Eigenvalue Problem and Nonlinear Programming Problem
For Economic Studies
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Keiko Nakayama

Eigenvalue Problem and Nonlinear Programming Problem

For Economic Studies

Springer
Students of economics are well aware that there is a close relationship between nonlinear programming and dynamic theory. Linear programming is insufficient for analyzing the complex dynamics of real economies, and the use of a nonlinear dynamic model is essential. In this publication, therefore, systematically summarizing nonlinear programming problems is the main subject. These have various economic applications. A secondary topic is Frobenius’s theorem related to nonlinear transformation. This is grounded in the fact that Frobenius roots in nonlinear transformations can be characterized as upper limits on real numbers that bring about non-negative solutions to nonlinear inequalities and that Frobenius’s theorem gives us useful information when trying to solve stability problems.

In this book, while trying to apply them to economic theory as much as possible, I have attempted to describe them systematically and self-contained to the fullest extent possible and summarize my modest research results until now in this area.

In this publication, I have made significant additions and revisions to the book entitled “Nonlinear Programming and Nonlinear Programming Eigenvalue Problems,” which was published in September 1995 as a research work in the Chukyo University Economics Research Book Series No. 5, by Keiso Shobo. There are several reasons for this first new publication in 20 years: nonlinear models are still relevant as a tool in economic issues today, and many areas of the original text required revision. In this publication, I have changed the title somewhat from the Japanese version to “Eigenvalue Problem and Nonlinear Programming Problem.”

The definition of the eigenvalue of a matrix is as follows: “When a linear transformation using a square matrix is represented by a simple constant multiple, the constant is called the eigenvalue of the square matrix”. The readers just may not understand what this is all about. However, in such multi-dimensional economic models as economic growth, business cycles, and input-output analysis, the eigenvalue of the matrix plays an important role in checking the conditions for stability of paths or feasibility of equilibrium solutions, although the meaning of the eigenvalue is slightly different from the above-mentioned definition. The eigenvalue problem dealt with in this book refers to various problems when applying the eigenvalues and
eigenvectors of a matrix to theoretical economics. Specifically speaking, this book includes the condition for achieving balanced growth in the von Neumann model and the condition on the input coefficients for obtaining a positive equilibrium output given a positive final demand in the static input-output analysis. And the Nonlinear Programming Problem is an optimization problem in which nonlinear expressions are included in the objective function or constraints. Both these have been researched and developed in various fields beyond mathematics, engineering, and economics. In this publication, therefore, I aim to describe both the close relationship between the two problems and introduce useful reference material in the final chapter of this book for those readers who wish to study this further.

I would like to express my deep gratitude to Mr. Shozo Miyamoto of Keiso Shobo for allowing me to contribute to this work. Additionally, I would like to thank Professor Emeritus Yoshiro Higano from Tsukuba University, Professor Kiyoshi Fujikawa from Aichi Gakuen University, and Professor Emeritus Akio Matsumoto from Chuo University for their many suggestions during the writing process, as well as Professor Emeritus Nozomu Matsubara who offered encouragement and advice when I wrote the initial version, and Mr. Yutaka Hirachi editor of Springer Japan, for his support while I was writing this book.

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Chapter 1
Possibilities in Linear Inequality Systems

1.1 Introduction

Knowledge relating to the severability of linear inequality systems is important in discussing constrained optimization problems, which is one of the primary topics of this book, as shown in subsequent chapters. Thus, in this chapter, the basic theorems relating to the solvability of many forms of linear inequality systems will be systematically discussed and proven.¹

1.2 Tucker’s Theorem

In this chapter, Tucker’s theorem, which is useful in deriving multiple theorems relating to the solvability of linear inequality equation systems, is shown.

[Lemma 1.1] Let $P$ be a given $m \times n$ matrix. Herein, let the set of solutions of the following inequality equation systems

$$\text{I} \quad [Px \geq 0]$$

and

¹The names of theorems in this chapter are mainly based on Mangasarian (1969).
II $[P \eta = 0, \eta \geq 0]$ 

be $S = \left\{ \begin{pmatrix} x \\ \eta \end{pmatrix} \in \mathbb{R}^{n+m} | \text{I and II} \right\}$.$^2$ There exists $x^1 \in \mathbb{R}^n$ and $\eta^1 \in \mathbb{R}^m$ that belong to $S$ and

$[p_1 x^1 + (\eta^1)_1 > 0]$

holds, where $p_1 = [p_{11}, \ldots, p_{1n}]$ is the first row of $P$ and $(\eta^1)_1$ is the first element of $\eta^1$.

**(Proof)** Prove using the induction method in respect of $m \times n$ matrix $P$'s matrix $m$.

(A) If we let $m = 1$, $P = p_1$.

In response to $p_1 = 0$ or $p_1 \neq 0$, $\begin{pmatrix} p_1^i \\ 1 \end{pmatrix}$ or $\begin{pmatrix} p_1^i \\ 0 \end{pmatrix}$ are the corresponding sought-after solutions.

(B) Let $m \geq 2$. To demonstrate the two stages induction method, split the $(m + 1) \times n$ matrix $\overline{P}$ into

$$\overline{P} = \begin{bmatrix} P \\ p_{m+1} \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{m+1} \end{bmatrix}.$$

If the assumptions of the induction method are applied to $P$, there exists $x^1$ and $\eta^1$ that satisfy

$P x^1 \geq 0$, $P^t \eta^1 = 0$, $\eta^1 \geq 0$ and $p_1 x^1 + (\eta^1)_1 > 0$. (1.1)

Next, consider whether $p_{m+1} x^1 \geq 0$ in respect of $x^1$.

(B-1) Letting $p_{m+1} x^1 \geq 0$ in respect of $\overline{\eta} = \begin{pmatrix} (\eta^1)^t, 0 \end{pmatrix}$ and $x^1$ by (1.1),

$$\overline{P} x^1 \geq 0, P^t \overline{\eta} = P^t \eta^1 = 0, \overline{\eta} \geq 0$$

and $p_1 x^1 + (\eta^1)_1 > 0$. Thus, the sought after conclusion has been obtained in respect of $\overline{P}$.

(B-2) When $p_{m+1} x^1 < 0$, define $\lambda_j = -\frac{p_j x^1}{p_{m+1} x^1} \geq 0$ ($j = 1, \ldots, m$), and set $m \times n$ matrix $Q$ to be

$^2S = \left\{ \begin{pmatrix} x \\ \eta \end{pmatrix} \in \mathbb{R}^{n+m} | P x \geq 0, P^t \eta = 0, \eta \geq 0 \right\}$. $x = 0$ and $\eta = 0$ clearly satisfy I and II, and thus $S \neq \phi$. 

\[ Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} p_1 + \lambda p_{m+1} \\ \vdots \\ p_m + \lambda_m p_{m+1} \end{bmatrix} \]

concerning all \( j \in \{1, \ldots, m\} \),

\[ q_j x^1 = p_j x^1 + \lambda_j p_{m+1} x^1 = 0, \]

i.e.,

\[ Q x^1 = 0. \]

Herein, if the assumptions in the induction method are applied to \( Q \), there exist \( v^1 \) and \( u^1 \) and

\[ Q v^1 \geq 0, \quad Q' u^1 = 0, \quad u^1 \geq 0 \quad \text{and} \quad q_j v^1 + (u^1)_j > 0 \]  

(1.2)

are satisfied. Finally, if we set \( \bar{u}^1 = \left( (u^1)^T, \sum_{j=1}^m \lambda_j (u^1)_j \right)^T \), \( w^1 = v^1 - \frac{p_{m+1} v^1}{p_{m+1} x^1} x^1 \), by (1.2) and definition of \( w^1 \),

\[ \bar{u}^1 \geq 0 \]

and

\[ p_{m+1} w^1 = p_{m+1} v^1 - p_{m+1} v^1 = 0. \]

Furthermore, from (1.2) and the direct calculation, the following is obtained.

\[
\bar{P} \bar{u}^1 = \bar{P} u^1 + p_{m+1} \sum_{j=1}^m \lambda_j (u^1)_j
= \left( Q' u^1 - \sum_{j=1}^m \lambda_j p_{m+1} (u^1)_j \right) + p_{m+1} \sum_{j=1}^m \lambda_j (u^1)_j
= Q' u^1
= 0,
\]

\[
\bar{P} w^1 = \begin{bmatrix} P \\ p_{m+1} \end{bmatrix} w^1
= \begin{bmatrix} P w^1 \\ 0 \end{bmatrix}
= \begin{bmatrix} (q_1 - \lambda_1 p_{m+1}) w^1 \\ (q_m - \lambda_m p_{m+1}) w^1 \\ \vdots \\ 0 \end{bmatrix}
\]
This concludes the proof. QED

This lemma can be generalized to [Theorem 1.1].

[Theorem 1.1] Tucker’s Theorem\textsuperscript{3} For a given $m \times n$ matrix $P$,

I \quad [Px \geq 0]

and

II \quad [P'\eta \geq 0, \eta \geq 0]

have solutions $x \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$ that satisfy

$$[Px + \eta > 0].$$

\textsuperscript{3}See Tucker (1956).
1.2 Tucker’s Theorem

(Proof) For any row number \( k \) of \( P \), define \( Q \) to be a matrix with \( m \)-order unit matrix with the first row and \( k \)-th row switched and apply the [Lemma 1.1] to the matrix \((QP)\), there exist \( y \) and \( \xi \) are

\[(QP)y \geq 0, (QP)^{i} \xi \geq 0, \xi \geq 0 \text{ and } (QP)_{1}y + (\xi^{i})_{1} > 0. \] (1.3)

From the definition of \( Q \),

\[ (QP)_{i} = p_{k} \text{ and } (Q^{i}\xi)_{i} = \xi_{k}, \quad (i = 1) \]

\[ (QP)_{i} = p_{1} \text{ and } (Q^{i}\xi)_{i} = \xi_{1}, \quad (i = k) \]

and

\[ (QP)_{i} = p_{i} \text{ and } (Q^{i}\xi)_{i} = \xi_{i}, \quad (i \neq 1, k). \]

Thus, \( Q^{i}\xi = \eta^{k} \) and letting \( y = x^{k} \), (1.3) can be rewritten as follows:

\[ Px^{k} \geq 0, P^{i}\eta^{k} = 0, \eta^{k} \geq 0 \text{ and } p_{i}x^{k} + (\eta^{k})_{k} > 0. \]

Hence, if we let \( x = \sum_{k=1}^{m} x^{k} \) and \( \eta = \sum_{k=1}^{m} \eta^{k} \), \( \begin{pmatrix} x \\ \eta \end{pmatrix} \in S \) and \( px + \eta > 0 \) are obtained. QED

If [Theorem 1.1] is written down as per the following series, it is useful for future use.

[Corollary 1.1] Let \( A, B, C, D \) be a given \( m_{1} \times n, m_{2} \times n, m_{3} \times n, m_{4} \times n \) matrix. Here,

\[
\text{I } [Ax \geq 0, Bx \geq 0, Cx \geq 0, Dx = 0] \\
\text{and} \\
\text{II } \begin{bmatrix}
A^{i}y^{1} + B^{i}y^{2} + C^{i}y^{3} + D^{i}y^{4} = 0 \\
y^{1} \geq 0, y^{2} \geq 0, y^{3} \geq 0
\end{bmatrix}
\]

has solution \( x \in R^{n}, y^{1} \in R^{m_{1}}, y^{2} \in R^{m_{2}}, y^{3} \in R^{m_{3}}, \text{ and } y^{4} \in R^{m_{4}} \), which satisfy

\[
\begin{bmatrix}
Ax + y^{1} > 0 \\
Bx + y^{2} > 0 \\
Cx + y^{3} > 0
\end{bmatrix}.
\]

(Proof) Let us define \( m \times n \) matrix \( P, \eta \in R^{m} \) as follows.
Possibilities in Linear Inequality Systems

\[
P = \begin{bmatrix}
A \\
B \\
C \\
D \\
-D
\end{bmatrix}, \quad \eta = \begin{bmatrix}
z^1 \\
z^2 \\
z^3 \\
z^4 \\
z^5
\end{bmatrix},
\]

where \( z^i \in R^{m_i} \) \((i = 1, 2, 3)\), \( z^4, z^5 \in R^{m_4}, m_1 + m_2 + m_3 + 2m_4 = m \).

From [Theorem 1.1],

\[
I' \quad [Ax \geq 0, Bx \geq 0, Cx \geq 0, Dx = 0, -Dx \geq 0]
\]

and

\[
II' \quad \begin{bmatrix}
A'z^1 + B'z^2 + C'z^3 + D'z^4 - D'z^5 = 0 \\
z^1 \geq 0, z^2 \geq 0, z^3 \geq 0, z^4 \geq 0, z^5 \geq 0
\end{bmatrix}
\]

have solutions \( x \) and \( z^i \) \((i = 1, 2, 3, 4, 5)\) that satisfy

\[
\begin{bmatrix}
Ax + z^1 > 0 \\
Bx + z^2 > 0 \\
Cx + z^3 > 0 \\
Dx + z^4 > 0 \\
-Dx + z^5 > 0
\end{bmatrix}
\]

Thus, while noting that \( I' \)’s \( Dx \geq 0 \) and \(-Dx \geq 0\) have the same value as \( Dx = 0 \), define \( y^i = z^i \geq 0 \) \((i = 1, 2, 3)\) and \( y^4 = z^4 - z^5 \), the conclusion is immediately obtained. QED

(Notes relating to [Corollary 1.1]): In future, if this corollary is to be used unless there is a particular reason not to, the solution of a corollary will be expressed as \( x_0, y_0^i \) \((i = 1, 2, 3, 4)\).

1.3 Theorem of Alternatives in Respect of Linear Inequality Systems

Generally, the alternative theorem indicates that two prepositions I and II are exclusive and that at least one of the prepositions always holds. Accordingly, to demonstrate the theorem of alternatives,

(a) \( I \Rightarrow \sim II^4 \) (equivalent to \( \sim I \leftrightarrow II \))

and

(b) \( \sim I \Rightarrow II \) (equivalent to \( I \leftrightarrow \sim II \))

must be proven.

\(^4\) (a) indicates that prepositions I and II do not both hold together. “\( \Rightarrow \)” is a logical symbol that shows implication and “\( \sim \)” is a sign that shows non-implication.
Originally, the description of [Theorem 1.2] shown below is correct. However, in the various theorems below ([Theorem 1.3] ~ [Theorem 1.11]), for simplification of notation, when only one of the prepositions I or II hold, it is simply expressed as “I or II.” Moreover, matrices $A$, $B$, $C$, and $D$ are considered to be real matrices with $n$ columns and $m_1$, $m_2$, $m_3$, $m_4$ rows, respectively, and $y'$ is defined to be a $m_i (i = 1, 2, 3, 4)$-dimension real vector. The number of order of a vector and the number of columns and rows of matrices other than this will be clarified only if they cannot be understood from the context.

Firstly, [Corollary 1.1] is used to explain [Theorem 1.2] and [Theorem 1.3]. In particular, [Theorem 1.2] is a versatile theorem of alternatives.

[Theorem 1.2] Slater’s Theorem of the Alternative\textsuperscript{5} Let $A$, $B$, $C$ and $D$ be given matrices, with $A$ and $B$ being nonvacuous. Then either

1. $[Ax > 0, Bx \geq 0, Cx = 0, Dx = 0]$ has a solution $x$,

or

2. \[
\begin{align*}
A'y^1 + B'y^2 + C'y^3 + D'y^4 &= 0 \\
\text{However, (i) } &y^1 \geq 0, \ y^2 \geq 0, \ y^3 \geq 0 \\
&\text{or} \\
\text{or (ii) } &y^1 \geq 0, \ y^2 > 0, \ y^3 \geq 0
\end{align*}
\]

has a solution $y^1, y^2, y^3, y^4$.

But never both.

(Proof)

(a) If the conclusion is contradicted and it is assumed that I and II holds,

$$(x'C')y^3 \geq 0 \text{ and } (x'D')y^4 = 0.$$ 

Furthermore, in response to II (i) or (ii),

$$(x'A')y^1 > 0 \text{ and } (x'B')y^2 \geq 0$$

Moreover,

$$(x'A')y^1 \geq 0 \text{ and } (x'B')y^2 > 0$$

are established. Thus, in any case,

\textsuperscript{5}See Slater (1951).

\((x' A') y^1 + (x' B') y^2 + (x' C') y^3 + (x' D') y^4 > 0\).

However, the left-hand side of the above inequality is
\[ x' (A y^1 + B y^2 + C y^3 + D y^4) = 0 \]

Accordingly, both these two equations do not hold together.

(b) If \(I\) is denied, in respect of any \(x\), at least one of the four equations in \(I\) no longer holds. However, if \([\text{Corollary 1.1}]\) is considered, \(C x_0 \geq 0\) and \(D x_0 = 0\). Thus, I may become untrue only in the case of (A) \(A x_0 \not> 0\) or (B) \(B x_0 \not< 0\).

(A) \(A x_0 \not> 0\) and \(A x_0 \geq 0\). Hence, there exists a number \(i_0 \in \{1, \ldots, m_4\}\) that satisfies \((A x_0)_{ia} = \sum_{j=1}^{n} a_{ij} x_0 = 0\). Moreover, from \(y_{i_0}^1 \geq 0\) and \(A x_0 + y_{i_0}^1 > 0\), \(y_{i_0}^1 \geq 0\). \(y_{i_0}^1 \geq 0\) \((i = 2, 3)\) is guaranteed, and therefore II(i) has been demonstrated.

(B) \(B x_0 \not< 0\) and \(B x_0 \geq 0\). Therefore, \(B x_0 = 0\). If this is substituted into \(B x_0 + y_{i_0}^2 > 0\), \(y_{i_0}^2 > 0\) is immediately obtained and \(y_{i_0}^2 \geq 0\) \((i = 1, 3)\) is self-apparent. Accordingly, in this case, II(ii) holds. \(\Box\)

(Notes on \([\text{Theorem 1.2}]\)): If the solution of \(I\) in \([\text{Theorem 1.2}]\) is considered to be \(x = -z\), I can be rewritten as follows:

\[ I' \quad [A z < 0, B z \leq 0, C z \geq 0, D z = 0] \]

has a solution \(z\).

It is the same for the different theorems below. Thus, when using the theorem, a more useful approach from the inequality system \(I\) or \(I'\) must be selected.

Below, the situation when a specific matrix as per \([\text{Theorem 1.2}]\) hence a vector which corresponds to it does not exist is considered. Thus, a matrix or a vector, according to Mangasarian (1969), when \(A\) or \(y^1\) does not exist, is expressed as \(A = [\phi]\) or \(y^1 = [\phi]\) and it is said that \(A\) is a vacuous matrix or that \(y^1\) is an empty vector. Henceforth, nonvacuous matrix \(A\) shall be written as \(A \neq [\phi]\).

[Theorem 1.3] Tucker’s Theorem of the Alternative\(^6\) Concerning the given matrices \(B(\neq [\phi]), C\) and \(D\)

\[ I \quad [B x \geq 0, C x \geq 0, D x = 0] \]

has a solution \(x\),

or

\[ \text{II} \quad \left[ B y^2 + C y^3 + D y^4 = 0 \right. \quad \left. y^2 > 0, \quad y^3 \geq 0 \right] \]

has solutions \(y^2, y^3, y^4\).

---

\(^6\)See Tucker (1956).
1.3 Theorem of Alternatives in Respect of Linear Inequality Systems

(Proof) In the inequality system of [Theorem 1.2], A, accordingly, if there exists no \( y^1 \), the inequality system of the theorem is obtained.

(a) It is evident from proof (a) of [Theorem 1.2].
(b) Shown in (B) in proof (b) of [Theorem 1.2]. QED

Furthermore, when there are no \( C \) and \( D \) for [Theorem 1.3], the following theorem can be obtained.

[Theorem 1.4] Stiemke’s Theorem 7 For a given \( B \),

\[ I \quad [Bx \geq 0] \text{ has a solution,} \]

or

\[ II \quad [B'y^2 = 0, \ y^2 > 0] \text{ has a solution } y^2. \]

(Proof) It is evident by letting [Theorem 1.3]’s \( C = [\phi] \) and \( D = [\phi] \). QED

[Theorem 1.5] Motzkin’s Theorem of the Alternative 8 Let \( A(\neq [\phi]) \), \( C \) and \( D \) be given matrices. Then

\[ I \quad [Ax > 0, \ Cx \geq 0, Dx = 0] \text{ has a solution } x, \]

or

\[ II \quad [A'y^1 + C'y^3 + D'y^4 = 0, \ y^1 \geq 0, y^3 \geq 0] \text{ has a solution } y^1, y^3, y^4. \]

(Proof) Just require to ignore [Theorem 1.2]’s \( B = [\phi] \) and thus \( y^2 \).

(a) It can be shown in the same as Proof (a) in [Theorem 1.2].
(b) It gets back to proof (b) of [Theorem 1.2]. QED

Next, let us rewrite this theorem such that it can be easily applied to an inhomogeneous system.

[Corollary 1.5] Let \( \beta, \gamma, \xi, \) and \( \eta \in R \) and define \( \bar{x} = \begin{bmatrix} x \\ \xi \end{bmatrix} \in R^{n+1}, \bar{y} = \begin{bmatrix} \eta \\ \gamma \end{bmatrix} \in R^2. \)

Here, corresponding to [Theorem 1.5]’s \( C = [\phi] \) or \( D = [\phi] \), (i) and (ii) hold.

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7See Stiemke (1915).
8See Motzkin (1936, 1951). Regarding Motzkin’s transposition theorem, there are many papers, but for example, Slater (1951), Ben (2001) and Nermirovski and Roos (2009) are recommended.
(i) Given $\tilde{A} = \begin{bmatrix} b' & -\beta \\ 0 & 1 \end{bmatrix}$, $\tilde{D} = [-A, c]$,

$$\begin{bmatrix} \tilde{A} \tilde{x} = \begin{bmatrix} b'x - \beta \xi \\ \xi \end{bmatrix} > 0, \tilde{D} \tilde{x} = [-Ax + \xi c] = 0 \end{bmatrix}$$

has a solution $\tilde{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}$, or

$$\begin{bmatrix} \tilde{A}' \tilde{y}^1 + \tilde{D}' y^4 = \begin{bmatrix} \eta b - A'y^4 \\ -\beta \eta + \gamma + C'y^4 \end{bmatrix} = 0 \\ \tilde{y}^1 = \begin{bmatrix} \eta \\ \gamma \end{bmatrix} \geq 0 \end{bmatrix}$$

has a solution $\tilde{y}^1, y^4$.

(ii) Given $\tilde{A} = \begin{bmatrix} b' & -\beta \\ 0 & 1 \end{bmatrix}$, $\tilde{C} = [-A, c]$,

$$\begin{bmatrix} \tilde{A} \tilde{x} = \begin{bmatrix} b'x - \beta \xi \\ \xi \end{bmatrix} > 0, \tilde{C} \tilde{x} = [-Ax + \xi c] \geq 0 \end{bmatrix}$$

has a solution $\tilde{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}$, or

$$\begin{bmatrix} \tilde{A}' \tilde{y}^1 + \tilde{C}' y^3 = \begin{bmatrix} \eta b - A'y^3 \\ -\beta \eta + \gamma + C'y^3 \end{bmatrix} = 0 \\ \tilde{y}^1 = \begin{bmatrix} \eta \\ \gamma \end{bmatrix} \geq 0, y^3 \geq 0 \end{bmatrix}$$

has a solution $\tilde{y}^1, y^3$.

Now, let us show below the various theorems that can be derived from [Theorem 1.5]. First, the case of [Theorem 1.5]’s $C = [\phi]$ will be discussed.

**[Theorem 1.6] Gale’s Theorem for Linear Equalities**

For a given $A$ and $c$,

$$\begin{bmatrix} Ax = c \end{bmatrix}$$

has a solution $x$, or

$$\begin{bmatrix} A'y = 0, c'y = 1 \end{bmatrix}$$

has a solution $y$.

**Proof** Let [Theorem 1.5]’s $C = [\phi]$. Accordingly, by letting it be system [1.5; (i)]’s $b = 0 \in R^n, \beta = -1, \eta = 0$; in other words, it can be rewritten as follows:

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See Gale (1960).
\[ \widetilde{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{D} = [A, c], \quad \tilde{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \tilde{y}^1 = \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \]

[Corollary 1.5; (i)],

\[ I' \quad [\xi > 0, \quad Ax = c\xi] \text{ has a solution } \begin{bmatrix} x \\ \xi \end{bmatrix}, \]

or

\[ II' \quad \begin{bmatrix} A' y^4 = 0, \quad \gamma + c'y^4 = 0 \\ 0 \geq 0 \end{bmatrix} \text{ has solution } \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \text{ and } y^4. \]

Clearly, I' and II' and II are mutually equivalent, and hence this concludes the proof. QED

[Theorem 1.7]–[Theorem 1.9] corresponds to the case when \( D \) does not exist in [Theorem 1.5]; thus, it is more directly obtained by applying [Corollary 1.5; (ii)].

[Theorem 1.7] Farkas’s Theorem\(^\text{10}\) For a given \( A \) and \( b \),

\[ I \quad [Ax \leq 0, \quad b'x > 0] \text{ has a solution } x, \]

or

\[ II \quad [A'y = b, y \geq 0] \text{ has a solution } y. \]

(Proof) If we define \( \widetilde{A} = \begin{bmatrix} b' & 0 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{C} = [ -A, 0], \quad \tilde{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \tilde{y}^1 = \begin{bmatrix} \eta \\ 0 \end{bmatrix} \]

it is no other than the case when we set [Series 1.5; (ii)]’s \( c = 0, \beta = \gamma = 0 \). Accordingly, I or II is equivalent to:

\[ I' \quad [b'x > 0, \quad \xi > 0, \quad Ax \leq 0] \text{ has a solution } x, \]

or

\[ II' \quad \begin{bmatrix} \eta b - A'y^3 = 0 \\ \eta > 0, y^3 \geq 0 \end{bmatrix} \text{ has a solution } \eta, y^3. \]

QED

\(^{10}\)See Farkas (1902).