(\omega) = \varphi(t, \omega)$ preserves the measure μ for all $t \in \mathbb{R}$;
(2) $\varphi^0 = \mathrm{Id}_M$ and $\varphi^{t+s} = \varphi^t \circ \varphi^s$ for all $t, s \in \mathbb{R}$.

Unless stated otherwise we will consider that the flow is ergodic in the usual sense that there exist no invariant sets except zero measure sets and their complements. Let $\mathcal{B}(X)$ be the Borel σ-algebra of a topological space X. A continuous-time linear *random dynamical system* on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, or a continuous-time *linear cocycle*, over φ is a $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{M}/\mathcal{B}(\mathrm{GL}(2,\mathbb{R}))$-measurable map

$$\Phi : \mathbb{R} \times M \to \mathrm{GL}(2,\mathbb{R})$$

such that the mappings $\Phi(t,\omega)$ form a cocycle over φ, that is:

(1) $\Phi^0(\omega) = \mathrm{Id}$ for all $\omega \in M$;
(2) $\Phi^{t+s}(\omega) = \Phi^t(\varphi^s(\omega)) \circ \Phi^s(\omega)$, for all $s, t \in \mathbb{R}$ and $\omega \in M$,

and $t \mapsto \Phi^t(\omega)$ is continuous for all $\omega \in M$. We recall that having $\omega \mapsto \Phi^t(\omega)$ measurable for each $t \in \mathbb{R}$ and $t \mapsto \Phi^t(\omega)$ continuous for all $\omega \in M$ implies that Φ is measurable in the product measure space. We also call these objects *linear differential systems*.

2.2 Kinetic Linear Cocycles

As we already said, in Sect. 1.3, the cocycles we consider are motivated by the non-autonomous linear homogeneous differential equation which describes a motion of the damped harmonic oscillator as the 'simple pendulum' along the path $(\varphi^t(\omega))_{t\in\mathbb{R}}$, with $\omega \in M$ described by the flow φ. Let $K \subset \mathbb{R}^{2\times 2}$ be the set of 2×2 matrices written in the form of

$$A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \tag{4}$$

for real numbers a, b, and denote by $\mathcal{G}$ the set of measurable applications $A : M \to \mathbb{R}^{2\times 2}$. Denote also by $\mathcal{K} \subset \mathcal{G}$ the set of *kinetic* measurable applications $A : M \to K$. We also identify two applications on $\mathcal{G}$ that coincide on μ-a.e. in M. Take measurable maps $\alpha \colon M \to \mathbb{R}$ and $\beta \colon M \to \mathbb{R}$. Consider the differential equation given in (2). Let $y(t) = \dot{x}(t)$ and rewrite (2) as the following vectorial linear system

$$\dot{X} = A(\varphi^t(\omega)) \cdot X, \tag{5}$$

where $X = X(t) = (x(t), y(t))^T = (x(t), \dot{x}(t))^T$ and $A \in \mathcal{K}$ is given by (3). It follows from [5, Thm. 2.2.2] (see also Lemma 2.2.5 and Example 2.2.8 in this reference) that if $A \in \mathcal{G}^1 =: \mathcal{G} \cap L^1(\mu)$, i.e. $\int_M \|A\|\, d\mu < \infty$, then it generates a unique linear differential system Φ_A satisfying

$$\Phi^t_A(\omega) = \mathrm{Id} + \int_0^t A(\varphi^s(\omega)) \cdot \Phi^s_A(\omega)\, ds. \tag{6}$$

The solution $\Phi^t_A(\omega)$ defined in (6) is called *mild solution* or *Carathéodory solution*. Given an initial condition $X(0) = v \in \mathbb{R}^2$, we say that $t \mapsto \Phi^t_A(\omega)v$ solves or is a solution of (5), or that (5) generates $\Phi^t_A(\omega)$. Note that $\Phi^0_A(\omega)v = v$ for all $\omega \in M$ and $v \in \mathbb{R}^2$. If the solution (6) is differentiable in time (i.e. with respect to t) and satisfies for all t

$$\frac{d}{dt}\Phi^t_A(\omega)v = A(\varphi^t(\omega)) \cdot \Phi^t_A(\omega)v \qquad \text{and} \qquad \Phi^0_A(\omega)v = v, \tag{7}$$

then it is called a *classical solution* of (5). Classical solutions arise when we consider $A : M \to K$ continuous. Of course that $t \mapsto \Phi^t_A(\omega)v$ is continuous for all ω and v. Due to (7) we call $A : M \to K$ a *kinetic infinitesimal generator* of Φ_A. Sometimes, due to the relation between A and Φ_A, we refer to both A and Φ_A as a kinetic linear cocycle or kinetic linear differential system. If (5) has initial condition $X(0) = v$ then $\Phi^0_A(\omega)v = v$ and $X(t) = \Phi^t_A(\omega)v$. Let $\mathcal{K}_0 \subset \mathcal{K}$ stand for the *traceless kinetic cocycles* induced from matrices as in (4) imposing the constraint $a = 0$. Let $\mathcal{K}^1 = \mathcal{K} \cap L^1(\mu) \subset \mathcal{G}^1$ and let $\mathcal{K}^1_0 = \mathcal{K}_0 \cap L^1(\mu) \subset \mathcal{K}^1$.

2.3 Topologies

Now we will define the topologies we are going to consider in the sequel.

2.3.1 The L^p Topology

We now define an L^p-like topology generated by a metric that compares the infinitesimal generators on $\mathcal{G}$. For $1 \le p < \infty$ and $A, B \in \mathcal{G}$ we set

$$\hat{\sigma}_p(A, B) := \begin{cases} \left(\int_M \|A(\omega) - B(\omega)\|^p \, d\mu(\omega)\right)^{\frac{1}{p}}, \\ \infty \;\; \text{if the above integral does not exists,} \end{cases}$$

and define

$$\sigma_p(A, B) := \begin{cases} \frac{\hat{\sigma}_p(A,B)}{1+\hat{\sigma}_p(A,B)}, & \text{if } \hat{\sigma}_p(A, B) < \infty \\ 1, & \text{if } \hat{\sigma}_p(A, B) = \infty \end{cases}.$$

Clearly, σ_p is a distance in $\mathcal{G}$. The following topological results were proved in [2].

Proposition 2.1 *Consider* $1 \le p < \infty$. *Then:*

(i) $\sigma_p(A, B) \le \sigma_q(A, B)$ *for all* $q \ge p$ *and all* $A, B \in \mathcal{G}$.
(ii) *If* $A \in \mathcal{G}^1$ *then for any* $B \in \mathcal{G}$ *satisfying* $\sigma_p(A, B) < 1$ *we have* $B \in \mathcal{G}^1$. *Therefore,* $\sup_{0 \le t \le 1} \log^+ \|\Phi_B^t(\omega)^{\pm 1}\| \in L^1(\mu)$.
(iii) *The sets* $(\mathcal{K}^1, \sigma_p)$ *and* $(\mathcal{K}_0^1, \sigma_p)$ *are closed, for all* $1 \le p < \infty$.
(iv) *For all* $1 \le p < \infty$, $(\mathcal{K}^1, \sigma_p)$ *and* $(\mathcal{K}_0^1, \sigma_p)$ *are complete metric spaces and, therefore Baire spaces.*

Next result is elementary in measure theory and captures the crucial idea which allows making huge perturbations on the uniform norm but small perturbations in the L^p-norm as long the support is small in measure. For the proof see [3].

Lemma 2.2 *Given* $A \in \mathcal{G}^1$ *and* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $\mathcal{F} \in \mathcal{M}$ *and* $\mu(\mathcal{F}) < \delta$, *then* $\int_{\mathcal{F}} \|A(\omega)\| \, d\mu(\omega) < \epsilon$.

2.3.2 Uniform Topologies

Now we consider that the kinetic infinitesimal generators $A : M \to \mathbb{R}^{2\times 2}$ are in L^∞ or are in C^0. The first is denoted by $L^\infty(M, \mathcal{K})$ and the second by $C^0(M, \mathcal{K})$. Clearly, $C^0(M, \mathcal{K}) \subset L^\infty(M, \mathcal{K}) \subset \mathcal{K}^1$. We also consider traceless infinitesimal generators $C_0^0(M, \mathcal{K}) \subset L_0^\infty(M, \mathcal{K}) \subset \mathcal{K}_0^1$.

We endow $L^\infty(M, \mathbb{R}^{2\times 2})$ with the L^∞ metric defined by

$$\|A - B\|_\infty = \operatorname*{ess\,sup}_{\omega \in M} \|A(\omega) - B(\omega)\|$$

where $A, B \in L^\infty(M, \mathbb{R}^{2\times 2})$. We also endow $C^0(M, \mathbb{R}^{2\times 2})$ with the C^0 metric defined by

$$\|A - B\|_0 = \max_{\omega \in M} \|A(\omega) - B(\omega)\|$$

where $A, B \in C^0(M, \mathbb{R}^{2\times 2})$. We also make use of the uniform operators norm to compare solutions given a fixed $t > 0$ like

$$\|\Phi_A^t - \Phi_B^t\|_0 = \max_{\omega \in M} \|\Phi_A^t(\omega) - \Phi_B^t(\omega)\| \quad \text{or} \quad \|\Phi_A^t - \Phi_B^t\|_\infty = \operatorname*{ess\,sup}_{\omega \in M} \|\Phi_A^t(\omega) - \Phi_B^t(\omega)\|.$$

Both $(C^0(M, \mathbb{R}^{2\times 2}), \|\cdot\|_0)$ and $(L^\infty(M, \mathbb{R}^{2\times 2}), \|\cdot\|_\infty)$ are complete metric spaces and, therefore, Baire spaces. The set $L^\infty(M, \mathcal{K})$ is L^∞-closed and the set $C^0(M, \mathcal{K})$ is C^0-closed. Moreover, the set $L^\infty(M, \mathcal{K}_0)$ is L^∞-closed and the set $C^0(M, \mathcal{K}_0)$ is C^0-closed.

2.3.3 The C^r Topology

Finally, we consider that the kinetic infinitesimal generators $A : M \to \mathbb{R}^{2\times 2}$ are $C^{r,\nu}$ (i.e. are in $C^{r+\nu}$) a set we denote by $C^{r,\nu}(M, \mathcal{K})$, where $r \in \mathbb{N} \cup \{0\}$ and $\nu \in [0, 1]$. We endow $C^{r,\nu}(M, \mathbb{R}^{2\times 2})$ with the $C^{r,\nu}$-topology defined using the norm

$$\|A\|_{r,\nu} = \sup_{0\le j\le r} \sup_{x\in M} \|D^j A(x)\| + \sup_{x\neq y} \frac{\|A(x) - A(y)\|}{\|x - y\|^\nu}, \tag{8}$$

where $A \in C^{r,\nu}(M, \mathbb{R}^{2\times 2})$ and $x, y \in M$. Let us also mention that it is enough to consider the case when $\nu = 1$ (i.e. A is Lipschitz). In fact, if A is ν-Hölder continuous with respect to the metric $d(\cdot, \cdot)$ then it is Lipschitz with respect to the metric $d(\cdot, \cdot)^\nu$. Hence, up to a change of metric we may assume that A is Lipschitz and we will do so throughout the presentation.

2.4 Lyapunov Exponents

Notice that if $A \in \mathcal{K}^1$ then the cocycle Φ_A satisfies the following *integrability condition*

$$\sup_{0\le t\le 1} \log^+ \|\Phi_A^t(\omega)^{\pm 1}\| \in L^1(\mu), \tag{9}$$

where $\log^+ = \max\{0, \log\}$. In fact, take ω in the full measure φ^t-invariant subset of M where $t \mapsto A(\varphi^t(\omega))$ is locally integrable. By (6) and by Grönwall's inequality (see [5]) we get

$$\sup_{0\le t\le T} \log^+ \|\Phi_A(t, \omega)^{\pm 1}\| \le \int_0^T \|A(\varphi^s(\omega))\|\, ds =: \psi(\omega, T). \tag{10}$$

By Arnold [5, Lemma 2.2.5] we have $\psi(\cdot, T) \in L^1(\mu)$, hence (9) holds. Fubini's theorem allow us also to obtain from (10) that:

$$\begin{aligned}\int_M \log^+ \|\Phi_A(t, \omega)^{\pm 1}\|\, d\mu(\omega) &\le \int_M \int_0^t \|A(\varphi^s(\omega))\|\, ds\, d\mu(\omega)\\ &= \int_0^t \int_M \|A(\varphi^s(\omega))\|\, d\mu(\omega)\, ds = t\|A\|_1.\end{aligned}$$

If $A \in \mathcal{G}^1$ then the integrability condition (9) holds and Oseledets theorem (see e.g. [5, 44]) gives that for μ-a.e. $\omega \in M$, there exists a Φ_A-invariant splitting called

Oseledets splitting of the fiber $\mathbb{R}^2_\omega = E^1_\omega \oplus E^2_\omega$ and real numbers called *Lyapunov exponents* $\lambda_1(A, \omega) \geq \lambda_2(A, \omega)$, such that:

$$\lambda_i(A, \omega) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|\Phi^t_A(\omega) v^i\|, \tag{11}$$

for any $v^i \in E^i_\omega \setminus \{\vec{0}\}$ and $i = 1, 2$. Furthermore, given subspaces E^1_ω and E^2_ω, the angle between them along the orbit has subexponential growth, meaning that

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \sin \measuredangle(E^1_{\varphi^t(\omega)}, E^2_{\varphi^t(\omega)}) = 0. \tag{12}$$

If the flow φ^t is ergodic, then the real numbers (11) and the dimensions of the associated subbundles are constant μ almost everywhere and we will denote them by $\lambda_1(A)$ and $\lambda_2(A)$. We say that A has *trivial Lyapunov spectrum* or *one-point Lyapunov spectrum* (respectively *simple Lyapunov spectrum*) if for μ a.e. $\omega \in M$, $\lambda_1(A, \omega) = \lambda_2(A, \omega)$ (respectively $\lambda_1(A, \omega) > \lambda_2(A, \omega)$).

2.5 The Search for Positive Lyapunov Exponents

A positive Lyapunov exponent gives us the average exponential rate of divergence of two neighboring orbits whereas a negative Lyapunov exponent gives us the average exponential rate of convergence of two neighboring orbits. Zero Lyapunov exponents gives us the lack of any kind of asymptotic exponential behaviour. The nonuniform hyperbolic theory [13] guarantees a invariant manifold theory in the presence of non-zero Lyapunov exponents. These geometric considerations are the basis of most of the central results in today's dynamical systems. Hence, there can be no doubt that pursuing non-zero Lyapunov exponents is an important feature in dynamics over the last 60 years (see e.g. [40]). Some criteria for the positivity of the Lyapunov exponents were obtained by Cornelis and Wojtkowski [26], and Ledrappier [35] and later Knill [42] and Nerurkar [41] showed that for a C^0-dense set of certain cocycles we have non-zero Lyapunov exponents. Arnold and Cong [8] proved the L^p-denseness of positive Lyapunov exponents and their technique was generalized in [20]. The use of a classical method developed by Moser and linked to the concept of rotation number allowed Fabbri and Johnson to obtain abundance of positive Lyapunov exponents for linear differential systems evolving on $SL(2, \mathbb{R})$ on the fiber and displaying a translation on the torus on the base (see [29, 31, 32] and also the work with Zampogni [33]). Due to area-preserving invariance, obtaining a positive Lyapunov exponent $\lambda > 0$ in $SL(2, \mathbb{R})$ allows us to obtain a negative Lyapunov exponent $-\lambda < 0$ and thus all Lyapunov exponents are different. A variety of results guaranteeing the positivity of Lyapunov exponents for strong topologies established recently bring out different new approaches [21, 24, 28, 47, 48]. As an example, in [11], Avila obtained prevalence of simple spectrum in a rather wide range of topologies and on the two dimensional case. The topology used to compare perturbations turn to be crucial to the conclusions.

3 The L^p Case

3.1 Towards Zero Lyapunov Exponents

The L^p-generic description of the Lyapunov spectrum for general linear differential systems was first studied in [20] by the author and Vilarinho following the pioneering approaches by Arnold-Cong and Arbieto-Bochi [4, 7]. In [20] was proved that the class of *accessible* (*twisting*) linear differential systems, a wider class that includes cocycles that evolve in $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$ and $Sp(2n, \mathbb{R})$, have a trivial Lyapunov spectrum L^p-generically. If we consider the stronger C^0-norm, then Millionshchikov's work [40] in the late sixties shows that the generic behaviour changes. We will consider this C^0 case in Sect. 4. For the time being we now describe the recent results by the author, Amaro and Vilarinho. In rough terms in [2] was obtained that for an L^p-generic choice of a kinetic linear differential system (as in (2)) and for almost every driving realization, no matter what position and momentum $(x(0), \dot{x}(0))$ we chose as initial conditions, the asymptotic exponential behaviour of the solutions will be the same.

In [2] was proved the following result which is the kinetic version of [20, Theorem 1].

Theorem 1 ([2]) *For all* $1 \leq p < \infty$ *there exists a* σ_p*-residual subset* $\mathcal{R} \in \mathcal{K}^1$ *such that any* $A \in \mathcal{R}$ *has one-point spectrum.*

The two main components to prove Theorem 1 are Propositions 3.1 and 3.2. Once we establish these two results the proof of Theorem 1 is easily obtained. To prove Proposition 3.1 we used Lemma 2.2 and through a perturbation we caused a rotational effect of Oseledets directions. Unfortunately, rotating in $\mathcal{K}^1$ is much more difficult and the way to overcome this problem is to induce rotations via translations in the projective plane. In summary we perform (fake) rotations but remain in the kinetic class. Once we know how to 'rotate', a classical Mañé argument (see e.g. [14, 15, 22, 23]) allows us to get:

Proposition 3.1 *Given* $A \in \mathcal{K}^1$ *and* $\varepsilon, \delta > 0$*, there exists* $B \in \mathcal{K}^1$ *such that* $\sigma_1(A, B) < \varepsilon$ *and*

$$\lambda_1(B) \leq \frac{\lambda_1(A) + \lambda_2(A)}{2} + \delta. \tag{13}$$

Inequality (13) is used to decrease the upper Lyapunov exponent of a perturbation of the original linear differential system.

Finally, in Proposition 3.2 we obtain the upper semi-continuity of the top Lyapunov exponent function with respect to the L^p topology. We define the function

$$\begin{aligned} \mathscr{L}\colon (\mathcal{G}^1,\sigma_p) &\longrightarrow \mathbb{R} \\ A &\longmapsto \int_M \lambda_1(A,\omega)\,d\mu(\omega). \end{aligned} \tag{14}$$

Clearly, if μ is ergodic for the flow φ^t we have $\mathscr{L}(A)=\lambda_1(A)$.

Proposition 3.2 *For all $1 \le p < \infty$, the function $\mathscr{L}$ is upper semicontinuous when we endow $\mathcal{G}^1$ with the σ_p-topology, that is, for all $A \in \mathcal{G}^1$ and $\varepsilon > 0$ there is $\delta > 0$ such that $\sigma_p(A,B) < \delta$ implies $\mathscr{L}(B) < \mathscr{L}(A) + \varepsilon$.*

In order to prove that $\mathscr{L}$ is upper semi-continuous when $\mathcal{G}^1$ is endowed with the σ_p metric defined in Sect. 2.3.1 we must deal with the two main continuity-like problems:

Step 1 The first had already appeared [4, 20]. Indeed, it was the main step in [4] in order to improve from L^p-dense (cf. [7]) to L^p-residual. We are talking about the way it is used a simple measure-theoretical result (in brief terms that L^1 functions are L^∞ in a large part of the domain) to still guarantee continuity properties even under L^p-regularity.

Step 2 The second one is also a problem on continuity but a bit more difficult. This time on continuous dependence of solutions of differential equations. Notice that the function $\mathscr{L}$ in (14) is defined using the Lyapunov exponent which in turn is defined using the solution Φ^t_A and not the infinitesimal generator which is precisely the input on the σ_p-topology. So we need to get that solutions Φ^t_A and Φ^t_B are σ_p-near if its corresponding infinitesimal generators A and B are σ_p-near.

3.2 *Towards Non-zero Lyapunov Exponents*

The L^p-dense characterization of the Lyapunov spectrum for general linear differential systems was also considered in [20] generalizing this time the work by Arnold-Cong [8]. In [20] was proved that the class of *accessible* (a *twisting* type of property) and *saddle-conservative* (a *pinching* type of property) linear differential systems, a wider class that includes again cocycles that evolve in $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R})$, have simple Lyapunov spectrum L^p-densely. Recently, in [3], was proved a corresponding result for kinetic linear differential systems. Hence, we get in particular that the residual of Theorem 1 cannot contain L^p-open sets. We state now this result which establishes the existence of a σ_p-dense subset of $\mathcal{K}^1$ displaying simple spectrum:

Theorem 2 *Let $\varphi^t : M \to M$ be ergodic. For any $A \in \mathcal{K}^1$, $1 \le p < \infty$ and $\epsilon > 0$, there exists $B \in \mathcal{K}^1$ exhibiting simple Lyapunov spectrum satisfying $\sigma_p(A,B) < \epsilon$.*