Kurt Marti

Stochastic Optimization Methods

Applications in Engineering and Operations Research

Fourth Edition

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Preface

Optimization problems in practice depend mostly on several model parameters, noise factors, uncontrollable parameters, etc., which are not given fixed quantities at the planning stage. Due to several types of stochastic uncertainties (physical uncertainty, economic uncertainty, statistical uncertainty, model uncertainty), these parameters must be modeled by random variables having a certain probability distribution. In most cases at least certain moments of this distribution are known.

In order to cope with these uncertainties, a basic procedure in the behavior of the structure/system from the prescribed performance (output, behavior), i.e., the *tracking error*, is compensated by (online) input corrections. However, the online correction of a system/structure is often time consuming and causes mostly increasing expenses (correction, repair, or recourse costs). Very large recourse costs may arise in case of damages or failures of the plant. This can be omitted to a large extent by taking into account already at the planning stage the possible consequences of the tracking errors and the known prior and sample information about the random data of the problem. Hence, instead of relying on ordinary deterministic parameter optimization methods—based on some nominal parameter values—and applying then just some correction actions, stochastic optimization methods should be applied: Incorporating the consequences of stochastic parameter variations into the optimization process, large and increasing recourse, repair, recovery costs can be omitted or at least reduced to a large extent.

Consequently, for the computation of robust optimal decisions/designs, i.e., optimal decisions which are insensitive with respect to random parameter variations, appropriate deterministic substitute problems must be formulated first. Based on decision theoretical principles, these substitute problems depend on probabilities of failure/success and/or on more general expected cost/loss terms. Since probabilities and expectations are defined by multiple integrals in general, the resulting often nonlinear and also non-convex deterministic substitute problems can be solved by approximate methods only. Two basic types of deterministic substitute problems occur mostly in practice:

- *Minimization of the expected primary costs subject to expected recourse cost constraints (reliability constraints) and remaining deterministic constraints, e.g., box constraints.*
- *Expected Total Cost Minimization Problems subject to deterministic constraints.*

In case of piecewise constant cost functions, probabilistic objective functions and/or probabilistic constraints occur.

Main analytical properties of the substitute problems have been examined in the first three editions of the book, where also appropriate deterministic and stochastic approximation and solution procedures can be found.

The aim of the present fourth edition is the presentation of updated methods for the transformation of actual technical and economic optimization problems with random parameters into appropriate deterministic substitute problems. Hence, updated analytical and numerical tools are provided for the approximate computation of robust optimal decisions/designs/control, as needed in concrete engineering/economic applications.

Last but not least I would like to thank Dipl. Math. Ina Stein, Munich, for her excellent support in the LaTeX-typesetting as well as in the final proofreading. Moreover, I am indebted to Springer Nature for inviting a new edition of the monograph *Stochastic Optimization Methods*. I would like to thank especially the Senior Editor for Business/Economics/Operations Research of Springer-Verlag Heidelberg, Germany, Christian Rauscher and the Springer Editors Yvonne Schwark-Reiber, Books Editorial Projects Management, and Jialin Yan, Book Editor Operations Research and Management, Information Systems and Applied Statistics, for their advice during the preparation of this new edition.

Munich, Germany Kurt Marti
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Contents

Chapter 1
Stochastic Optimization Methods

Abstract Basic methods for treating stochastic optimization problems (SOP), hence, optimization problems with random data are presented: Optimization problems in practice depend mostly on several model parameters, noise factors, uncontrollable parameters, etc., which are not given fixed quantities at the planning stage. Typical examples from engineering and economics/operations research are: Material parameters (e.g., elasticity moduli, yield stresses, allowable stresses, moment capacities, specific gravity), external loadings, friction coefficients, moments of inertia, length of links, mass of links, location of the center of gravity of links, manufacturing errors, tolerances, noise terms, demand parameters, technological coefficients in input-output functions, cost factors, interest rates, exchange rates, etc. Due to several types of stochastic uncertainties (physical uncertainty, economic uncertainty, statistical uncertainty, model uncertainty) these parameters must be modeled by random variables having a certain probability distribution. In most cases at least certain moments of this distribution are known.n order to cope with these uncertainties, a basic procedure in the engineering/economic practice is to replace first the unknown parameters by some chosen nominal values, e.g., estimates, guesses, of the parameters. Then, the resulting and mostly increasing deviation of the performance (output, behavior) of the structure/system from the prescribed performance (output, behavior), i.e., the *tracking error*, is compensated by (online) input corrections. However, the online correction of a system/structure is often time consuming and causes mostly increasing expenses (correction or recourse costs). Very large recourse costs may arise in case of damages or failures of the plant. This can be omitted to a large extent by taking into account already at the planning stage the possible consequences of the tracking errors and the known prior and sample information about the random data of the problem. Hence, instead of relying on ordinary deterministic parameter optimization methods - based on some nominal parameter values—and applying then just some correction actions, stochastic optimization methods should be applied: Incorporating stochastic parameter variations into the optimization process, expensive and increasing online correction expenses can be omitted or at least reduced to a large extent. Consequently, for the computation of robust optimal decisions/designs, i.e., optimal decisions which are insensitive with respect to random parameter variations, appropriate deterministic substitute problems must be formulated first. Based on decision theoretical principles, these substitute problems depend on probabilities

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of failure/success and/or on more general expected cost/loss terms. Two basic types of deterministic substitute problems occur mostly in practice:

- *Reliability-Based Optimization Problems*: primary cost minimization subject to expected recourse (correction) cost constraints: Minimization of the expected primary costs subject to expected recourse cost constraints (reliability constraints) and remaining deterministic constraints, e.g., box constraints. In case of piecewise constant cost functions, probabilistic objective functions and/or probabilistic constraints occur;
- *Expected Total Cost Minimization Problems*: Minimization of the expected total costs (costs of construction, design, recourse/correction, repair costs, etc.) subject to the remaining deterministic constraints.

Since probabilities and expectations are defined by multiple integrals in general, the resulting often nonlinear and also non-convex deterministic substitute problems can be solved by approximate methods only.

1.1 Introduction

Many concrete problems from engineering, economics, operations research, etc., can be formulated by an optimization problem of the type

$$\min \quad f_0(a,x) \tag{1.1a}$$

$$\text{s.t.}$$

$$f_i(a,x) \le 0, \qquad i = 1,\dots,m_f \tag{1.1b}$$

$$g_i(a,x) = 0, \qquad i = 1,\dots,m_g \tag{1.1c}$$

$$x \in D_0. \tag{1.1d}$$

Here, the objective (goal) function $f_0 = f_0(a,x)$ and the constraint functions $f_i = f_i(a,x), i = 1,\dots,m_f$ and $g_i = g_i(a,x), i = 1,\dots,m_g$, defined on a joint subset of $\mathbb{R}^\nu \times \mathbb{R}^r$, depend on a decision, design, control orinput vector $x = (x_1, x_2, \dots, x_r)^T$ and a vector $a = (a_1, a_2, \dots, a_\nu)^T$ ofmodel parameters. Typical model parameters in technical applications, operations research, and economics are material parameters, external load parameters, cost factors, technological parameters in input-output operators, demand factors. Furthermore, manufacturing and modeling errors, disturbances or noise factors, etc., may occur. Frequent decision, control, or input variables are material, topological, geometrical and cross-sectional design variables in structural optimization [23], forces and moments in optimal control of dynamic systems and factors of production in operations research and economic design.

The objective function (1.1a) to be optimized describes the aim, the goal of the modeled optimal decision/design problem or the performance of a technical, economic system or process to be controlled optimally. Furthermore, the constraints

(1.1b)–(1.1d) represent the operating conditions guaranteeing a safe structure, a correct functioning of the underlying system, process, etc. Note that the constraint (1.1d) with a given, fixed convex subset $D_0 \subset \mathbb{R}^r$ summarizes all (deterministic) constraints being independent of unknown model parameters a, as, e.g., box constraints:

$$x^L \leq x \leq x^U \tag{1.1e}$$

with given bounds x^L, x^U.

Important concrete optimization problems, which may be formulated, at least approximate, this way, are problems from optimal design of mechanical structures and structural systems [1, 23, 43, 48], adaptive trajectory planning for robots [2, 3, 14, 30, 37, 45], adaptive control of dynamic system [46, 47], optimal design of economic systems [22], production planning, manufacturing [26, 38] and sequential decision processes [34], etc.

In *optimal control*, cf. Chap. 3, the input vector $x := u(\cdot)$ is interpreted as a function, a *control or input function* $u = u(t)$, $t_0 \leq t \leq t_f$, on a certain given time interval $[t_0, t_f]$. Moreover, see Chap. 3, the objective function $f_0 = f_0(a, u(\cdot))$ is defined by a certain integral over the time interval $[t_0, t_f]$. In addition, the constraint functions $f_j = f_j(a, u(\cdot))$ are defined by integrals over $[t_0, t_f]$, or $f_j = f_j(t, a, u(t))$ may be functions of time t and the control input $u(t)$ at time t.

A basic problem in practice is that the vector of model parameters $a = (a_1, \ldots, a_\nu)^T$ is not a given, fixed quantity. Model parameters are often unknown, only partly known and/or may vary randomly to some extent.

Several techniques have been developed in the recent years in order to cope with uncertainty with respect to model parameters a. A well-known basic method, often used in engineering practice, is the following two-step procedure [3, 14, 37, 45, 46]:

(I) **Parameter Estimation and Approximation**:
First, replace first the ν-vector a of the unknown or stochastic varying model parameters $a_1, \ldots, a_\nu$ by some estimated/chosen fixed vector a_0 of so-called *nominal* values a_{0l}, $l = 1, \ldots, \nu$.
Then, apply an optimal decision (control) $x^* = x^*(a_0)$ with respect to the resulting approximate optimization problem

$$\min \quad f_0(a_0, x) \tag{1.2a}$$

s.t.

$$f_i(a_0, x) \leq 0, \qquad i = 1, \ldots, m_f \tag{1.2b}$$

$$g_i(a_0, x) = 0, \qquad i = 1, \ldots, m_g \tag{1.2c}$$

$$x \in D_0. \tag{1.2d}$$

Due to the deviation of the actual parameter vector a from the nominal vector a_0 of model parameters, deviations of the actual state, trajectory or performance of the system from the prescribed state, trajectory, goal values occur.

(II) **Compensation or correction**:
Then, the deviation of the actual state, trajectory or performance of the system from the prescribed values/functions is compensated by online measurement and correction actions (decisions or controls). Consequently, in general, increasing measurement and correction expenses result in course of time.

Considerable improvements of this standard procedure can be obtained by taking into account already at the planning stage, i.e., offline, the mostly available a priori (e.g., the type of random variability) and sample information about the parameter vector a. Indeed, based, e.g., on some structural insight, or by parameter identification methods, regression techniques, calibration methods, etc., in most cases information about the vector a of model parameters can be extracted. Repeating this information gathering procedure at some later time points $t_j > t_0$ (= initial time point), $j = 1, 2, \ldots,$ adaptive decision/control procedures occur [34].

Based on the inherent random nature of the parameter vector a, the observation or measurement mechanism, resp., or adopting a Bayesian approach concerning unknown parameter values [6], here we make the following basic assumption:

Stochastic (Probabilistic) Uncertainty : The unknown parameter vector a is a realization

$$a = a(\omega)\omega \in \Omega, \tag{1.3}$$

of a random ν-vector $a(\omega)$ on a certain probability space $(\Omega, \mathcal{A}_0, P)$, where the probability distribution $P_{a(\cdot)}$ of $a(\omega)$ is known, or it is known that $P_{a(\cdot)}$ lies within a given range W of probability measures on $\mathbb{R}^\nu$. Using a Bayesian approach, the probability distribution $P_{a(\cdot)}$ of $a(\omega)$ may also describe the subjective or personal probability of the decision maker, the designer.

Hence, in order to take into account the stochastic variations of the parameter vector a, to incorporate the a priori and/or sample information about the unknown vector a, resp., the standard approach "insert a certain nominal parameter vector a_0, and correct then the resulting error", must be replaced by a more appropriate deterministic substitute problem for the basic optimization problem (1.1a)–(1.1d) under stochastic uncertainty.

1.2 Deterministic Substitute Problems: Basic Formulation

The proper selection of a deterministic substitute problem is a decision theoretical task, see [27]. Hence, for (1.1a)–(1.1d) we have first to consider the *outcome map*

$$\begin{aligned} e &= e(a, x) \\ &:= \Big(f_0(a, x), f_1(a, x), \ldots, f_{m_f}(a, x), g_1(a, x), \ldots, g_{m_g}(a, x)\Big)^T, \end{aligned} \tag{1.4a}$$

$$a \in \mathbb{R}^\nu, x \in \mathbb{R}^r, (x \in D_0),$$

and to evaluate then the outcomes $e \in \mathcal{E} \subset \mathbb{R}^{1+m_0}$, $m_0 := m_f + m_g$, by means of certain loss or cost functions

$$\gamma_i : \mathcal{E} \to \mathbb{R}, \qquad i = 0, 1, \ldots, m \tag{1.4b}$$

with an integer $m \geq 0$. For the processing of the numerical outcomes $\gamma_i\big(e(a, x)\big)$, $i = 0, 1, \ldots, m$, there are two basic concepts:

1.2.1 Minimum or Bounded Expected Costs

Consider the vector of (conditional) expected losses or costs

$$\mathbf{F}(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix} := \begin{pmatrix} E\gamma_0(e(a(\omega), x)) \\ E\gamma_1(e(a(\omega), x)) \\ \vdots \\ E\gamma_m(e(a(\omega), x)) \end{pmatrix}, \qquad x \in \mathbb{R}^r, \tag{1.5}$$

where the (conditional) expectation "E" is taken with respect to the time history $\mathfrak{A} = \mathfrak{A}_t, (\mathfrak{A}_j) \subset \mathfrak{A}_0$ up to a certain time point t or stage j. A short definition of expectations is given in Sect. 1.3, for more details, see, e.g., [5, 18, 40].

Having different expected cost or performance functions $F_0, F_1, \ldots, F_m$ to be minimized or bounded, as a basic deterministic substitute problem for (1.1a)–(1.1d) with a random parameter vector $a = a(\omega)$ we may consider the multi-objective expected cost minimization problem

$$\text{"min"} \quad \mathbf{F}(x) \tag{1.6a}$$

$$\text{s.t.} \quad x \in D_0. \tag{1.6b}$$

Obviously, a good compromise solution x^* of this vector optimization problem should have at least one of the following properties [13, 41]:

Definition 1.1

(a) A vector $x^0 \in D_0$ is called a **functional-efficient** or **Pareto optimal** solution of the vector optimization problem (1.6a), (1.6b) if there is no $x \in D_0$ such that

$$F_i(x) \leq F_i(x^0), \qquad i = 0, 1, \ldots, m \tag{1.7a}$$

and

$$F_{i_0}(x) < F_{i_0}(x^0) \qquad \text{for at least one } i_0,\ 0 \leq i_0 \leq m. \tag{1.7b}$$

(b) A vector $x^0 \in D_0$ is called a **weak functional-efficient** or **weak Pareto optimal** solution of (1.6a)–(1.6b) if there is no $x \in D_0$ such that

$$F_i(x) < F_i(x^0), \quad i = 0, 1, \ldots, m \tag{1.8}$$

(Weak) Pareto optimal solutions of (1.6a)–(1.6b) may be obtained now by means of scalarizations of the vector optimization problem (1.6a)–(1.6b). Three main versions are stated in the following:

(I) *Minimization of primary expected cost/loss under expected cost constraints*

$$\min \quad F_0(x) \tag{1.9a}$$

s.t.

$$F_i(x) \le F_i^{\max}, \quad i = 1, \ldots, m \tag{1.9b}$$

$$x \in D_0. \tag{1.9c}$$

Here, $F_0 = F_0(x)$ is assumed to describe the primary goal of the design/decision-making problem, while $F_i = F_i(x)$, $i = 1, \ldots, m$, describe secondary goals. Moreover, $F_i^{\max}$, $i = 1, \ldots, m$, denote given upper cost/loss bounds.

Remark 1.1 An optimal solution x^* of (1.9a)–(1.9c) is a weak Pareto optimal solution of (1.6a)–(1.6b).

(II) *Minimization of the total weighted expected costs*
Selecting certain positive weight factors $c_0, c_1, \ldots, c_m$, the expected weighted total costs are defined by

$$\tilde{F}(x) := \sum_{i=0}^{m} c_i F_i(x) = Ef\Big(a(\omega), x\Big), \tag{1.10a}$$

where

$$f(a, x) := \sum_{i=0}^{m} c_i \gamma_i\Big(e(a, x)\Big). \tag{1.10b}$$

Consequently, minimizing the expected weighted total costs $\tilde{F} = \tilde{F}(x)$ subject to the remaining deterministic constraint (1.1d), the following deterministic substitute problem for (1.1a)–(1.1d) occurs

$$\min \quad \sum_{i=0}^{m} c_i F_i(x) \tag{1.11a}$$

$$\text{s.t.} \quad x \in D_0. \tag{1.11b}$$

Remark 1.2 Let $c_i > 0, i = 1, 1, \ldots, m$, be any positive weight factors. Then, an optimal solution x^* of (1.11a)–(1.11b) is a Pareto optimal solution of (1.6a)–(1.6b).

(III) *Minimization of the maximum weighted expected costs*
Instead of adding weighted expected costs, we may consider the maximum of the weighted expected costs:

$$\tilde{F}(x) := \max_{0 \le i \le m} c_i F_i(x) = \max_{0 \le i \le m} c_i E\gamma_i\Big(e\big(a(\omega), x\big)\Big). \tag{1.12}$$

Here again, $c_0, c_1, \ldots, c_m$, are positive weight factors.
Thus, minimizing $\tilde{F} = \tilde{F}(x)$ we have the deterministic substitute problem

$$\min \quad \max_{0 \le i \le m} c_i F_i(x) \tag{1.13a}$$

$$\text{s.t.} \quad x \in D_0. \tag{1.13b}$$

Remark 1.3 Let $c_i, i = 0, 1, \ldots, m$, be any positive weight factors. An optimal solution of x^* of (1.13a)–(1.13b) is a weak Pareto optimal solution of (1.6a)–(1.6b).

1.2.2 Minimum or Bounded Maximum Costs (Worst Case)

Instead of taking expectations, we may consider the worst case with respect to the cost variations caused by the random parameter vector $a = a(\omega)$. Hence, the random cost function

$$\omega \to \gamma_i\Big(e\big(a(\omega), x\big)\Big) \tag{1.14a}$$

is evaluated by means of

$$F_i^{\sup}(x) := \text{ess sup}\, \gamma_i\Big(e\big(a(\omega), x\big)\Big), \quad i = 0, 1, \ldots, m. \tag{1.14b}$$

Here, ess sup $(\ldots)$ denotes the (conditional) essential supremum with respect to the random vector $a = a(\omega)$, given information $\mathfrak{A}$, i.e., the infimum of the supremum of (1.14a) on sets $A \in \mathfrak{A}_0$ of (conditional) probability one, see, e.g., [40].

Consequently, the vector function $\mathbf{F} = \mathbf{F}^{\sup}(x)$ is then defined by

$$\mathbf{F}^{\sup}(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix} := \begin{pmatrix} \text{ess sup}\, \gamma_0\Big(e\big(a(\omega), x\big)\Big) \\ \text{ess sup}\, \gamma_1\Big(e\big(a(\omega), x\big)\Big) \\ \vdots \\ \text{ess sup}\, \gamma_m\Big(e\big(a(\omega), x\big)\Big) \end{pmatrix}. \tag{1.15}$$

Working with the vector function $\mathbf{F} = \mathbf{F}^{\mathrm{sup}}(x)$, we have then the vector minimization problem

$$\text{"min"} \quad \mathbf{F}^{\mathrm{sup}}(x) \tag{1.16a}$$

$$\text{s.t.} \quad x \in D_0. \tag{1.16b}$$

By scalarization of (1.16a)–(1.16b) we then obtain deterministic substitute problems for (1.1a)–(1.1d) related to the substitute problem (1.6a)–(1.6b) introduced in Sect. 1.2.1.

More details on the selection and solution of appropriate deterministic substitute problems for (1.1a)–(1.1d) are given in the next sections. Deterministic substitute problems for optimal control problems under stochastic uncertainty are considered in Chap. 3.

1.3 Optimal Decision/Design Problems with Random Parameters

In the optimal design of technical or economic structures/systems, in optimal decision problems arising in technical or economic systems, resp., two basic classes of criteria appear.

First there is a primary cost function

$$G_0 = G_0(a, x). \tag{1.17a}$$

Important examples are the total weight or volume of a mechanical structure, the costs of construction, design of a certain technical or economic structure/system, or the negative utility or reward in a general decision situation. Basic examples in optimal control, cf. Chap. 3, are the total run time, the total energy consumption of the process or a weighted mean of these two cost functions.

For the representation of the structural/system safety or failure, for the representation of the admissibility of the state, or for the formulation of the basic operating conditions of the , certain **state, performance or response functions**

$$y_i = y_i(a, x), \; i = 1, \ldots, m_y \tag{1.17b}$$

are chosen. In structural design these functions are also called "limit state functions" or "safety margins". Frequent examples are some displacement, stress, load (force and moment) components in structural design, or more general system output functions in engineering design. Furthermore, production functions and several cost functions are possible performance functions in production planning problems, optimal mix problems, transportation problems, allocation problems and other problems of economic decision.

In (1.17a,b), the design or input vector x denotes the r-vector of design or input variables, $x_1, x_2, \ldots, x_r$, as, e.g., structural dimensions, sizing variables, such as cross-sectional areas, thickness in structural design, or factors of production, actions in economic decision problems. For the decision, design or input vector x one has mostly some basic deterministic constraints, e.g., nonnegativity constraints, box constraints, represented by

$$x \in D, \tag{1.17c}$$

where D is a given convex subset of $\mathbb{R}^r$. Moreover, a is the ν-vector of model parameters. In optimal structural/engineering design

$$a = \begin{pmatrix} p \\ R \end{pmatrix} \tag{1.17d}$$

is composed of the following two subvectors: R is the m-vector of the acting external loads or structural/system inputs, e.g., wave, wind loads, payload, etc. Moreover, p denotes the $(\nu - m)$-vector of the further model parameters, as, e.g., material parameters, like strength parameters, yield/allowable stresses, elastic moduli, plastic capacities, etc., of the members of a mechanical structure, parameters of an electric circuit, such as resistances, inductances, capacitances, the manufacturing tolerances and weight or more general cost coefficients.

In linear programming, as, e.g., in production planning problems,

$$a = (A, b, c) \tag{1.17e}$$

is composed of the $m \times r$ matrix A of technological coefficients, the demand m-vector b and the r-vector c of unit costs.

Based on the m_y-vector of state functions

$$y(a, x) := \Big(y_1(a, x), y_2(a, x), \ldots, y_{m_y}(a, x)\Big)^T, \tag{1.17f}$$

the admissible or safe states of the structure/system can be characterized by the condition

$$y(a, x) \in B, \tag{1.17g}$$

where B is a certain subset of $\mathbb{R}^{m_y}$; $B = B(a)$ may depend also on some model parameters.

In production planning problems, typical operating conditions are given, cf. (1.17e), by

$$y(a, x) := Ax - b \ge 0 \qquad \text{or} \qquad y(a, x) = 0, \ x \ge 0. \tag{1.18a}$$

In mechanical structures/structural systems, the safety (survival) of the structure/system is described by the operating conditions

$$y_i(a,x) > 0 \quad \text{for all} \quad i = 1, \ldots, m_y \tag{1.18b}$$

with state functions $y_i = y_i(a,x)$, $i = 1, \ldots, m_y$, depending on certain response components of the structure/system, such as displacement, stress, force, moment components.

Hence, a failure occurs if and only if the structure/system is in the i-th failure mode (failure domain)

$$y_i(a,x) \leq 0 \tag{1.18c}$$

for at least one index i, $1 \leq i \leq m_y$.

Note 1.1 The number m_y of safety margins or limit state functions $y_i = y_i(a,x)$, $i = 1, \ldots, m_y$, may be very large. For example, in optimal plastic design the limit state functions are determined by the extreme points of the admissible domain of the dual pair of static/kinematic LPs related to the equilibrium and linearized convex yield condition, see [32, 33].

Basic problems in optimal decision/design are

(I) *Primary (construction, planning, investment, etc.) cost minimization under operating or safety conditions*

$$\min \quad G_0(a,x) \tag{1.19a}$$

s.t.

$$y(a,x) \in B \tag{1.19b}$$

$$x \in D. \tag{1.19c}$$

Obviously we have $B = (0, +\infty)^{m_y}$ in (1.18b) and $B = [0, +\infty)^{m_y}$ or $B = \{0\}$ in (1.18a).

(II) *Failure or recourse cost minimization under primary cost constraints*

$$\text{“min”} \quad \gamma\Big(y(a,x)\Big) \tag{1.20a}$$

s.t.

$$G_0(a,x) \leq G^{\max} \tag{1.20b}$$

$$x \in D. \tag{1.20c}$$

In (1.20a) $\gamma = \gamma(y)$ is a scalar or vector valued cost/loss function evaluating violations of the operating conditions (1.19b). Depending on the application, these costs are called “failure” or “recourse” costs [20, 21, 31, 39, 43, 44]. As already discussed in Sect. 1.1, solving problems of the above type, a basic difficulty is the uncertainty about the true value of the vector a of model parameters or the (random) variability of a. In practice, due to several types of uncertainties such as, see [49],

- physical uncertainty (variability of physical quantities, like material, loads, dimensions, etc.)
- economic uncertainty (trade, demand, costs, etc.)
- statistical uncertainty (e.g., estimation errors of parameters due to limited sample data)
- model uncertainty (model errors).

The ν-vector a of model parameters must be modeled by a random vector

$$a = a(\omega), \omega \in \Omega, \tag{1.21a}$$

on a certain probability space $(\Omega, \mathfrak{A}_0, P)$ with sample space Ω having elements ω, see (1.3). For the mathematical representation of the corresponding (conditional) probability distribution $P_{a(\cdot)} = P_{a(\cdot)}^{\mathfrak{A}}$ of the random vector $a = a(\omega)$ (given the time history or information $\mathfrak{A} \subset \mathfrak{A}_0$), two main distribution models are taken into account in practice:

(i) Discrete probability distributions,
(ii) Continuous probability distributions.

In the first case there is a finite or countably infinite number $l_0 \in \mathbb{N} \cup \{\infty\}$ of realizations or scenarios $a^l \in \mathbb{R}^\nu, \quad l = 1, \ldots, l_0$,

$$P\Big(a(\omega) = a^l\Big) = \alpha_l, \quad l = 1, \ldots, l_0, \tag{1.21b}$$

taken with probabilities $\alpha_l, \ l = 1, \ldots, l_0$.

In the second case, the probability that the realization $a(\omega) = a$ lies in a certain (measurable) subset $B \subset \mathbb{R}^\nu$ is described by the multiple integral

$$P\Big(a(\omega) \in B\Big) = \int_B \varphi(a)\, da \tag{1.21c}$$

with a certain probability density function $\varphi = \varphi(a) \geq 0, \ a \in \mathbb{R}^\nu, \ \int \varphi(a) da = 1$.

The properties of the probability distribution $P_{a(\cdot)}$ may be described—fully or in part—by certain numerical characteristics, called parameters of $P_{a(\cdot)}$. These distribution parameters $\theta = \theta_h$ are obtained by considering expectations

$$\theta_h := Eh\Big(a(\omega)\Big) \tag{1.22a}$$

of some (measurable) functions

$$(h \circ a)(\omega) := h\Big(a(\omega)\Big) \tag{1.22b}$$

composed of the random vector $a = a(\omega)$ with certain (measurable) mappings

$$h : \mathbb{R}^{\nu} \longrightarrow \mathbb{R}^{s_h}, \quad s_h \geq 1. \tag{1.22c}$$

According to the type of the probability distribution $P_{a(\cdot)}$ of $a = a(\omega)$, the expectation $Eh\Big(a(\omega)\Big)$ is defined, cf. [4, 5], by

$$Eh\Big(a(\omega)\Big) = \begin{cases} \sum\limits_{l=1}^{l_0} h\Big(a^l\Big)\, \alpha_l, & \text{in the discrete case (1.21b)} \\ \int\limits_{\mathbb{R}^{\nu}} h(a)\varphi(a)\, da, & \text{in the continuous case (1.21c).} \end{cases} \tag{1.22d}$$

Further distribution parameters θ are functions

$$\theta = \Psi(\theta_{h_1}, \ldots, \theta_{h_s}) \tag{1.23}$$

of certain "h-moments" $\theta_{h_1}, \ldots, \theta_{h_s}$ of the type (1.22a). Important examples of the type (1.22a), (1.23), resp., are the expectation

$$\overline{a} = Ea(\omega) \quad (\text{for } h_1(a) := \overline{a},\ \overline{a} \in \mathbb{R}^{\nu}) \tag{1.24a}$$

and the covariance matrix

$$Q := E\Big(a(\omega) - \overline{a}\Big)\Big(a(\omega) - \overline{a}\Big)^T = Ea(\omega)a(\omega)^T - \overline{a}\,\overline{a}^T \tag{1.24b}$$

of the random vector $a = a(\omega)$.

Due to the stochastic variability of the random vector $a(\cdot)$ of model parameters, and since the realization $a(\omega) = a$ is not available at the decision-making stage, the optimal design problem (1.19a)–(1.19c) or (1.20a)–(1.20c) under stochastic uncertainty cannot be solved directly.

Hence, appropriate deterministic substitute problems must be chosen taking into account the randomness of $a = a(\omega)$, cf. Sect. 1.2.

1.4 Deterministic Substitute Problems in Optimal Decision/Design

According to Sect. 1.2, a basic deterministic substitute problem in optimal design under stochastic uncertainty is the minimization of the total expected costs including the expected costs of failure

$$\min \quad c_G \cdot EG_0\Big(a(\omega), x\Big) + c_f \cdot p_f(x) \tag{1.25a}$$

$$\text{s.t.} \quad x \in D. \tag{1.25b}$$

Here,

$$p_f = p_f(x) := P\left(y\Big(a(\omega), x\Big) \notin B\right) \tag{1.25c}$$

is the probability of failure or the probability that a safe function of the structure, the system is not guaranteed. Furthermore, c_G is a certain weight factor, and $c_f > 0$ describes the failure or recourse costs. In the present definition of expected failure costs, constant costs for each realization $a = a(\omega)$ of $a(\cdot)$ are assumed. Obviously, it is

$$p_f(x) = 1 - p_s(x) \tag{1.25d}$$

with the probability of safety or survival

$$p_s(x) := P\left(y\Big(a(\omega), x\Big) \in B\right). \tag{1.25e}$$

In case (1.18b) we have

$$p_f(x) = P\left(y_i\Big(a(\omega), x\Big) \le 0 \text{ for at least one index } i,\ 1 \le i \le m_y\right). \tag{1.25f}$$

The objective function (1.25a) may be interpreted as the Lagrangian (with given cost multiplier c_f) of the following reliability-based optimization (RBO) problem, cf. [1, 29, 39, 43, 49]:

$$\min \quad EG_0\Big(a(\omega), x\Big) \tag{1.26a}$$

s.t.

$$p_f(x) \le \alpha^{\max} \tag{1.26b}$$

$$x \in D, \tag{1.26c}$$

where $\alpha^{\max} > 0$ is a prescribed maximum failure probability, e.g., $\alpha^{\max} = 0.001$, cf. (1.19a)–(1.19c).

The "dual" version of (1.26a)–(1.26c) reads

$$\min \quad p_f(x) \tag{1.27a}$$

s.t.

$$EG_0\Big(a(\omega), x\Big) \le G^{\max} \tag{1.27b}$$

$$x \in D \tag{1.27c}$$

with a maximal (upper) cost bound $G^{\max}$, see (1.20a)–(1.20c).

1.4.1 Expected Cost or Loss Functions

Further substitute problems are obtained by considering more general expected failure or recourse cost functions

$$\Gamma(x) = E\gamma\left(y\Big(a(\omega), x\Big)\right) \tag{1.28a}$$

arising from structural systems weakness or failure, or because of false operation. Here,

$$y\Big(a(\omega), x\Big) := \left(y_1\Big(a(\omega), x\Big), \ldots, y_{m_y}\Big(a(\omega), x\Big)\right)^T \tag{1.28b}$$

is again the random vector of state or performance functions, and

$$\gamma : \mathbb{R}^{m_y} \to \mathbb{R}^{m_\gamma} \tag{1.28c}$$

is a scalar or vector valued cost or loss function. In case $B = (0, +\infty)^{m_y}$ or $B = [0, +\infty)^{m_y}$ it is often assumed that $\gamma = \gamma(y)$ is a non-increasing function, hence,

$$\gamma(y) \geq \gamma(z), \quad \text{if} \quad y \leq z, \tag{1.28d}$$

where inequalities between vectors are defined component-by-component.

Example 1.1 If $\gamma(y) = 1$ for $y \in B^c$ (complement of B) and $\gamma(y) = 0$ for $y \in B$, then $\Gamma(x) = p_f(x)$.

Example 1.2 Suppose that $\gamma = \gamma(y)$ is a nonnegative measurable scalar function on $\mathbb{R}^{m_y}$ such that

$$\gamma(y) \geq \gamma_0 > 0 \text{ for all } y \notin B \tag{1.29a}$$

with a constant $\gamma_0 > 0$. Then for the probability of failure we find the following upper bound

$$p_f(x) = P\left(y\Big(a(\omega), x\Big) \notin B\right) \leq \frac{1}{\gamma_0} E\gamma\left(y\Big(a(\omega), x\Big)\right), \tag{1.29b}$$

where the right-hand side of (1.29b) is obviously an expected cost function of type (1.28a)–(1.28c). Hence, the condition (1.26b) can be guaranteed by the expected cost constraint

$$E\gamma\Big(y\big(a(\omega), x\big)\Big) \leq \gamma_0 \alpha^{\max}. \tag{1.29c}$$

Example 1.3 If the loss function $\gamma(y)$ is defined by a vector of individual loss functions γ_i for each state function $y_i = y_i(a, x)$, $i = 1, \ldots, m_y$, hence,

$$\gamma(y) = \Big(\gamma_1(y_1), \ldots, \gamma_{m_y}(y_{m_y})\Big)^T, \tag{1.30a}$$

then

$$\Gamma(x) = \left(\Gamma_1(x), \ldots, \Gamma_{m_y}(x)\right)^T, \ \Gamma_i(x) := E\gamma_i\left(y_i\left(a(\omega), x\right)\right), 1 \le i \le m_y, \tag{1.30b}$$

i.e., the m_y state functions $y_i, i = 1, \ldots, m_y$, will be treated separately.

Working with the more general expected failure or recourse cost functions $\Gamma = \Gamma(x)$, instead of (1.25a)–(1.25c), (1.26a)–(1.26c) and (1.27a)–(1.27c) we have the related substitute problems:

(I) *Expected total cost minimization*

$$\min \quad c_G EG_0\left(a(\omega), x\right) + c_f^T \Gamma(x), \tag{1.31a}$$

$$\text{s.t.} \quad x \in D. \tag{1.31b}$$

(II) *Expected primary cost minimization under expected failure or recourse cost constraints*

$$\min \quad EG_0\left(a(\omega), x\right) \tag{1.32a}$$

s.t.

$$\Gamma(x) \le \Gamma^{\max} \tag{1.32b}$$

$$x \in D, \tag{1.32c}$$

(III) *Expected failure or recourse cost minimization under expected primary cost constraints*

$$\min \quad \Gamma(x) \tag{1.33a}$$

s.t.

$$EG_0\left(a(\omega), x\right) \le G^{\max} \tag{1.33b}$$

$$x \in D. \tag{1.33c}$$

Here, c_G, c_f are (vectorial) weight coefficients, $\Gamma^{\max}$ is the vector of upper loss bounds, and "min" indicates again that $\Gamma(x)$ may be a vector valued function.

1.5 Basic Properties of Deterministic Substitute Problems

As can be seen from the conversion of an optimization problem with random parameters into a deterministic substitute problem, cf. Sect. 1.4.1, a central role is played by expectation or mean value functions of the type

$$\Gamma(x) = E\gamma\left(y\Big(a(\omega), x\Big)\right), \qquad x \in D_0, \tag{1.34a}$$

or more general

$$\Gamma(x) = Eg\Big(a(\omega), x\Big), \qquad x \in D_0. \tag{1.34b}$$

Here, $a = a(\omega)$ is a random ν-vector, $y = y(a, x)$ is an m_y-vector valued function on a certain subset of $\mathbb{R}^\nu \times \mathbb{R}^r$, and $\gamma = \gamma(z)$ is a real-valued function on a certain subset of $\mathbb{R}^{m_y}$.

Furthermore, $g = g(a, x)$ denotes a real-valued function on a certain subset of $\mathbb{R}^\nu \times \mathbb{R}^r$. In the following we suppose that the expectation in (1.34a)–(1.34b) exists and is finite for all input vectors x lying in an appropriate set $D_0 \subset \mathbb{R}^r$, cf. [7].

The following basic properties of the mean value functions Γ are needed in the following again and again.

Lemma 1.1 (Convexity) *Suppose that $x \to g\Big(a(\omega), x\Big)$ is convex a.s. (almost sure) on a fixed convex domain $D_0 \subset \mathbb{R}^r$. If $Eg\Big(a(\omega), x\Big)$ exists and is finite for each $x \in D_0$, then $\Gamma = \Gamma(x)$ is convex on D_0.*

Proof This property follows [20, 21, 27] directly from the linearity of the expectation operator. □

If $g = g(a, x)$ is defined by $g(a, x) := \gamma\Big(y(a, x)\Big)$, see (1.34a), then the above theorem yields the following result:

Corollary 1.1 *Suppose that γ is convex and $E\gamma\left(y\Big(a(\omega), x\Big)\right)$ exists and is finite for each $x \in D_0$.*

(a) If $x \to y\Big(a(\omega), x\Big)$ is linear a.s., then $\Gamma = \Gamma(x)$ is convex.

(b) If $x \to y\Big(a(\omega), x\Big)$ is convex a.s., and γ is a convex, monotoneous nondecreasing function, then $\Gamma = \Gamma(x)$ is convex.

It is well known [25] that a convex function is continuous on each open subset of its domain. A general sufficient condition for the continuity of Γ is given next.

Lemma 1.2 (Continuity) *Suppose that $Eg\Big(a(\omega), x\Big)$ exists and is finite for each $x \in D_0$, and assume that $x \to g\Big(a(\omega), x\Big)$ is continuous at $x_0 \in D_0$ a.s.. If there is a function $\psi = \psi\Big(a(\omega)\Big)$ having finite expectation such that*

$$\left|g\Big(a(\omega), x\Big)\right| \le \psi\Big(a(\omega)\Big) \text{ a.s. for all } x \in U(x_0) \cap D_0, \tag{1.35}$$

where $U(x_0)$ is a neighborhood of x_0, then $\Gamma = \Gamma(x)$ is continuous at x_0.

Proof The assertion can be shown by using Lebesgue's dominated convergence theorem, see, e.g., [27]. □

For the consideration of the differentiability of $\Gamma = \Gamma(x)$, let D denote an open subset of the domain D_0 of Γ.

Lemma 1.3 (Differentiability) *Suppose that*

(i) $Eg\Big(a(\omega), x\Big)$ *exists and is finite for each* $x \in D_0$,

(ii) $x \to g\Big(a(\omega), x\Big)$ *is differentiable on the open subset* D *of* D_0 *a.s. and*

(iii)

$$\left\| \nabla_x g\Big(a(\omega), x\Big) \right\| \le \psi\Big(a(\omega)\Big), \quad x \in D, \ a.s., \tag{1.36a}$$

where $\psi = \psi\Big(a(\omega)\Big)$ *is a function having finite expectation. Then the expectation of* $\nabla_x g\Big(a(\omega), x\Big)$ *exists and is finite,* $\Gamma = \Gamma(x)$ *is differentiable on* D *and*

$$\nabla\Gamma(x) = \nabla_x Eg\Big(a(\omega), x\Big) = E\nabla_x g\Big(a(\omega), x\Big), \quad x \in D. \tag{1.36b}$$

Proof Considering the difference quotients $\frac{\Delta\Gamma}{\Delta x_k}$, $k = 1, \ldots, r$, of Γ at a fixed point $x_0 \in D$, the assertion follows by means of the mean value theorem, inequality (1.36a) and Lebesgue's dominated convergence theorem, cf. [20, 21, 27]. □

Example 1.4 In case (1.34a), under obvious differentiability assumptions concerning γ and y we have $\nabla_x g(a, x) = \nabla_x y(a, x)^T \nabla\gamma\Big(y(a, x)\Big)$, where $\nabla_x y(a, x)$ denotes the Jacobian of $y = y(a, x)$ with respect to a. Hence, if (1.36b) holds, then

$$\nabla\Gamma(x) = E\nabla_x y\Big(a(\omega), x\Big)^T \nabla\gamma\left(y\Big(a(\omega), x\Big)\right). \tag{1.36c}$$

1.6 Approximations of Deterministic Substitute Problems in Optimal Design/Decision

The main problem in solving the deterministic substitute problems defined above is that the arising probability and expected cost functions $p_f = p_f(x)$, $\Gamma = \Gamma(x)$, $x \in \mathbb{R}^r$, are defined by means of multiple integrals over a ν-dimensional space.

Thus, the substitute problems may be solved, in practice, only by some approximative analytical and numerical methods [16, 20, 27, 33]. In the following we consider possible approximations for substitute problems based on general expected recourse cost functions $\Gamma = \Gamma(x)$ according to (1.34a) having a real-valued convex loss function $\gamma(z)$. Note that the probability of failure function $p_f = p_f(x)$ may be approximated from above, see (1.29a)–(1.29b), by expected cost functions $\Gamma = \Gamma(x)$ having a nonnegative function $\gamma = \gamma(z)$ being bounded from below on the failure domain B^c. In the following several basic approximation methods are presented.

1.6.1 Approximation of the Loss Function

Suppose here that $\gamma = \gamma(y)$ is a continuously differentiable, convex loss function on $\mathbb{R}^{m_y}$. Let then denote

$$\overline{y}(x) := Ey\Big(a(\omega), x\Big) = \Big(Ey_1\Big(a(\omega), x\Big), \ldots, Ey_{m_y}\Big(a(\omega), x\Big)\Big)^T \tag{1.37}$$

the expectation of the vector $y = y\Big(a(\omega), x\Big)$ of state functions $y_i = y_i\Big(a(\omega), x\Big)$, $i = 1, \ldots, m_y$.

For an arbitrary continuously differentiable, convex loss function γ we have

$$\gamma\left(y\Big(a(\omega), x\Big)\right) \geq \gamma\Big(\overline{y}(x)\Big) + \nabla\gamma\Big(\overline{y}(x)\Big)^T \left(y\Big(a(\omega), x\Big) - \overline{y}(x)\right). \tag{1.38a}$$

Thus, taking expectations in (1.38a), we find Jensen's inequality

$$\Gamma(x) = E\gamma\left(y\Big(a(\omega), x\Big)\right) \geq \gamma\Big(\overline{y}(x)\Big) \tag{1.38b}$$

which holds for any convex function γ. Using the mean value theorem, we have

$$\gamma(y) = \gamma(\overline{y}) + \nabla\gamma(\hat{y})^T(y - \overline{y}), \tag{1.38c}$$

where $\hat{y}$ is a point on the line segment $\overline{y}y$ between $\overline{y}$ and y. By means of (1.38b), (1.38c) we get

$$0 \leq \Gamma(x) - \gamma\Big(\overline{y}(x)\Big) \leq E\left\|\nabla\gamma\left(\hat{y}\Big(a(\omega), x\Big)\right)\right\| \cdot \left\|y\Big(a(\omega), x\Big) - \overline{y}(x)\right\|. \tag{1.38d}$$

(a) Bounded gradient

If the gradient $\nabla\gamma$ is bounded on convex hull $R^{conv}(y(\cdot, \cdot))$ of the range of $y = y\Big(a(\omega), x\Big)$, $\omega \in \Omega, x \in D$, i.e., if

$$\|\nabla\gamma(y)\| \leq \vartheta^{\max} \quad \text{for each } y \in R^{conv}(y(\cdot, \cdot)), \tag{1.39a}$$

with a constant $\vartheta^{\max} > 0$, then