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Fractional Calculus

High-Precision Algorithms and Numerical Implementations

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ISBN 978-981-99-2069-3 ISBN 978-981-99-2070-9 (eBook)
https://doi.org/10.1007/978-981-99-2070-9

Translation from the Chinese Simplified language edition: "分数阶微积分学——数值算法与实现" by Xue Dingyü and Bai Lu. © Tsinghua University Press, Beijing 2023. Published by Tsinghua University Press, Beijing. All Rights Reserved.

Part of the translation was done with the help of an artificial intelligence machine translation tool. A subsequent human revision was done primarily in terms of content.

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

Fractional calculus and applications are rapidly developing research directions in science and engineering at present. This book provides a systematic introduction to fractional calculus, focusing on the use of computer tools to directly solve problems in the field of fractional calculus and applications. The structure and general idea of this book are as follows: ▸ Chap. 1 gives a review of the related fields; ▸ Chap. 2 introduces some special functions commonly used in this field; ▸ Chaps. 3 and 4 introduce the evaluation of fractional-order derivatives and integrals of known functions or sampling points; ▸ Chap. 5 introduces the use of specially designed filters to find the fractional-order derivatives and integrals of signals when the signals are unknown in advance. Chapters 3 to 5 can also be understood as offline and online methods for evaluating fractional-order derivatives and integrals. ▸ Chapter 6 introduces the analytical and numerical solutions of linear fractional-order differential equations (FODEs), ▸ Chap. 7 introduces the "command-driven" solution of nonlinear FODEs, ▸ Chap. 8 introduces the block diagram-based solution of FODEs, and ▸ Chap. 9 introduces the solution of special FODEs (including implicit ones, delayed differential equations, differential equation boundary value problems, and partial differential equations) that were previously difficult or impossible to solve. Each part in the book is equipped with MATLAB general solution functions written by the author, whereas a new FOTF Toolbox is released with the book. The readers can directly use these reusable codes to reproduce the results in the book, and more importantly, use them to creatively solve practical problems and explore new knowledge.

In 2015, I was invited by Prof. Li Changpin of the Department of Mathematics, Shanghai University, an internationally renowned scholar in the field of fractional calculus, to write a related monograph for his series "Fractional Calculus in Applied Science and Engineering". In 2017, my monograph Fractional-Order Control Systems: Fundamentals and Numerical Implementations was fortunately published as the first volume of the series in de Gruyter Publishing. The following year, the corresponding monograph in Chinese, Fractional-Order Calculus and Fractional-Order Control, was officially published by Science Press.

This book was completed in collaboration with Dr. Bai Lu, School of Information Engineering of Shenyang University. It incorporates many of our new results in recent years. In this book, there are many original research results which are published for the first time, including analytical solutions in fractional calculus, high-precision algorithms for high-order fractional-order derivatives, simulation and stability analysis of irrational systems, new FOTF Toolbox and FOTF Blockset, unified framework for solving FODEs, solution methods for fractional-order delay differential equations, solution methods for FODEs in boundary value problems, and time-fractional partial differential equations, benchmark problems for more types of algorithms for solving various FODEs, and so on.

In around 2000, I was encouraged and even persuaded by a long-time collaborator, Prof. YangQuan Chen, now at the University of California, Merced, to start research in the field of fractional-order control. But it was not until 2003, when I started working with Prof. Chen on the first edition of MATLAB Solutions for

Advanced Applied Mathematics Problems, that I really took the time to study the literature in this area and began to study fractional calculus, I began my research on numerical implementations in fractional calculus. In that work, a lot of work on fractional calculus computation, filter approximation, closed-form solution algorithms for linear FODEs, and block diagram-based solution of nonlinear FODEs were systematically introduced, and many of these codes and models are still widely used by the researchers in the field of fractional calculus. Therefore, I must first thank Prof. YangQuan Chen here.

We would also like to thank a number of prominent scholars and active researchers in the field of fractional calculus, including Profs. Igor Podlubny and Ivo Petráš of Technical University of Košice, Slovakia, Li Changpin of Shanghai University, Li Yan of Shandong University, Wang Yong of University of Science and Technology China, Lu Junguo of Shanghai Jiaotong University, Li Donghai of Tsinghua University, Yu Yongguang of Beijing Jiaotong University, Sun Guanghui of Harbin Institute of Technology, Chen Wen and Sun Hongguang of Hohai University, Zeng Caibin of South China University of Technology, Wang Chunyang of Changchun University of Technology, Wei Yiheng of Southeast University, Liu Dayan and Driss Boutat of INSA Centre Val de Loire, France (in no particular order), and others. I am also grateful to my former co-authors, namely, Profs. Blas Vinagre, Concepción Monje, and Vicente Feliu, when I published my monograph at Springer Publishing House in 2010, and my discussions and idea exchanges with them have generated many new ideas and research results in this field, which have enriched the content of this book.

My in-depth discussions with my colleagues at Northeastern University, especially Profs. Pan Feng, Chen Dali, and Zhang Xuefeng, have also brought much meaningful content to this book. I also thank my former students for their results and contributions to this book and related research, specifically Dr. Zhao Chunna for her contributions to the numerical computation of fractional calculus and differential equations, Drs. Zhao Chunna and Meng Li for their contributions to filter design, Drs. Zhao Chunna, Meng Li, MSc student Wang Weinan, Dr. Liu Lu, and Dr. Li Tingxue for their contributions to controller design, and also in other related areas Dr. Yang Yang, Dr. Zhang Yanzhu, Dr. Liu Yanmei, Dr. Chen Zhen, Dr. Chen Lanfeng and Ph.D. students Cui Xinshu, Liu Yitong and Wang Zhe for their contributions.

We thank the National Natural Science Foundation of China for the Natural Science Foundation projects (project numbers: 61174145 and 61673094) for the research work on this book.

Last but not least, I would like to thank my wife, Yang Jun, and my daughter, Xue Yang, for their great help and encouragement in my life and career. Without their encouragement and continued support, this book and my other books would not have come out successfully, and I would like to dedicate this book to them.

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Shenyang, China
November 2022

Contents

Introduction to Fractional Calculus

Contents

D. Y. Xue and L. Bai, *Fractional Calculus*, https://doi.org/10.1007/978-981-99-2070-9_1

1

1.1 Historic Review of Fractional Calculus

At the beginning of the development of the theory of classical calculus (called integer-order calculus in this book), the British scientist Isaac Newton and the German mathematician Gottfried Wilhelm Leibniz used different symbols for different orders of derivatives. For example, Newton used the notation $\dot{y}(x)$, $\ddot{y}(x)$ and $\dddot{y}(x)$, while Leibniz used the notation $\mathrm{d}^n y(x)/\mathrm{d}x^n$, where n is a positive integer. A natural question is how to extend n into fractions or even complex numbers. In a letter written by the French mathematician Marquis de l'Hôpital to Leibniz in 1695, he asked question "what would be the meaning if $n = 1/2$ in the $\mathrm{d}^n y(x)/\mathrm{d}x^n$ notation". In a letter dated 30 September 1695, Leibniz replied, "Thus it follows that $\mathrm{d}^{1/2}x$ will be equal to $x\sqrt{\mathrm{d}x : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn" [1]. The question and answer between these two mathematicians is widely considered to be the beginning of fractional calculus.

In 1819, the French mathematician Sylvestre François Lacroix used the Gamma function to study the power function

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n+1}, \quad m \geqslant n, \tag{1.1}$$

and found that

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}x = \frac{\Gamma(2)}{\Gamma(3/2)}\sqrt{x} = 2\sqrt{\frac{x}{\pi}}. \tag{1.2}$$

This conclusion is in full agreement with the later results of Riemann–Liouville fractional-order derivatives.

Now it seems that the derivative notation invented by Newton is not suitable for extension to the field of fractional calculus, while the notation invented by Leibniz can be used directly in fractional calculus.

More than three centuries have passed and until a few decades ago research in the field of fractional calculus has focused on purely mathematical theoretical aspects of the work. Some of the better historical reviews in the field of fractional calculus can be found in References [1] and [2]. In Reference [1], Profs. Kenneth Miller and Bertram Ross gave a good historical review of fractional calculus from a purely mathematical point of view, and in Reference [2], Keith Oldham and Jerome Spanier quoted the chronicles of fractional calculus from its inception to the year 1975, summarized by Prof. Ross. An overview of historical figures and their contributions in the field of fractional calculus is presented in Reference [3].

Starting from 1960, the study of fractional calculus began to be extended to the field of science and engineering. To solve the nonzero initial value problems in fractional calculus, Prof. Michele Caputo, an Italian scholar, proposed a new definition of fractional calculus, which was later called Caputo's definition [4]. He presented a dissipation model based on fractional-order derivatives with Prof. Francesco Mainardi [5], which established the foundation for the engineering applications of fractional calculus. In Japan, Prof. Shunji Manabe extended the study of non-integer-order to the application of control systems and introduced the concept of non-integer-order control systems [6]. In Slovakia, Prof. Igor Podlubny proposed the fractional-order PID controller [7]. The research group led by Prof. Alain Oustaloup in France proposed the concept and technique of fractional-order robust control and successfully applied it to suspension control in the automotive industry [8, 9], which is regarded as a milestone for the applications of fractional calculus in the real world.

Since about 2000, a number of monographs devoted to fractional calculus and its applications have appeared in various fields of specialization, among which the more influential ones are Prof. Podlubny's 1999 book on fractional-order differential equations and their applications in the field of automatic control [10], Prof. Rudolf Hilfer's 2000 book in the field of physics [11], Prof. Richard Magin's 2006 book in the field of bioengineering [12], and so on.

Several works have also been published in recent years on the theory and numerical computation in fractional calculus, such as the works of Prof. Kai Diethelm's in 2010 [13], Prof. Das's in 2010 [14], Prof. Uchaikin's in 2013 [15], and Prof. Li Changpin and Dr. Zeng Fanhai's in 2015 [16]. Also, Prof. Xue Dingyü's monograph in 2017 on numerical implementation of fractional-order systems and control [17].

In 2017, de Gruyter published the series "Fractional Calculus in Applied Sciences and Engineering", edited by Prof. Li Changpin, and the "Handbook of Calculus in Applied Sciences and Engineering", edited by Profs. Anatoly Kochubei and Yuri Luchko.

In the field of automatic control, some monographs have been published in recent years, such as those by Profs. Caponetto, Dongola, Fortuna, and Petráš in 2010 [18]; Profs. Monje, Chen, Vinagre, Xue, and Feliu in 2010 [19]; Prof. Petráš in 2011 [20]; Dr. Luo Ying and Prof. Chen YangQuan in 2012 [21]; and Prof. Oustaloup in 2014 [22]. Prof. Uchaikin gave a good introduction to the applications of fractional calculus in various fields in 2013 [23].

Chinese scholars have also published textbooks and monographs on fractional calculus and its applications. There is a special chapter on fractional calculus and its computation in the book published by Profs. Xue Dingyü and Chen YangQuan in 2004 [24]. The following works are related to the research of this book, including the books by Profs. Chen Wen, Sun Hongguang, and Li Xicheng in 2010 [25]; Prof. Wang Jifeng in 2010 [26]; Profs. Zhao Chunna, Li Yingshun, and Lu Tao in 2011 [27]; Profs. Wang Chunyang, Li Mingqiu, and Jiang Shuhua in 2014 [28]; Profs. Li Wen and Zhao Huimin in 2014 [29]; Profs. Xiaozhong Liao and Zhe Gao in 2016 [30]; Profs. Qiang Wu and Jianhua Huang in 2016 [31]; Prof. Xue Dingyü in 2018 [32]; and so on.

Chinese scholars in the field of fractional calculus and applications established a special committee on fractional-order systems and control under the Chinese Association of Automation in July 2018 [33]. Special sessions or special issues in the field of fractional-order systems have also appeared one after another in many international conferences or international journals.

It is important to note that the term "fractional-order" is a misused one. The correct name should be "non-integer-order" or even "arbitrary order", because the order can be irrational or even complex. In addition to fractions (rational numbers), for example, $\mathrm{d}^{\sqrt{2}}y(t)/\mathrm{d}t^{\sqrt{2}}$ can be considered as the $\sqrt{2}$nd-order derivative of the signal $y(t)$. The complex order is beyond the scope of this book. However, the term "fractional-order" has been used by the huge majority of researchers in the vast reference text. So the term will be used in this book, but in essence it includes irrational orders and even irrational system structures.

Integer-order calculus has a concise and clear physical meaning. For example, displacement, velocity, and acceleration can be used to explain the relationship between a signal and its integer-order derivatives well. However, the fractional calculus does not have such a concise and understandable physical interpretation, although many scholars have tried to do so. One meaningful interpretation was given by Prof. Podlubny as "a shadow moving on a fence" [34], but it still seems to lack such a concise interpretation as the integer-order calculus.

Examples are given below to demonstrate fractional calculus operations for commonly used functions.

▶ Example 1.1

Consider the sinusoidal signal $\sin t$. It is known that the first-order derivative of this signal is $\cos t$. If we then find the high-order derivative of this signal, the result will be nothing more than $\pm\sin t$ and $\pm\cos t$. No other signal can be derived. What will happen if we introduce the concept of fractional calculus?

Solutions From the famous Cauchy integral formula

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\sin t = \sin\left(t + n\frac{\pi}{2}\right).$$

In fact, the above formula holds when n is any non-integer, so the following MATLAB statements can be used to draw the surface of the fractional-order derivatives of the function, as shown in Fig. 1.1.

```
>> n0=0:0.1:1.5; t=0:0.2:2*pi; Z=[];
   for n=n0, Z=[Z; sin(t+n*pi/2)]; end, surf(t,n0,Z)
```

It can be seen that, in addition to the four known results $\pm\sin t$ and $\pm\cos t$, other information can be obtained and the results are asymptotic. Therefore, the fractional-order derivative of a function may provide richer information than the integer-order ones. In practical applications, if the world is viewed from the perspective of fractional calculus, it may reveal more things that were invisible from the perspective of integer-order calculus. ◀

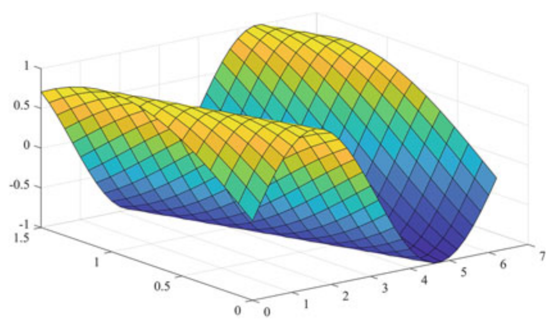

Fig. 1.1 Representation of surfaces with different order derivatives

1.2 Fractional Calculus Phenomena and Modeling Examples in Nature

There are many examples about the applications of fractional calculus in References [11, 12, 14]. Some relevant typical examples are listed here, which often cannot be described well in the framework of integer-order calculus. They must be described with the help of fractional calculus. Thus, fractional-order phenomena are actually ubiquitous.

▶ Example 1.2

In polymer materials and elastic material research, according to the suggestions in Reference [11], the rheological constitutive equation should be more precisely described as fractional-order differential equations (FODEs)

$$\sigma(t) + \tau^{\alpha-\beta}\frac{\mathrm{d}^{\alpha-\beta}\sigma(t)}{\mathrm{d}t^{\alpha-\beta}} = E\tau^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\sigma(t),$$

where $0 < \alpha, \beta < 1$. ◀

▶ Example 1.3

Consider the driving-point impedance problem of a semi-infinite length lossy transmission line, whose standard voltage equation satisfies the integer-order partial differential equation. The boundary value conditions are known

$$\frac{\partial v(x,t)}{\partial t} = \alpha\frac{\partial^2 v(x,t)}{\partial x^2},\ \ v(0,t) = v_{\mathrm{I}}(t),\ \ v(\infty,t) = 0.$$

After a series of direct mathematical formulations [14], the voltage–current equation for the impedance at the driving point can be derived as the following FODE with zero initial value:

$$i(t) = \frac{1}{R\sqrt{\alpha}}\frac{\mathrm{d}^{1/2}v(t)}{\mathrm{d}t^{1/2}} \ \text{ or } \ v(t) = R\sqrt{\alpha}\frac{\mathrm{d}^{-1/2}i(t)}{\mathrm{d}t^{-1/2}}.$$

◀

▶ Example 1.4

The Bagley–Torvik equation [35, 36] describing the vibration of an oscillator in a viscoelastic medium is

$$A\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + B\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}y(t) + Cy(t) = C(t+1). \tag{1.3}$$

The initial value conditions are $y(0) = y'(0) = 1$, with solution $y(t) = t + 1$. ◀

▶ Example 1.5

Ionic polymer metal composite (IPMC) is a new type of smart material that has a wide range of applications in areas such as robotics actuators and artificial muscles. In order

1

to identify the model of IPMC, a set of frequency domain response data can be measured experimentally. The identification can be attempted by the well-established method of linear system model identification, but Reference [18] shows that no good identification model can be obtained in the framework of integer-order system. If the idea of fractional calculus is introduced, it may result in the following identification model:

$$G(s) = \frac{340}{s^{0.756}(s^2 + 3.85s + 5880)^{1.15}}.$$

It is clear that the identified model is a special form of fractional-order model. ◀

▶ Example 1.6

In a standard heat diffusion process, the temperature of a heat source rod at coordinate x can be directly described by the one-dimensional linear partial differential equation given below:

$$\frac{\partial c}{\partial t} = K\frac{\partial^2 c}{\partial x^2}.$$

If a constant temperature source C_0 is added at $x = 0$, the Laplace transform expression for the temperature under thermal diffusion can be deduced [37]

$$c(x, s) = \frac{C_0}{s} e^{-x\sqrt{s/k}}.$$

◀

▶ Example 1.7

The memristor (resistor with memory) was pointed out as the fourth basic circuit element (the first three are the familiar and physically present resistor, capacitor, and inductor) by Prof. Chua in 1971 [38]. In 2008, researchers claimed to have found such a missing element [39]. Since the fractional calculus has the capabilities of describing memory, its resistance can be expressed in the fractional calculus sense as follows:

$$R_{\rm m} = \left[R_{\rm in}^{\alpha+1} \mp 2kR_{\rm d} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\alpha}} {\rm d}\tau \right]^{1/(\alpha+1)},$$

where the integral function is the basic expression of the fractional calculus [40]. ◀

In controller design, it is often necessary to deliberately introduce fractional-order actions in order to achieve a certain control performance index. An example of a controller is given below.

▶ Example 1.8

Consider the fractional-order quantitative feedback theory (QFT) given in the literature [19]. The controller model is as follows, which contains fractional-order operations.

$$G_c(s) = 1.8393 \left(\frac{s+0.011}{s}\right)^{0.96} \left(\frac{8.8\times 10^{-5}s+1}{8.096\times 10^{-5}s+1}\right)^{1.76} \frac{1}{(1+s/0.29)^2}.$$

◄

In integer-order calculus, the signal in the system exhibits the form of an exponential function, while it can be observed in real life that certain phenomena exhibit results that may be power functions of time, often referred to as power-law phenomenon. This phenomenon is easier to understand under the fractional calculus framework. With the perspective of fractional calculus, one can better understand the complex world [41].

1.3 Historic Review of Fractional Calculus Computations

1.3.1 Numerical Computing in Fractional Calculus

With the maturity of computer technology, the theory of numerical computation in fractional calculus has developed rapidly. The first problem that was extensively studied was how to evaluate fractional-order derivatives and integrals numerically. Many classical algorithms emerged. A concise approach is to remove the limit sign from the definition of Grünwald–Letnikov fractional calculus, which leads to an approximate formula. References [10, 42] proved that this method has $O(h)$ accuracy. For most functions, the Grünwald–Letnikov definition is equivalent to the Riemann–Liouville definition, so this method can also be applied to evaluate the Riemann–Liouville fractional-order derivatives and integrals. Profs. Meerschaert and Tadjeran proposed the shifted approximation formula, which shifts the discrete points backward, so as to increase the convergence speed of the formula [43].

Prof. Lubich and co-workers, in their study of the Abel–Volterra integral equation, proposed the linear fractional-order multi-step method [44, 45], which introduced higher accuracy formulas for evaluating fractional-order derivatives and integrals. Reference [44] proved that the accuracy of this algorithm is not only related to the order of the generating function, but also affected by the initial value conditions of the original function. The linear fractional-order multi-step method is a classical algorithm that has had a great influence on the development of fractional-order numerical algorithms. Prof. Podlubny introduced the fast Fourier transform-based method to evaluate w_j on this basis. This algorithm has a faster computational speed [10], although it is possible to introduce greater errors as a result.

The finite-part integral (FPI) method proposed by Prof. Kai Diethelm and co-workers [13, 46–48] allows the evaluation of not only Riemann–Liouville fractional-order derivatives and integrals, but also Caputo fractional-order derivatives. This method converts the fractional-order derivative or integral into a Hadamard integral by dividing equidistant grid points on the integration interval, finding a simple function to replace the original function at the grid points, and then deriving the computational formula. Reference [46] proves that this method has $O(h^2)$ accuracy.

Prof. Igor Podlubny proposed the matrix method [49], which represents fractional-order derivatives and integrals in the form of matrices. Applying this method, it is also possible to convert FODEs into matrix equations, which are easier to find the numerical solutions.

1.3.2 Numerical Computing in Fractional-Order Ordinary Differential Equations

With the vigorous development of fractional calculus theory, more and more FODEs appear in engineering, and how to solve FODEs becomes a hot topic of research. From the perspective of the demand of applied science, the solution of FODEs under Caputo's definition is widely concerned. For some simple FODEs, the analytical solutions can be found directly. Commonly used methods include the integral transform method [50, 51], the Green's function method [10, 52, 53], and the Adomian decomposition method [54]. However, for most of the FODEs, finding analytical solutions is an impractical matter. Therefore, numerical algorithms for solving FODEs have become a hot research topic in this field.

The Adams–Bashforth–Moulton algorithm [55, 56] is an effective method for solving one-term FODEs. This algorithm first derives the integral form of the equation, then applies the first-order extrapolation to calculate the predictor solution, then substitutes the predictor solution into the original equation and applies the second-order extrapolation to calculate the corrector solution. The Adams–Bashforth–Moulton algorithm has $O(h^2)$ accuracy. Even if the FODE is nonlinear, it can also be solved by this algorithm.

It is also possible to introduce Prof. Igor Podlubny's matrix algorithm into the numerical solution process of linear and nonlinear FODEs. It should be noted in particular that the matrix algorithm can also solve certain fractional-order implicit differential equations.

The finite partial integration algorithm proposed by Prof. Kai Diethelm and his collaborators can be used to solve the one-term FODEs [46]. This algorithm writes the solution of the original equation in the form of Hadamard integral, applies the interpolation algorithm to find a simple function to replace the integrand, numerically computes the integral to obtain the predictor solution, and substitutes the predictor solution into the original equation to calculate the corrector solution. Reference [46] proves that the finite partial integration algorithm has $O(h^2)$ accuracy. Based on the finite partial integration algorithm, Diethelm proposed the predictor–corrector (PECE) algorithm [48, 57, 58]. This algorithm first applies extrapolation to solve the predictor solution of the equation and then substitutes the predictor solution into the original equation to find the corrector solution. In Reference [59], FODEs with nonzero initial value conditions were investigated. An auxiliary function was introduced to transform the original equation into an equation with zero initial value condition before solving it efficiently. The iterative algorithm was embedded into a corrector process in References [17, 32], which substantially improved the solution accuracy of the predictor–corrector algorithm.

Chinese scholars have made many contributions in the numerical algorithms about fractional calculus, including the closed-form solutions of linear FODEs proposed by Prof. Xue Dingyü and co-workers [24, 60]; high-precision algorithms in fractional calculus [61, 62]; high-precision numerical algorithms for Caputo ordinary differential equations [17, 32]; block diagram-based unified modeling and simulation framework for Caputo ordinary differential equations [63]; the high-precision numerical algorithms for fractional calculus and differential equations proposed by Prof. Li Changpin and co-workers [64, 65]; the block-by-block algorithm proposed by Profs. Wang Ziqiang and Cao Junying [66]; and so on. The algorithm in Reference [65] has a high-precision accuracy. With this algorithm it is possible to evaluate the Caputo derivative of orders in the (0, 1) interval and to find the numerical solution of the equation. Reference [65] proves that this method has an accuracy of $O(h^{r+1-\alpha})$. In the block-by-block algorithm, the FODE is transformed into the Volterra integral equation, and the integral interval is divided into equally spaced intervals. The quadratic Lagrange basis function is applied to replace the integrand in each interval to derive the formula for solving the numerical solution of the original equation.

To evaluate the algorithms for solving FODEs, the authors of this book proposed a series of benchmark problems for Caputo equations [67, 68]. These problems are highly general and can be used for a fair comparison of various FODE algorithms. Based on a general modeling and simulation approach based on block diagrams, this book further explores fractional-order state space equations with orders larger than 1, implicit FODEs, fractional-order delay differential equations (FODDEs), and FODEs with known boundary value conditions. Many of them are extremely rare or even completely non-existent in the literature, but the general modeling and simulation methods presented in this book make it easy to derive high-precision numerical solutions.

1.3.3 Numerical Computing in Fractional-Order Partial Differential Equations

Fractional-order partial differential equations (FOPDEs) can be broadly classified into time-fractional partial differential equations, space-fractional partial differential equations, time–space-fractional partial differential equations, and so on. There are some mature methods for solving FOPDEs, including finite difference algorithm, Adomian decomposition method, and variational iteration method.

Prof. Liu Fawang and co-workers [69] and Prof. Sun Zhizhong and co-workers [70] have discussed finite difference methods in detail in their respective monographs, given difference format algorithms for different kinds of FOPDEs. The accuracy, stability, solvability, and convergence of each difference format are discussed. More results and literature on finite difference methods are also listed and the development of finite difference methods is summarized. The authors of this book give algorithms for the solution of the unified two time-fractional difference formats [71].

The decomposition method proposed by Prof. Adomian is a method to apply the series to find the approximate analytical solution of equation [72]. Some scholars have applied the Adomian decomposition method to solve FOPDEs [73, 74]. The Adomian decomposition method avoids the discretization of FODEs and thus is with less computational load, but this method requires the calculation of complicated fractional-order integrals, so it increases the computational difficulty.

Many numerical methods for solving other forms of FOPDEs, such as variational iterative methods, finite element methods, spectral methods, and so on, have also appeared in the literature. A summary of numerical algorithms for fractional calculus and FOPDEs is given in the works of Prof. Liu Fawang and co-workers [69], Prof. Guo Boling and co-workers [75], and Prof. Li Changpin and Dr. Zeng Fanhai [16], and interested readers can find the related literature.

1.4 Tools in Fractional Calculus and Fractional-Order Control

There are several MATLAB toolboxes that are widely used in the field of fractional calculus and fractional-order control. A comparative review of the commonly used toolboxes is given in Reference [76]. In fact, many of the tools compared in Reference [76] are only single MATLAB functions, among which only four can be called toolboxes. These toolboxes happen to be in the field of fractional-order control. A brief comparison is presented here in the chronological order of their introduction.

(1) **CRONE Toolbox** [8] is the result of the CRONE research group led by the famous French scholar Prof. Oustaloup. The work started around 1990 as a practical tool for solving fractional-order system identification and robust controller design. The disadvantage is that it is distributed in MATLAB pseudo-code encrypted form, and there is no way for the user to modify or extend any of the features of the toolbox. CRONE is abbreviation of a French words *commande robuste d'ordre non entier*, meaning non-integer-order robust control.

(2) **Ninteger Toolbox** [77] was developed by a Portuguese scholar, Prof. Valério, in 2001. The earliest version mainly implemented the CRONE controller, and in the more mature version 2.3, the toolbox provided a set of functions, models, and interfaces for the design and analysis of fractional-order system and controller design based on MATLAB and Simulink. The toolbox has two core functions, one is the identification of fractional-order systems and the other is the approximation of fractional-order systems with integer-order models. No new versions of this toolbox have been released since 2009.

(3) **FOTF Toolbox** [78, 79] is a MATLAB toolbox for fractional calculus and fractional-order control system research written by Prof. Xue Dingyü in China. It was first made public under the name of FOTF in 2006, and since 2004, the author has successively released many MATLAB functions and Simulink models for fractional calculus and control. In 2017, all the programs and models were rewritten to support the analysis and design of fractional-order multivariable systems in conjunction with the publication of the monographs [17, 32]. In addition, the underlying fractional calculus calculations are replaced by the high-precision algorithms proposed by the authors, which are usually many orders of magnitude more accurate than the existing ones, making the toolbox itself more efficient and reliable. A brief introduction and demonstration of the main functions of the toolbox are given in Reference [80]. With the writing and publication of this book, the FOTF Toolbox has undergone another major revision, which is more powerful and more suitable for practical applications. This book will introduce the theoretical knowledge of the system and the details of the toolbox software development based on this toolbox in detail. The code is also all open source, which is also useful for readers to learn the numerical implementation in the field of fractional-order systems.

(4) **The FOMCON Toolbox** [81] was developed by the Estonian scholar Prof. Aleksie Tepljakov in his master's research. The toolbox was developed initially by copying

and integrating the facilities of the FOTF Toolbox and the Ninteger Toolbox. Later, some of the procedures were rewritten under the framework established by the original authors to form a set of procedures and models for solving the identification, analysis, and design of fractional-order systems. Since the toolbox was based on an earlier version of the FOTF class, the toolbox is limited to solving problems for univariate fractional-order systems, and the new version provides limited functionality for solving problems in multivariate systems [82].

In addition, readers are advised to download user contributed toolboxes and utility functions that may be available from the “File Exchange” section of the MathWorks’ website, but because of the varying levels of programming on this site, some tools may be of poor quality and may sometimes lead to incorrect results. Users should be especially careful when choosing to download toolboxes and utility functions.

1.5 Structures in the Book

1.5.1 Main Contents

This book systematically introduces the fundamentals related to numerical computational problems in the field of fractional calculus, provides readers with directly usable computer tools that can enhance their understanding of the content and improve the manipuility of individual topics.

In ▸ Chap. 1, a brief review of the development of fractional calculus and its application areas is given. An explanation of why the fractional calculus perspective on the world is introduced through some real-world fractional-order phenomena, and a summary of several internationally available MATLAB toolboxes for fractional calculus and fractional-order control are presented. This book will make extensive use of the FOTF Toolbox developed by the authors.

▸ Chapter 2 focuses on the definition, properties, and computation of various special functions commonly used in the field of fractional calculus, and provides a foundation for introducing the definition and computation of fractional calculus.

▸ Chapter 3 introduces various common definitions of fractional calculus, such as the Cauchy integral formula, the Grünwald–Letnikov definition, the Riemann–Liouville definition, and the Caputo definition; summarizes the relationships between the definitions; and gives concise analytical and numerical operations. The chapter takes the fractional-order integral as an example, its geometric interpretation is given.

▸ Chapter 4 addresses the problem of numerical computation in fractional calculus by proposing and implementing a series of numerical algorithms with high accuracy, several orders of magnitude higher than conventional algorithms, which can be considered as the basis of numerical computation for the subsequent contents of the book. This chapter also explores a new path to efficiently derive solutions to fractional-order derivative problems of higher orders.

▸ Chapter 5 introduces filter approximation methods for fractional-order behaviors and discusses various continuous and discrete filter design methods, including filter implementation methods for fractional-order operators, fractional-order transfer functions (FOTFs), irrational transfer functions, and so on. An online implementation method for fractional-order derivative of unknown signals is given, as well as suggestions for the selection of filter design parameters.

▸ Chapter 6 introduces the format of linear FODEs, discusses the analytical and numerical algorithms, proposes high-precision numerical solutions of linear FODEs with zero and nonzero initial values, and gives the stability assessment methods of linear fractional-order systems and irrational systems. In particular, this chapter also systematically investigates the simulation method of irrational systems based on numerical Laplace transform and its inverse, which can be extended to the study of feedback control of irrational systems.

▸ Chapter 7 discusses the numerical methods for nonlinear fractional-order systems, focusing mainly on the command-driven solution algorithm and MATLAB implementation. Firstly, the numerical methods for explicit FODEs and fractional-order state space equations are introduced, and the computer code of the traditional numerical algorithm is given; in addition, a high-precision numerical method for nonlinear explicit FODEs is proposed and implemented, which significantly improves the computational accuracy and efficiency of the algorithm.

▸ Chapter 8 continues the discussion on the numerical methods for nonlinear FODEs, focusing mainly on block diagram-based differential equation solving methods, which significantly extend the solution capabilities and improve the efficiency. This chapter gives a brief introduction to the FOTF Toolbox, introduces the practical blocks that can be used for Simulink modeling and simulation, and presents a general method based on integrator chains for modeling and solving the Riemann–Liouville and the Caputo equations, and evaluates the accuracy and efficiency of the methods. Such methods can theoretically be used to deal with the solution of nonlinear FODEs with arbitrary complexity.

▸ Chapter 9 introduces algorithms for solving special FODEs, including implicit FODEs that are difficult or even impossible to solve in the traditional sense, fractional-order delay differential equations with nonzero history functions and boundary value problems of FODEs, and so on. It also introduces numerical methods for solving time-fractional partial differential equations. This chapter provides the ideas and implementation for a comprehensive study about the solution of FODEs.

In order to test and evaluate the accuracy and the efficiency of the algorithms for solving FODEs, Appendix A designs and proves some benchmark problems composed of various FODEs, which can be used to fairly compare the accuracy, efficiency, and other superiority of various numerical algorithms. Appendix B gives the common special functions and Laplace transforms related to fractional calculus. Appendix C lists the function tables of the FOTF Toolbox for the readers' reference and review, so that the readers can better use the FOTF Toolbox to complete the research in the field of fractional-order systems.

1.5.2 Reading Suggestions

An important feature of this book is that each topic is supported by MATLAB codes and models written by the authors, so that the reader can also better understand the relevant contents of this book from the supporting codes. More importantly, the reader can directly use these codes to solve relevant problems and even creatively solve some unknown problems that may be associated with them. The FOTF Toolbox provided in this book can solve basic problems in fractional calculus operations and can be used throughout the process of modeling, analysis, and design of fractional-order systems.

The FOTF Toolbox can be downloaded for free at the following URL.

► http://cn.mathworks.com/matlabcentral/fileexchange/60874-fotf-toolbox

This book can also be used as a textbook and reference book for general readers to learn fractional calculus and fractional-order control. In particular, the following study suggestions are given for reference to the following three groups of readers, i.e., control engineers who want to introduce fractional-order control methods into their own research fields, general researchers who want to study fractional calculus and fractional-order control systematically, and researchers in the field of mathematics who want to study fractional calculus.

(1) For the control engineers who only want to introduce the concept of fractional-order control in their practical applications, they can skip all the theoretical contents of this book and only need to understand the basic definition of fractional calculus, and then learn the direct calculation in fractional calculus—for known functions and data, calculating directly by the functions of the MATLAB toolbox; for unknown signals, reconstructing its fractional-order derivative and integral by the filters of the MATLAB toolbox. Such readers should also learn the methods for solving linear and nonlinear FODEs, especially those based on the Simulink environment. They should also learn how to use the two classes FOTF and FOSS written by the authors and will find the process of modeling and analyzing linear fractional-order systems as easy and convenient as integer-order systems. With these tools as a basis, users can introduce the concept of fractional calculus into their practical work as they wish and try out a class of controllers that may yield better results.

(2) For non-mathematical researchers, they can make full use of the special writing format of this book, use the provided MATLAB program as the main tool to systematically learn the necessary knowledge in the field of fractional calculus and fractional-order control, and reproduce the results in the book to better understand the relevant technical contents. Further, one can make full use of such tools to solve similar problems directly, to investigate more complicated and unknown problems creatively, and try to apply fractional-order systems theory to one's own research field to find more useful results. The proofs of the theorems can be skipped directly when reading this book.

(3) For the researchers with mathematical backgrounds, the proofs of the theorems covered in this book are not very comprehensive. It is recommended to find the omitted parts from other related works or to give the proofs by yourself. The programming details and techniques given in this book are worthy of reference and learning. It is recommended that such readers fully study the MATLAB programming techniques in this book to improve their programming ability and increase the efficiency of their code implementation. In addition, the authors sincerely hope that the computational examples which are selected carefully in this book can be used as the comparisons for such readers to test their algorithms. A set of benchmark problems for comparing the performance of numerical algorithms for solving FODEs are also constructed in Appendix A. These benchmark problems can be solved directly by the algorithms given in the book, and many of them have already yielded numerical solutions with

considerable accuracy and speed. In addition, the reusable codes and models in this book can also reproduce the numerical solutions of these problems, and readers can also write codes for their own or others' algorithms to find the results which are better than those in this book in terms of speed, accuracy, and adaptability. Readers can also develop their own benchmark problems to challenge the algorithms in this book. As the old Chinese saying goes, "Try it to show you are a horse not a mule", and only through such fair comparisons and solutions can meaningful progress be made in this area of research in the right direction.

All the results in this book can be reproduced by the corresponding MATLAB statements given in the book and the open-source FOTF Toolbox. It is believed that readers can use this book and the toolbox to gain more knowledge in the field of fractional calculus.

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Contents

D. Y. Xue and L. Bai, ***Fractional Calculus***, https://doi.org/10.1007/978-981-99-2070-9_2