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Applied Linear Algebra, Probability and Statistics

A Volume in Honour of C. R. Rao and Arbind K. Lal

Springer

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Dedicated to
A great statistician, linear algebraist and living legend who is in his 103rd year and still inspiring the young generation

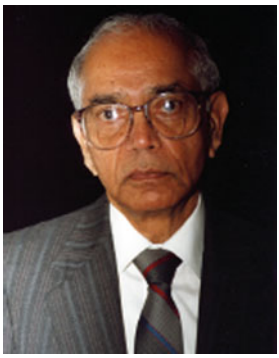
Padma Vibhushan
Calyampudi Radhakrishna Rao
(September 10, 1920–)

and our beloved friend who left us behind due to his untimely demise

Lal da
Arbind K. Lal
(January 01, 1966–March 07, 2021)

About Rao and Lal

Prof. Calyampudi Radhakrishna Rao



Calyampudi Radhakrishna Rao (1920–). Courtesy: Simo Puntanen

Rao[1] was born on September 10, 1920, in a small town in south India called Hoovina Hadagali, then in the integrated Madras Province of British India, but now in the Bellary district of state Karnataka. He was the eighth child (out of ten: six boys and four girls) of his parents, C. D. Naidu and Laxmikanthamma.

CRR joined the course 'training in statistics' offered by the ISI in January 1941. Six months later, Rao joined Calcutta University, which had just started a Master's degree program in statistics, the first of its kind in India. He completed the course with first-class honors, first rank and a gold medal in 1943. In August 1946, CRR visited England on the invitation of J. C. Trever to work in the Anthropological

[1] Thanks to Simo Puntanen for the contents of this section of Rao.

Museum and registered for a Ph.D. degree under Ronald Aylmer Fisher (1890–1962). Based on the work he did at the Museum, CRR received the Ph.D. degree from Cambridge University in 1948 with the thesis entitled 'Statistical Problems of Biological Classifications'.

Rao returned to India in August 1948, marrying Bhargavi (April 28, 1925–July 24, 2017) on September 09, 1948. Professor Mahalanobis then offered Rao a Professorship at ISI as well as the headship of the Research and Training School in 1949. He served in this position till 1963. Being an inspiring teacher, over the years about fifty students have completed their Ph.D. thesis under his guidance. CRR was the Director of ISI from 1963 to 1972, and, later, he was the Director and Secretary of the Indian Statistical Institute till 1976. He then continued in the ISI as a Jawaharlal Nehru Professor till 1984. During this period, he took leave from ISI in 1979 and went to the United States to accept a University Professorship at the University of Pittsburgh. He moved to The Pennsylvania State University in 1988, where he was offered several chair positions. He retired from Penn State at the age of 81, but continued doing research as the Director of the Center for Multivariate Analysis at Penn State until 2008. After that, he has the position as a Research Professor at the University at Buffalo.

In 1954, C. R. Rao collected some data from Japan in order to study the long-term effects of radiation caused by the Hiroshima and Nagasaki atom bomb explosions. In the statistical analysis, he encountered the situation of finding a matrix to replace the inverse of $X'X$, where X is the design matrix appearing in the linear model; here the matrix X is not a full column rank matrix and $X'X$ is singular and so the inverse was not defined. This led to a pseudo-inverse, which Rao introduced in his 1955 paper 'Analysis of dispersion for multiply classified data with an unequal number of cells' in Sankhya. This was the same year that R. A. Penrose published his paper on the generalized inverse. [Proc. Cambridge Philos. Soc. 51: 406–413 (1955)]. An interesting note to be made here is that the notion of pseudo-inverse, initially defined for the operators, was introduced by Moore in 1920, the year in which Rao was born. C. R. Rao then discovered the key condition for a generalized inverse G of a matrix A satisfying the equation $AGA = A$ and introduced the notation $G = A^{-}$. The calculus of g-inverses and the unified theory of linear estimation were then presented in 1962. The subject of g-inverses was further developed with the help of his colleagues at the ISI, leading to the full-length monograph with Sujith Kumar Mitra, Generalized Inverse of Matrices and its Applications, published in 1971. Using the concept of the g-inverse, Rao developed a unified theory for linear estimation, noting that g-inverses were particularly helpful with explicit expressions for projectors. He introduced the Inverse Partitioned Matrix (IPM) method [Unified theory of linear estimation, Sankhya Ser. A 33: 371–394 (1972)] for computing summary statistics for the general linear model. Linear models, which have wide applications in statistics, have also provided outlets for some basic research in Linear Algebra: the special issues on linear algebra and statistics of Linear Algebra and its Applications [Vols. 67 (1985),

70 (1985), 82 (1986), 127 (1990), 176 (1992), 210 (1994)] bear witness to this. Some of Rao's contributions are a general matrix inequality, separations for singular values and eigenvalues, matrix approximations and reduction of dimensionality, and Kantorovich type inequalities.

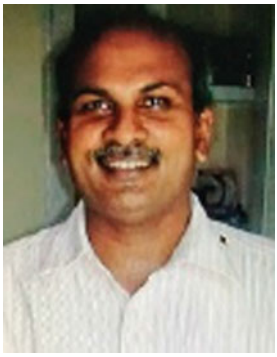
C. R. Rao has developed statistical estimation theory in small samples, extending the scope of statistical methods in practice. Many results in this area bear his name, such as the Cramer-Rao lower bound, Rao-Blackwell theorem, Rao-Blackwellization, Fisher-Rao theorem, Rao's second order efficiency and the Geary-Rao theorem on Pitman closeness criterion. Rao's orthogonal arrays were a major contribution to research in coding theory and in experimental design—especially in industrial experimentation as developed by Genichi Taguchi. Rao has made pioneering contributions to the development of multivariate statistical analysis, especially with the concept of Rao's quadratic entropy, Rao's U-test, Rao's F-approximation to the likelihood-ratio criterion and canonical coordinates. He has also introduced a broad class of asymptotic tests of hypothesis, including Rao's score statistic and the Neyman-Rao statistic; he has used differential-geometric techniques in discussing problems of statistical inference, based on Rao's distance function.

2023 International Prize in Statistics.

On 1 April 2023, it was announced on the website https://statprize.org/2023-International-Prize-in-Statistics-Awarded-to-C-R-Rao.cfm that C. R. Rao, a professor whose work more than seventy five years ago continues to exert a profound influence on science, has been awarded the 2023 International Prize in Statistics. The Award citation further reads: "In his remarkable 1945 paper published in the Bulletin of the Calcutta Mathematical Society, C. R. Rao demonstrated three fundamental results that paved the way for the modern field of statistics and provided statistical tools heavily used in science today". Here is the complete bibliographical information of the paper: Information and accuracy attainable in the estimation of statistical parameters, Bulletin of Calcutta Mathematical Society, Vol 37, No 3, 81-91, 1945.

The Prize was established in 2016 and is awarded once every two years to an individual or team "for major achievements using statistics to advance science, technology and human welfare". The Prize is managed by a foundation comprising representatives of the five major statistical organizations working cooperatively to develop this prestigious award which are American Statistical Association; Institute of Mathematical Statistics; International Biometric Society; International Statistical Institute; and Royal Statistical Society. The Prize is being considered as the statistics' equivalent of the Nobel Prize.

Arbind Kumar Lal



Arbind Kumar Lal (1966–2021). Courtesy: Sukanta Pati

Lal[2] known as Lalda for the younger generation, was in long association with CARAMS, MAHE and its activities. He is known for his nature to help everyone (students, teaching and nonteaching staff) in trouble, and for his ability to resolve issues within the department. Professor Lal was popularly tagged as the 'permanent HOD' by his colleagues. Born in Uttara (Madhubani district, Bihar) to Smt. Sushila Devi and Shri. Lekh Narayan Lal, his upbringing had made him a thorough gentleman and a person with a huge heart.

He completed his schooling at Sainik School, Tilaiyaa, in 1983 and obtained his BA with Mathematics, Honors in 1986 from Hansraj College, Delhi. He went to do his MStat (1986–1988) at ISI Delhi, and this is where his growth as a mathematician started. Because of his frank nature, being a sports-loving person and being a thoughtful person, he created many friends for life here. He completed his Ph.D. (1988–1993) from the same place, ISI Delhi. He was a visiting fellow at TIFR Mumbai for a year during which he worked on Combinatorics. After that, he became a visiting fellow at HRI Allahabad for two years during which he worked on coding theory. After that, he joined IIT Kanpur as a faculty in the Department of Mathematics in 1996 and served the department in various capacities till his untimely demise in 2021.

He guided seven Ph.D. students and his contribution 'Aq-analogue of the Distance Matrix of a Tree', Linear Algebra and its Applications, 416, 2–3, 2006, 799–814, has motivated many researchers to work on various aspects of the distance matrix. He was an editor of the journal 'Proceedings Mathematical Sciences' published by Springer and Indian Academy of Sciences.

CARAMS, MAHE salutes him and documents his tribute, besides praying to the Almighty to bless Rao with good health, prosperity and undiminished energy to inspire young generations of the future.

[2] Thanks to Sukanta Pati for the contents of this section on Lal.

Foreword

It gives me immense pleasure that a special volume dedicated to the conferences organized by the 'Center for Advanced Research in Applied Mathematics & Statistics, MAHE' is being published by Springer Nature's Indian Statistical Institute series in the form of a book-volume in honor of Profs. Calyampudi Radhakrishna Rao and Arbind K. Lal. CARAMS, MAHE initiated the activities in honor of Padma Vibhushan C. R. Rao, a great statistician and a living legend, in the year 2020. Due to the prevailing Covid situation, the events could not be organized in the normal physical format, but Rao's hundred years of life was celebrated by organizing the 'International Conference on Applied Linear Algebra, Probability and Statistics' in the online format. However, the conferences IWMS 2021 and ICLAA 2021, which were earlier scheduled in December 2020, have been rescheduled and organized in the hybrid format in December 2021. I congratulate the entire team led by the scientific advisory committee consisting of Ravindra B. Bapat, Manjunatha Prasad Karantha, Steve Kirkland, and Simo Puntanen for the great success of the events IWMS 2021 and ICLAA 2021 which brought several collaborators, disciples, and the followers of Rao together and connected with the young minds of India and abroad, remembering the achievements of Rao and further developments. Arbind K. Lal, an eminent young mathematician serving at IIT Kanpur, who had been continuously associated with the ICLAA series of conferences of MAHE, left us behind due to his untimely demise, and CARAMS, MAHE dedicated a session of ICLAA 2021 to remember him.

The book-volume 'Applied Linear Algebra, Probability, and Statistics—A Volume in Honor of C. R. Rao and Arbind K. Lal' is a part of several outcomes of the series of activities held by CARAMS, MAHE in honor of Rao. I understand that several associates of Rao and Lal and young scholars have contributed their work in the form

of book-chapters. I thank the entire editorial team and the authors for their efforts and contribution in the preparation of the book-volume ALAPS being published in the Indian Statistical Institute series of Springer Nature.

April 2023

Narayana Sabhahit
Pro Vice Chancellor—Technology and Science
Manipal Academy of Higher Education
Manipal, India

Preface

Linear Algebra, Probability Theory and Statistics are subjects having applications in almost every branch of science including Biology, Economics, Engineering and Physics, which are consistent with the focus research activities of CARAMS, MAHE. ICLAA 2021 is the fourth in its sequel, which started with CMTGIM 2012, and the fifth is ICLAA 2023 scheduled in December 2023. The initial intention was to dedicate the common day between IWMS 2020 and ICLAA 2020 to honor C. R. Rao, a living legend among Indian Statisticians, who was celebrating the birth centenary in the year 2020. Unfortunately, the events could not be organized as per schedule in the normal format due to the prevailing situation. With the hope of having a normal life in 2021, the events in the physical formats have been rescheduled to December 2021, and CARAMS organized ALAPS 2020 (International Conference on Applied Linear Algebra, Probability, and Statistics) in the online format (164 participants—25 invited, 40 contributory), in which many collaborators, disciples and followers of Rao participated. However, IWMS 2021 (192 participants—20 invited, 14 contributory) and ICLAA 2021 (242 participants—25 invited, 28 contributory) were organized successfully in the hybrid format and decided to publish a special issue in 'AKCE International Journal of Graphs and Combinatorics', and a book-volume 'Applied Linear Algebra, Probability, and Statistics—A volume in honor of Rao and Lal' (ALAPS) in the ISI series of Springer Nature.

The present volume is dedicated to Prof. Rao, who completed 100 years of legendary life and continues to inspire all of us, and Prof. Arbind Lal, who sadly departed us too early. The book focuses on, but is not limited to, linear algebra, statistics, matrices, graphs and applications—the areas to which Rao and Lal contributed. The volume is rich with contributions from collaborators, students, colleagues and admirers of Profs. Rao and Lal.

Many chapters feature new findings due to applications of matrix and graph methods, while others present rediscoveries of the subject using new methods. Chapters from Simo Puntanen, Steve Haslet and their collaborators reflect the continuation of Rao's legacy in the use of matrix methods in statistics, whereas two from TES Raghavan describe the use of matrix-graph methods in the computation of the nucleolus and Shapley value, which appear in game theory. The contributions from

Raghavan are based on his lectures delivered at the workshop and conferences held in honor of Rao.

Some chapters are expository, summarizing the development of subjects and presenting new open problems. BLS Prakasa Rao's 'On some matrix versions of covariance, harmonic mean and other inequalities: an overview' is one such article. Jeffery Hunter's contribution 'The impact of Prof. Rao's research used in solving problems in applied probability' describes how Rao's work influenced the author's research.

Neogy et al. (On some special matrices and its applications in linear complementarity problem) and Eagambaram (Characterization of Q-matrices using bordered matrix algorithm) discuss special classes of matrices having applications in the linear complementarity problem. The chapters by Saumyadipta Pyne and Sudeep Bapat present key applications of statistical data analysis to real-life problems. The book also includes one of Lal's last works 'Some observations on algebraic connectivity of graphs'.

With many more chapters on Generalized Inverses, Matrix Analysis, Matrices and Graphs, and the History of Ancient Mathematics, this volume offers a diverse array of mathematical results, techniques, applications and perspectives. Linear Algebra, Probability and Statistics are important branches of mathematics having applications in every dimension of science. The book promises to be especially rewarding for readers with an interest in the focus areas of ALAPS.

Manipal, Karnataka, India
April 2023

Ravindra B. Bapat
Manjunatha Prasad Karantha
Stephen J. Kirkland
Samir Kumar Neogy
Sukanta Pati
Simo Puntanen

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The editors acknowledge the incredible contribution of all the authors (see the list of contributors), which made the book-volume so rich. Also, the editors acknowledge the association and the support of

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at different stages of editorial work. The selfless efforts of the reviewers helped the editorial team and the authors to a great extent in enhancing the quality of work. The editors acknowledge the following reviewers, other than the editors, for spending their quality time in reading the articles and offering their comments.

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Manipal, Karnataka, India
April 2023

Ravindra B. Bapat
Manjunatha Prasad Karantha
Stephen J. Kirkland
Simo Puntanen

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Chapter 1
On Some Matrix Versions of Covariance, Harmonic Mean and Other Inequalities: An Overview

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1.1 Introduction

C. R. Rao has extensively used matrix algebra in his study of unified theory of linear models. He has shown that some of the important results in matrix algebra can be derived from statistical results in Rao [19]. Dey et al. [6] gave proofs of some results in matrix algebra using ideas derived from statistics. Mitra [13] gave statistical proofs of some results for non-negative definite matrices. Rao [18] derived seven inequalities in statistical estimation theory one of which deals with the harmonic mean of positive random variables. A generalization of these results for random matrices was derived in Prakasa Rao [17]. Rao [19] gave an alternate proof of this result using ideas from statistics. Kimeldorf and Sampson [11] derived a class of covariance inequalities. Matrix versions of these inequalities were obtained in Prakasa Rao [16]. We will now give a short overview of some of these results dealing with the harmonic mean for positive random variables and some covariance inequalities and the recent work of Gibilisco and Hansen [9] generalizing the results of Rao [18] and Prakasa Rao [17] and present some open problems.

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1.2 Covariance Inequalities

Let $\mathcal{A}$ be an open convex subset of R^2 and (X, Y) be a bivariate random vector with support $\mathcal{A}$. Further suppose that $E(X^2)$ and $E(Y^2)$ are finite. Suppose $g(x, y)$ is a function satisfying the condition

$$(x_2 - x_1)(y_2 - y_1) \leq [g(x_2, y_2) - g(x_1, y_1)]^2, (x_i, y_i) \in \mathcal{A}, i = 1, 2.$$

Kimeldorf and Sampson [11] proved that

$$Cov(X, Y) \leq Var[g(X, Y)].$$

A matrix version of this result was proved in Prakasa Rao [16]. We will now review this result and give some applications.

Hereafter, we consider only random matrices with real entries.

Suppose ξ and η are two random matrices of order $r \times s$ and further suppose that all the entries of these matrices have finite second moments. Let $E(\xi)$ be the matrix of expectations of the entries of the matrix ξ. Let ξ' denote the transpose of the matrix ξ. For any two random matrices ξ and η, define the covariance and variance by the following relations:

$$Cov(\xi, \eta) = \frac{1}{2}\{E[(\xi - E(\xi))(\eta - E(\eta))')] + E[(\eta - E(\eta))(\xi - E(\xi))']\} \quad (1.1)$$

and

$$Var[\xi] = E[(\xi - E(\xi))(\xi - E(\xi))']. \quad (1.2)$$

It is clear from the definitions given above that the matrix $Var(\xi)$ is a non-negative definite matrix and that the covariance matrix $Cov(\xi, \eta)$ is a symmetric matrix. If the matrices ξ and η are random commuting symmetric matrices (and hence $r = s$), then

$$\begin{aligned} Cov(\xi, \eta) &= E[(\xi - E(\xi))(\eta - E(\eta))'] \\ &= E(\xi\eta) - E(\xi)E(\eta) \end{aligned} \quad (1.3)$$

which is analogous to the standard definition of the covariance for two random variables. For any two square matrices A and B of the same order, we say that $A \leq B$ if the matrix $B - A$ is non-negative definite. We say that $A < B$ if $B - A$ is positive definite.

The following result is proved in Prakasa Rao [16].

Theorem 1.1 *Let ξ and η be random matrices of order $r \times s$ such that $P[(\xi, \eta) \in \mathcal{A}] = 1$ where $\mathcal{A}$ is a subset of $R^{r\times s} \times R^{r\times s}$. Let $g(A, B)$ be a function defined over the space $R^{r\times s} \times R^{r\times s}$ taking values in $R^{r\times \ell}$. Then*

$$Cov(\xi, \eta) \leq Var[g(\xi, \eta)] \tag{1.4}$$

if and only if, for all $(A_i, B_i) \in \mathcal{A}, i = 1, 2,$

$$\begin{aligned} &\frac{1}{2}[(A_2 - A_1)(B_2 - B_1)' + (B_2 - B_1)(A_2 - A_1)'] \\ &\leq [g(A_2, B_2) - g(A_1, B_1)][g(A_2, B_2) - g(A_1, B_1)]'. \end{aligned} \tag{1.5}$$

For proof, see Prakasa Rao [16]. We now consider some interesting special cases of this theorem.

Corollary 1.1 *Suppose* ξ *and* η *are random matrices of order* $r \times r$. *Let* $g(A, B) = \frac{A+B}{2}$. *It is easy to check that the condition (1.5) holds using the fact that*

$$[\frac{A_2 - A_1}{2} - \frac{B_2 - B_1}{2}][\frac{A_2 - A_1}{2} - \frac{B_2 - B_1}{2}]' \geq 0.$$

Hence, an application of Theorem 1.1 shows that

$$Cov(\xi, \eta) \leq Var[\frac{\xi + \eta}{2}]. \tag{1.6}$$

Corollary 1.2 *Suppose* ξ *and* η *are positive definite matrices of order* $r \times r$. *Then* $A^{\frac{1}{2}}$ *and* $B^{\frac{1}{2}}$ *are symmetric matrices. Suppose they commute. Define* $g(A, B) = A^{\frac{1}{2}}B^{\frac{1}{2}} = B^{\frac{1}{2}}A^{\frac{1}{2}}$ *by commutativity. It can be checked that the condition (1.5) holds and hence, by Theorem 1.1,*

$$Cov(\xi, \eta) \leq Var[\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}]. \tag{1.7}$$

This inequality can also be obtained as a consequence of the following matrix version of the Cauchy-Schwartz inequality (cf. Ibragimov and Khasminskii [10, p. 73]).

Theorem 1.2 *Let* ξ *and* η *be two random matrices of order* $r \times s$ *with the entries of these matrices having finite variances. Further suppose that* $E(\xi) = E(\eta) = 0$ *and* $E(\eta\eta') > 0$. *Then*

$$E(\xi\xi') \geq E(\xi\eta')[E(\eta\eta')]^{-1}E(\eta\xi'). \tag{1.8}$$

Example 1

It is easy to see that, for any positive definite matrix D,

$$D + D^{-1} \geq 2I \tag{1.9}$$

since $(D^{\frac{1}{2}} - D^{-\frac{1}{2}})^2 \geq 0$. In particular, for any two positive definite matrices A_1 and A_2,

$$(A_1 - A_2)(A_1^{-1} - A_2^{-1}) = 2I - A_2A_1^{-1} - A_1A_2^{-1} = 2I - D - D^{-1} \leq 0 \quad (1.10)$$

where $D = A_2A_1^{-1}$. Let $g(A, B) = AB$. It can be checked that the condition (1.5) holds for the function $g(A, B)$ and hence, by Theorem 1.1,

$$Cov[\xi, \xi^{-1}] \leq Var[g(\xi, \xi^{-1})] = 0. \quad (1.11)$$

This in turn implies that

$$E[\xi]E[\xi^{-1}] \geq I \quad (1.12)$$

for any positive definite random matrix ξ. This was proved by alternate arguments in Babrovsky et al. [4].

Remark 1.1 For any two positive definite matrices A and B, define

$$A \triangle B = \frac{A + B}{2}, \quad (1.13)$$

$$A \# B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}, \quad (1.14)$$

and

$$A ! B = [\frac{1}{2}(A^{-1} + B^{-1})]^{-1}. \quad (1.15)$$

Ando [2] proved that

$$A ! B \leq A \# B \leq A \triangle B \quad (1.16)$$

generalizing the arithmetic mean-geometric mean-harmonic mean inequality for positive real numbers. It was also proved by Ando [1] that

$$A \triangle B \geq [\frac{1}{2}(A^q + B^q)]^{1/q} \quad (1.17)$$

for $\frac{1}{2} \leq q \leq 1$. Under the commutativity assumption on $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, it can be checked that $A \# B = (AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}$. It was shown earlier that Theorem 1.1 holds for the function $g_1(A, B) = A \triangle B$ and for the function $g_2(A, B) = A^{\frac{1}{2}}B^{\frac{1}{2}}$ under the commutativity condition. Theorem 1.1 does not hold the function $g_3(A, B) = A ! B$ even for real-valued random variables from Kimeldorf and Sampson [11]. They have proved that Theorem 1.1 holds for the function

$$g(x, y) = [\frac{1}{2}(x^\beta + y^\beta)]^{1/\beta}, 0 < \beta \leq 1$$

for real-valued random variables.

It is an open problem to find whether Theorem 1.1 holds for the function

$$g_4(A, B) = [\frac{1}{2}(A^q + B^q)]^{1/q}, \frac{1}{2} \leq q \leq 1,$$

and for the function

$$g_5(A, B) = A\#B$$

in view of Ando's inequality (1.17).

For other remarks on such inequalities for random matrices, see Prakasa Rao [16].

1.3 Harmonic Mean Inequalities

Rao [18] proved the following result.

Theorem 1.3 *If X and Y are positive-valued random variables with finite expectations, then*

$$E[(\frac{1}{X} + \frac{1}{Y})^{-1}] \leq (\frac{1}{E(X)} + \frac{1}{E(Y)})^{-1}. \tag{1.18}$$

Rao's proof involves the use of Holder's inequality and the dominated convergence arguments. We have extended this result to positive definite random matrices in Prakasa Rao [17].

Theorem 1.4 *If A and B are almost surely positive definite random matrices with $E(A)$, $E(A^{-1})$, $E(B)$ and $E(B^{-1})$ finite and either $E(A^{-1}) - (E[A])^{-1}$ or $E(B^{-1}) - (E[B])^{-1}$ is non-singular, then*

$$E((A^{-1} + B^{-1})^{-1}) < ((E[A])^{-1} + (E[B])^{-1})^{-1}. \tag{1.19}$$

Theorem 1.4 can be proved in several ways. The first method uses the Jensen inequality for random matrices. For the notion of convexity for functions defined on the space of matrices, see Davis [5].

Proof For any fixed positive definite matrix $\mathbf{M}$, the matrix-valued function $g(\mathbf{X}) = (\mathbf{M} + \mathbf{X}^{-1})^{-1}$ is concave in $\mathbf{X}$ over the class of positive definite matrices $\mathbf{X}$, and hence

$$E[(\mathbf{M} + \mathbf{X}^{-1})^{-1}] < [\mathbf{M} + E(\mathbf{X})^{-1}]^{-1}.$$

Taking expectations with respect to the probability measure of the random matrix $\mathbf{M}$ on both sides, the result follows.

Other proofs of Theorem 1.4 are based on the following lemmas which are of independent interest.