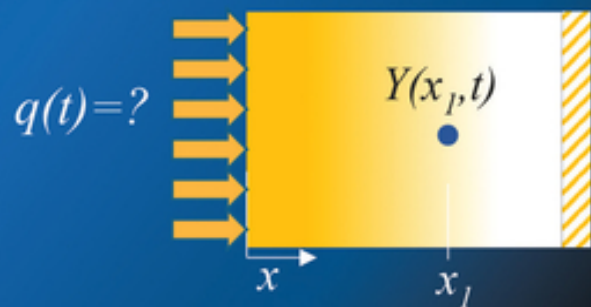


Second Edition

# Inverse Heat Conduction

Ill-Posed Problems

Keith A. Woodbury • Hamidreza Najafi  
Filippo de Monte • James V. Beck



$$\mathbf{q} = \mathbf{F}\mathbf{Y} \quad \leftarrow \quad \begin{aligned} \mathbf{T} &= \mathbf{X}\mathbf{q} \\ \mathbf{Y} &= \mathbf{T} + \boldsymbol{\varepsilon} \end{aligned}$$

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# Table of Contents

[Cover](#)

[Title Page](#)

[Copyright Page](#)

[Dedication Page](#)

[Memorium Page](#)

[List of Figures](#)

[Nomenclature](#)

[Preface to First Edition](#)

[Preface to Second Edition](#)

[1 Inverse Heat Conduction Problems](#)

[1.1 Introduction](#)

[1.2 Basic Mathematical Description](#)

[1.3 Classification of Methods](#)

[1.4 Function Estimation Versus Parameter Estimation](#)

[1.5 Other Inverse Function Estimation Problems](#)

[1.6 Early Works on IHCPs](#)

[1.7 Applications of IHCPs: A Modern Look](#)

[1.8 Measurements](#)

[1.9 Criteria for Evaluation of IHCP Methods](#)

[1.10 Scope of Book](#)

[1.11 Chapter Summary](#)

[References](#)

[2 Analytical Solutions of Direct Heat Conduction Problems](#)

[2.1 Introduction](#)

[2.2 Numbering System](#)

[2.3 One-Dimensional Temperature Solutions](#)

[2.4 Two-Dimensional Temperature Solutions](#)

[2.5 Chapter Summary](#)

[Problems](#)

[References](#)

### [3 Approximate Methods for Direct Heat Conduction Problems](#)

[3.1 Introduction](#)

[3.2 Superposition Principles](#)

[3.3 One-Dimensional Problem with Time-Dependent Surface Temperature](#)

[3.4 One-Dimensional Problem with Time-Dependent Surface Heat Flux](#)

[3.5 Two-Dimensional Problem with Space-Dependent and Constant Surface Heat Flux](#)

[3.6 Two-Dimensional Problem with Space- and Time-Dependent Surface Heat Flux](#)

[3.7 Chapter Summary](#)

[Problems](#)

[References](#)

### [4 Inverse Heat Conduction Estimation Procedures](#)

[4.1 Introduction](#)

[4.2 Why is the IHCP Difficult?](#)

[4.3 Ill-Posed Problems](#)

[4.4 IHCP Solution Methodology](#)

[4.5 Sensitivity Coefficients](#)

[4.6 Stolz Method: Single Future Time Step Method](#)

[4.7 Function Specification Method](#)

[4.8 Tikhonov Regularization Method](#)

[4.9 Gradient Methods](#)

[4.10 Truncated Singular Value Decomposition Method](#)

[4.11 Kalman Filter](#)

[4.12 Chapter Summary](#)

[Problems](#)

[References](#)

## [5 Filter Form of IHCP Solution](#)

[5.1 Introduction](#)

[5.2 Temperature Perturbation Approach](#)

[5.3 Filter Matrix Perspective](#)

[5.4 Sequential Filter Form](#)

[5.5 Using Second Temperature Sensor as Boundary Condition](#)

[5.6 Filter Coefficients for Multi-Layer Domain](#)

[5.7 Filter Coefficients for Non-Linear IHCP: Application for Heat Flux Measurement Using Directional Flame Thermometer](#)

[5.8 Chapter Summary](#)

[Problems](#)

[References](#)

## [6 Optimal Regularization](#)

[6.1 Preliminaries](#)

[6.2 Two Conflicting Objectives](#)

[6.3 Mean Squared Error](#)

[6.4 Minimize Mean Squared Error in Heat Flux  \$R\_q^2\$](#)

[6.5 Mean Squared Error in Temperature  \$R\_T^2\$](#)

[6.6 The L-Curve](#)

[6.7 Generalized Cross Validation](#)

[6.8 Chapter Summary](#)

[Problems](#)

[References](#)

## [7 Evaluation of IHCP Solution Procedures](#)

[7.1 Introduction](#)

[7.2 Test Cases](#)

[7.3 Function Specification Method](#)

[7.4 Tikhonov Regularization](#)

[7.5 Conjugate Gradient Method](#)

[7.6 Truncated Singular Value Decomposition](#)

[7.7 Kalman Filter](#)

[7.8 Chapter Summary](#)

[Problems](#)

[References](#)

## [8 Multiple Heat Flux Estimation](#)

[8.1 Introduction](#)

[8.2 The Forward and the Inverse Problems](#)

[8.3 Examples](#)

[8.4 Chapter Summary](#)

[Problems](#)

[References](#)

## [9 Heat Transfer Coefficient Estimation](#)

[9.1 Introduction](#)

[9.2 Sensitivity Coefficients](#)

[9.3 Lumped Body Analyses](#)

[9.4 Bodies with Internal Temperature Gradients](#)

[9.5 Chapter Summary](#)

[Problems](#)

[References](#)

## [10 Temperature Measurement](#)

[10.1 Introduction](#)

[10.2 Correction Kernel Concept](#)

[10.3 Unsteady Surface Element Method](#)

[10.4 Chapter Summary](#)

[Problems](#)

[References](#)

## [Appendix A: Numbering System](#)

[A.1 Dimensionality, Coordinate System, and Types of Boundary Conditions](#)

[A.2 Boundary Condition Information](#)

[A.3 Initial Temperature Distribution](#)

[References](#)

## [Appendix B: Exact Solution](#)

[B.1 Exact Analytical Solution: Short-Time Form](#)

[B.2 Exact Analytical Solution. Large-Time Form](#)

[References](#)

## [Appendix C: Green's Functions Solution Equation](#)

[C.1 Introduction](#)

[C.2 One-Dimensional Problem with Time-Dependent Surface Temperature](#)

[C.3 One-Dimensional Problem with Time-Dependent Surface Heat Flux](#)

[C.4 Two-Dimensional Problem with Space- and Time-Dependent Surface Heat Flux](#)

[Reference](#)

[Index](#)

## End User License Agreement

# List of Tables

### Chapter 2

[Table 2.1 Comparison between the total number of terms required from the tw...](#)

[Table 2.2 Number of terms required from the three different summations appe...](#)

### Chapter 3

[Table 3.1 Comparison of exact and approximate temperatures for three differ...](#)

[Table 3.2 Comparison of exact and approximate temperatures \(limited to the ...](#)

[Table 3.3 Exact and approximate variations of the surface temperature into ...](#)

[Table 3.4 Comparison of exact and approximate temperatures for three differ...](#)

[Table 3.5 Comparison of exact and approximate temperatures \(limited to the ...](#)

[Table 3.6 Exact and approximate variation of the surface heat flux into the...](#)

[Table 3.7 Surface heat flux components and building block temperatures at t...](#)

[Table 3.8 Comparison of approximate temperatures for six different space st...](#)

[Table 3.9 Building block temperatures at the surface point  \$y\_P = 10\$  cm for a...](#)

[Table 3.10 Surface heat flux components and sensitivity coefficients at the...](#)

## Chapter 4

[Table 4.1 Dimensionless gain coefficients for Eqs. \(4.71a\)-\(4.71d\) for sens...](#)

[Table 4.2 Dimensionless gain coefficients for Eqs. \(4.75a\) and \(4.75b\) for ...](#)

[Table 4.3 Dimensionless gain coefficients for Eq. \(4.71a,b\) for sensor at  \$x\$](#)

[Table 4.4 Condition number of coefficient matrix in Eq. \(4.112\) for piecewi...](#)

[Table 4.5 Iteration results for steepest descent](#)

[Table 4.6 Iteration results for Fletcher-Reeves conjugate gradient.](#)

[Table 4.7 Iteration results for adjoint method variables.](#)

[Table 4.8 Iteration results for adjoint method updates using Fletcher-Reeve...](#)

[Table 4.9 TSVD results when  \$\Delta t = 0.125\$  deleting one singular value.](#)

## Chapter 5

[Table 5.1 Filter coefficients for the function specification method X22B10T...](#)

[Table 5.2 Values of filter coefficients \(middle-column\) for Tikhonov regula...](#)

[Table 5.3 Values of filter coefficients \(middle-column\) for CGM \( \$\tilde{x} = 0.5, \Delta \tilde{t} = 0.05\$ \).](#)

[Table 5.4 Characteristics of the layers of a directional flame thermometer ...](#)

[Table 5.5 Calculated RMS error between the estimated heat fluxes using the ...](#)

[Table 5.6 Calculated RMS error between the estimated heat fluxes using the ...](#)

## Chapter 6

[Table 6.1 Expected error values for Example 6.1 for  \$\alpha\_0 = 1E-4\$  for vario...](#)

[Table 6.2 Optimal  \$\alpha\_0\$  and expected error values for three different hea...](#)

[Table 6.3 Optimal  \$r\$  and expected error values for three different heat flux...](#)

[Table 6.4 Data for Example 6.4.](#)

## Chapter 7

[Table 7.1 Temperatures at an insulated surface of a finite plate heated wit...](#)

[Table 7.2 Temperatures at an insulated surface of a finite plate heated wit...](#)

[Table 7.3 Random numbers used to generate simulated measurements for  \$nt = 3...\$](#)

[Table 7.4 Table of materials,  \$L\$ , and  \$\alpha\_0\$  for putting the dimensionless test ...](#)

[Table 7.5 Summary of results for step heat flux case using function specifi...](#)

[Table 7.6 Summary of results for triangular heat flux case using function s...](#)

[Table 7.7 Summary of results for quartic heat flux case using function spec...](#)

[Table 7.8 Summary of results for step heat flux case using Tikhonov regular...](#)

[Table 7.9 Summary of results for triangular and quartic heat flux cases usi...](#)

[Table 7.10 Summary of results for step heat flux case using Conjugate Gradi...](#)

[Table 7.11 Summary of results for triangular and quartic heat flux cases us...](#)

[Table 7.12 Summary of results for step heat flux case using Truncated SVD m...](#)

[Table 7.13 Summary of results for triangular and quartic heat flux cases us...](#)

[Table 7.14 Summary of results for step heat flux case using Kalman Filter. ...](#)

[Table 7.15 Summary of results for triangular and quartic heat flux cases us...](#)

[Table 7.16 Filter coefficient for Example 7.1.](#)

[Table 7.17 Measurement data for Example 7.1.](#)

[Table 7.18 Summary of results for step heat flux case for  \$\Delta\tilde{t} = 0.060\$  and with  \$\sigma\$](#)

[Table 7.19 Summary of results for triangular heat flux case for  \$\Delta\tilde{t} = 0.060\$  and with](#)

[Table 7.20 Summary of results for quartic heat flux case for  \$\Delta\tilde{t} = 0.060\$  and with  \$\sigma...\$](#)

## Chapter 8

[Table 8.1 Temperature measurement data for Problem 8.2.](#)

## Chapter 9

[Table 9.1 Data for the copper billet example and exact matching results.](#)

[Table 9.2 Convergence history for Example 9.3.](#)

[Table 9.3 Converged results for  \$h\_M\$  for Example 9.4 using  \$r = 2\$ .](#)

[Table 9.4 Measured data \( \$^{\circ}\text{C}\$ \) for Example 9.5.](#)

[Table 9.5 Estimated heat flux components \( \$\text{W}/\text{m}^2\$ \) for Example 9.5.](#)

[Table 9.6 Estimated surface temperatures \( \$^{\circ}\text{C}\$ \) for Example 9.5.](#)

[Table 9.7 Estimated heat transfer coefficients \( \$\text{W}/\text{m}^2\text{-}^{\circ}\text{C}\$ \) for Example 9.5.](#)

## Chapter 10

[Table 10.1 Simulation parameters for Example 10.1.](#)

[Table 10.2 Sensor and undisturbed temperature data for heating with  \$q = \text{con}...\$](#)

[Table 10.3 Sensor and undisturbed temperature data for heating with Eq. \(10...](#)

[Table 10.4 Temperatures for Example 10.4.](#)

[Table 10.5 Temperature measurements for Example 10.5.](#)

[Table 10.6 Corrected Temperatures for Example 10.5.](#)

## Appendix A

[Table A.1 Types of boundary conditions included in the heat conduction numb...](#)

[Table A.2 Cases \(XIJ\) for one-directional rectangular bodies.](#)

[Table A.3 Types of time boundary condition function on the  \$i\$ -th bound...](#)

[Table A.4 Types of space boundary condition function for a 2D problem invol...](#)

[Table A.5 Types of space-variable initial condition.](#)

# List of Illustrations

## Chapter 1

[Figure 1.1 Schematic of 1D slab subject to heat flux at  \$x = 0\$ . \(a\) Dir...](#)

[Figure 1.2 \(a\) Measured temperatures at discrete times. \(b\) Representations ...](#)

[Figure 1.3 Subdivision of a single interior sensor IHCP into inverse and dir...](#)

[Figure 1.4 Composite plate with multiple temperature sensors.](#)

[Figure 1.5 Surface heat flux as a function of position for a flat plate.](#)

[Figure 1.6 Examples of studies involving solution of IHCP in manufacturing. ...](#)

[Figure 1.7 Examples of studies involving solution of IHCP in manufacturing. ...](#)

[Figure 1.8 Example of reentering vehicle for which the surface heat flux is ...](#)

[Figure 1.9 Schematic of a three-layer thermal protection system.](#)

## Chapter 2

[Figure 2.1 Schematic of the 1D transient X32B-0T1 problem.](#)

[Figure 2.2 Nonhomogeneous constant boundary condition of the first kind at  \$x\$](#)

[Figure 2.3 Dimensionless temperature of the X12B10T0 problem: \(a\) versus spa...](#)

[Figure 2.4 Nonhomogeneous linearly-in-time variable boundary condition of th...](#)

[Figure 2.5 Dimensionless temperature of the X12B20T0 problem for  \$\tilde{t}\_{ref} = 1\$ : \(a\) vers...](#)

[Figure 2.6 Nonhomogeneous constant boundary condition of the second kind at](#)

[Figure 2.7 Dimensionless temperature of the X22B10T0 problem: \(a\) versus spa...](#)

[Figure 2.8 Nonhomogeneous linearly-in-time variable boundary condition of th...](#)

[Figure 2.9 Dimensionless temperature of the X22B20T0 problem for  \$\tilde{t}\_{ref} = 1\$ : \(a\) vers...](#)

[Figure 2.10 Schematic of the 2D transient X22B\(y1pt1\)0Y22B00T1 problem.](#)

[Figure 2.11 Dimensionless temperature of the X22B\(y1pt1\)0Y22B00T0 problem ve...](#)

## Chapter 3

[Figure 3.1 Piecewise-constant approximation for  \$\theta\_0\(t\)\$ , where the  \$i\$ th co...](#)

[Figure 3.2 Component “ \$i\$ ” of the piecewise-constant approximation for the su...](#)

[Figure 3.3 Piecewise-linear approximation for  \$\theta\_0\(t\)\$ , where the  \$i\$ th compo...](#)

[Figure 3.4 Component “ \$i\$ ” of the piecewise-linear approximation for the surfa...](#)

[Figure 3.5 Plot of exact and approximate variations of the surface temperatu...](#)

[Figure 3.6 Piecewise-constant approximation for  \$q\_0\(t\)\$  where the  \$i\$ th componen...](#)

[Figure 3.7 Piecewise-linear approximation for  \$q\_0\(t\)\$  where the  \$i\$ th component,...](#)

[Figure 3.8 Plot of exact and approximate variations of the surface heat flux...](#)

[Figure 3.9 Piecewise-uniform approximation for the surface heat flux where  \$t\$ ...](#)

[Figure 3.10 Superposition in space applied to the  \$T\_j\(x, y, t\)\$  temperature du...](#)

[Figure 3.11 Building blocks differences: \(a\) backward difference \(BD\) in spa...](#)

## Chapter 4

[Figure 4.1 Piecewise constant \(dashed lines\) and piecewise linear approximat...](#)

[Figure 4.2 Heat flux sensitivity coefficients at insulated surface of flat p...](#)

[Figure 4.3 Heat flux sensitivity coefficients at insulated surface of flat p...](#)

[Figure 4.4 Sensitivity coefficients for temperature \(X12\) compared to those ...](#)

[Figure 4.5 Sensitivity coefficients for estimating a constant  \$Bi\$  for  \$Bi = 1\$ ,...](#)

[Figure 4.6 Finite two-dimensional slab with piecewise constant heat flux pul...](#)

[Figure 4.7 Two-dimensional sensitivity coefficients \(a\)  \$W/L = 4\$  and \(b\)  \$W/L\$  ...](#)

[Figure 4.8 Scarpa and Milano results for varying process noise parameter. No...](#)

[Figure 4.9 Effect of post-Kalman estimation smoothing on estimated heat flux...](#)

## Chapter 5

[Figure 5.1 Columns 7-15 of the filter matrix for function specification meth...](#)

[Figure 5.2 Surface plot of the filter coefficients for the function specific...](#)

[Figure 5.3 Filter coefficients for zeroth order Tikhonov regularization \( \$\tilde{x} = 0.5\$ \).](#)

[Figure 5.4 Surface plot of the rows of the filter matrix for Tikhonov regula...](#)

[Figure 5.5 Values of the filter coefficients along the main diagonal, diagon...](#)

[Figure 5.6 Filter coefficients for the mid column and plus and minus 2 adjac...](#)

[Figure 5.7 Surface plot of the rows of the filter matrix for SVD method \( \$\tilde{x} = 0.5\$  f...](#)

[Figure 5.8 Filter coefficients for the mid-column and plus and minus 1 adjac...](#)

[Figure 5.9 Surface plot of the columns of the CGM filter matrix \( \$\tilde{x} = 0.5\$ ,  \$\Delta\tilde{t} = 0.05\$  for 31 ...](#)

[Figure 5.10 Mid-column of the TR-filter matrix for various dimensionless loc...](#)

[Figure 5.11 Temperature response for X22B10T0 case at mid-plane \( \$\tilde{x} = 0.5\$  and  \$\Delta\tilde{t} = 0.02\$ \).](#)

[Figure 5.12 Calculated surface heat flux using filter-based TR versus exact ...](#)

[Figure 5.13 Schematic of the physical domain with second temperature sensor ...](#)

[Figure 5.14 Dimensionless filter coefficients  \$f\$  for the surface heat flux at...](#)

[Figure 5.15 Dimensionless filter coefficients  \$g\$  for the surface heat flux at...](#)

[Figure 5.16 Temperatures for triangular heat flux profile \( \$\Delta\tilde{t} = 0.02\$ \).](#)

[Figure 5.17 Comparison of estimated surface heat flux values by the whole-ti...](#)

[Figure 5.18 Schematic of the IHCP in a two-layer medium.](#)

[Figure 5.19 Dimensionless temperature for X1C11B10T0 case \( \$t\_d = 0.01\$ \).](#)

[Figure 5.20  \$f\$  filter coefficients for the two-layer domain \(X1C11B10T0,  \$t\_d = \dots\$](#)

[Figure 5.21  \$g\$  filter coefficients for the two-layer domain \(X1C11B10T0,  \$t\_d = \dots\$](#)

[Figure 5.22 Calculated surface heat flux for \(X1C11B10T0,  \$t\_d = 0.01\$ ,  \$\alpha\_t\$](#)

[Figure 5.23 Schematic of a directional flame thermometer \(DFT\).](#)

[Figure 5.24 The two-layer IHCP associated with DFT application.](#)

[Figure 5.25  \$f\$  filter coefficients for different temperatures.](#)

[Figure 5.26  \$g\$  filter coefficients for different temperatures.](#)

[Figure 5.27 Temperature measurements on front \( \$Y\$ \) and back \( \$y\$ \) side of the s...](#)

[Figure 5.28 Surface heat flux evaluated for 5.2 with constant material prope...](#)

[Figure 5.29 Calculated surface heat flux for Example 5.2 accounting for temp...](#)

[Figure 5.30 Temperature measurement data for DFT field test \(Example 5.5\)](#)

[Figure 5.31 Calculated surface heat flux for DFT field data-constant materia...](#)

[Figure 5.32 Comparison of calculated surface heat flux for DFT field data by...](#)

## Chapter 6

[Figure 6.1 The true heat flux  \$q\_M\$  at time  \$t\_M\$ , the estimated value  \$\hat{q}\_M\$ , and the...](#)

[Figure 6.2 Expected value of the average sum-squared error in heat flux with...](#)

[Figure 6.3 Expected value of the average sum-squared error in heat flux with...](#)

[Figure 6.4 Comparison of results from three regularization techniques applie...](#)

[Figure 6.5 Histogram of results from 200 trials using the Discrepancy Princi...](#)

[Figure 6.6 Typical L-curve for 1-D IHCP.](#)

[Figure 6.7 Various “optimal” regularizations presented on the L-curves of Fi...](#)

[Figure 6.8 GCV function values for SVD regularization in Example 6.6.](#)

[Figure 6.9 GCV function values for CGM regularization in Example 6.6.](#)

## Chapter 7

[Figure 7.1 Various geometries, boundary conditions, and interface conditions...](#)

[Figure 7.2 Heat flux test case for a step change in surface heat flux.](#)

[Figure 7.3 Triangular heat flux for test case. Finite insulated plate.](#)

[Figure 7.4 Heated and insulated surface temperatures for triangular heat flu...](#)

[Figure 7.5 Quartic heat flux pulse for test case. Finite insulated plate.](#)

[Figure 7.6 Heated and insulated surface temperatures for quartic heat flux p...](#)

[Figure 7.7 Estimated heat fluxes using function specification for step chang...](#)

[Figure 7.8 Estimated heat fluxes using function specification for step chang...](#)

[Figure 7.9 Estimated heat fluxes using function specification for triangular...](#)

[Figure 7.10 Estimated heat fluxes using function specification for triangula...](#)

[Figure 7.11 Estimated heat fluxes using function specification for quartic t...](#)

[Figure 7.12 Estimated heat fluxes using function specification for quartic t...](#)

[Figure 7.13 Filter coefficients for function specification obtained from tem...](#)

[Figure 7.14 Filter coefficients for function specification obtained from tem...](#)

[Figure 7.15 Filter coefficients for function specification obtained from tem...](#)

[Figure 7.16 Filter coefficients for function specification obtained from tem...](#)

[Figure 7.17 Optimal estimated heat fluxes using Tikhonov Regularization for ...](#)

[Figure 7.18 Optimal estimated heat fluxes using Tikhonov Regularization for ...](#)

[Figure 7.19 Optimal estimated heat fluxes using Tikhonov Regularization for ...](#)

[Figure 7.20 Filter coefficients for Tikhonov Regularization obtained from te...](#)

[Figure 7.21 Filter coefficients for Tikhonov Regularization with  \$\Delta\tilde{t} = 0.03\$  using pie...](#)

[Figure 7.22 Filter coefficients for Tikhonov Regularization with  \$\Delta\tilde{t} = 0.03\$  using pie...](#)

[Figure 7.23 Optimal estimated heat fluxes using Conjugate Gradient Method fo...](#)

[Figure 7.24 Optimal estimated heat fluxes using conjugate gradient method fo...](#)

[Figure 7.25 Optimal estimated heat fluxes using conjugate gradient method fo...](#)

[Figure 7.26  \$E\(R\_q^2\)\$  versus iteration number for Steepest Descent in the triangular...](#)

[Figure 7.27  \$E\(R\_q^2\)\$  versus iteration number for Fletcher-Reeves CGM in the triangu...](#)

[Figure 7.28 Filter coefficients for conjugate gradient methods obtained from...](#)

[Figure 7.29 Filter coefficients for Steepest Descent iterations with  \$\Delta\tilde{t} = 0.03\$  using...](#)

[Figure 7.30 Filter coefficients for Fletcher-Reeves Conjugate Gradient itera...](#)

[Figure 7.31 Optimal estimated heat fluxes using Truncated Singular Value Dec...](#)

[Figure 7.32 Optimal estimated heat fluxes using Truncated Singular Value Dec...](#)

[Figure 7.33 Optimal estimated heat fluxes using Truncated Singular Value Dec...](#)

[Figure 7.34 Filter coefficients for Truncated SVD method with  \$\Delta\tilde{t} = 0.06\$  using either...](#)

[Figure 7.35 Filter coefficients for Truncated SVD with  \$\Delta\tilde{t} = 0.03\$  using piecewise con...](#)

[Figure 7.36  \$E\(R\_q^2\)\$  and its random and bias error components as  \$n\_{sing}\$  is increased...](#)

[Figure 7.37 Optimal estimated heat fluxes using Kalman Filter for step chang...](#)

[Figure 7.38 Optimal estimated heat fluxes using Kalman Filter for triangular...](#)

[Figure 7.39 Optimal estimated heat fluxes using Kalman Filter for quartic te...](#)

[Figure 7.40 Filter coefficients for Kalman Filter solution with  \$\Delta\tilde{t} = 0.06\$  optimized ...](#)

[Figure 7.41 Filter coefficients for Kalman Filter solution with  \$\Delta\tilde{t} = 0.06\$  optimized ...](#)

[Figure 7.42 Filter coefficients for Kalman Filter solution with  \$\Delta\tilde{t} = 0.06\$  and  \$r = 0\$  ...](#)

[Figure 7.43 Filter coefficients for Kalman Filter solution with  \$\Delta\tilde{t} = 0.06\$  and  \$Q\_q = 6...\$](#)

[Figure 7.44 Estimated and exact heat fluxes for Example 7.1.](#)

[Figure 8.1 Schematic of a rectangular domain, partially heated from the bott...](#)

[Figure 8.2 Schematic of partial heating of a rectangular plate from the bott...](#)

[Figure 8.3 Schematic of an IHCP with four unknown heat fluxes and four tempe...](#)

[Figure 8.4 Rows of the filter matrix for Example 8.1 \( \$\alpha\_t = 10^{-4}\$  a...](#)

[Figure 8.5 Filter coefficients for Example 8.1 \( \$\alpha\_t = 10^{-4}\$  and  \$\alpha...\$](#)

[Figure 8.6 Input surface heat fluxes for Example 8.1](#)

[Figure 8.7 Temperature data at four locations for Example 8.1.](#)

[Figure 8.8 Estimated surface heat flux values for Example 8.1 \( \$q\_1, q\_2, q\_3, a...\$](#)

[Figure 8.9 Estimated heat flux values for Example 8.1 in the presence of 0.5...](#)

[Figure 8.10 Heat fluxes applied on the bottom surface of the domain \(Example...](#)

[Figure 8.11 Calculated temperatures for the sensors at different locations \(...\)](#)

[Figure 8.12 Filter coefficients for Example 8.2 \( \$\alpha\_t = 10^{-4}\$  and  \$\alpha...\$](#)

[Figure 8.13 Estimated heat flux values for Example 8.2 with 0.5% error prese...](#)

[Figure 8.14 Schematic of an IHCP with two unknown heat fluxes and two temper...](#)

[Figure 9.1 Electrically heated flat plate.](#)

[Figure 9.2 Temperatures and sensitivity coefficients for convective heat tra...](#)

[Figure 9.3 Heat transfer coefficient history approximated by constant segmen...](#)

[Figure 9.4 Heat transfer sensitivity coefficients for  \$h\_i\$  constant over segme...](#)

[Figure 9.5 Filter coefficient for zeroth-order Tikhonov regularization for h...](#)

[Figure 9.6 Effect of regularization parameter for zeroth-order Tikhonov on h...](#)

[Figure 9.7 Estimate heat transfer coefficients for Example 9.5 for various r...](#)

## Chapter 10

[Figure 10.1 Isotherms around an idealized temperature sensor: \(a\) sensor imb...](#)

[Figure 10.2 Schematic of idealized subsurface thermocouple installation and ...](#)

[Figure 10.3 ANSYS mesh and typical simulation results.](#)

[Figure 10.4 Correction kernel  \$\mathbf{H}\$  for various regularization coefficients.](#)

[Figure 10.5 Heat flux and reconstructions for varying values of  \$\alpha\_{Tik}\$  in...](#)

[Figure 10.6 Measured and corrected sensor readings in Example 10.3 using exa...](#)

[Figure 10.7 Measured and corrected sensor readings in Example 10.3 using noi...](#)

[Figure 10.8 Three-dimensional geometry used by Woolley et al. \(2008\).](#)

[Figure 10.9 Comparison of axisymmetric and three-dimensional sensor models: ...](#)

[Figure 10.10 Intrinsic thermocouple geometry.](#)

[Figure 10.11 Temperatures for Example 10.4.](#)

[Figure 10.12  \$\mathbf{H}\$  vector for Example 10.4.](#)

[Figure 10.13 Simulated measurements and corrected temperatures for Example 1...](#)

## Appendix C

[Figure C.1 Principle of superposition applied to the constant surface temper...](#)

[Figure C.2 Principle of superposition applied to the linear-in-time surface ...](#)

[Figure C.3 Basic functions deriving from superposition applied to the linear...](#)

[Figure C.4 Basic functions deriving from superposition applied to the linear...](#)

# **Inverse Heat Conduction**

## **Ill-Posed Problems**

Second Edition

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in two-dimensional transient approximate solution [90-93](#), [250](#)  
in using second sensor as boundary condition [167](#)  
surface element method [77](#), [209](#), [287](#)

## ***t***

temperature dependent material properties [177](#)  
temperature perturbation [157](#), [210](#), [217](#)  
for Conjugate Gradient method [232-234](#), [233](#), [234](#)  
for Function Specification [221-223](#), [222](#), [223](#)  
for Kalman Filter [241-243](#), [242](#), [243](#)  
for Tikhonov regularization [226-228](#), [227](#), [228](#)  
for Truncated SVD [237-238](#), [237](#), [238](#)  
test cases  
for IHCP [210](#)  
Quartic Heat Flux [213](#), [247](#)  
results for Conjugate Gradient Method [229-232](#)  
results for Function Specification Method [217-221](#)  
results for Kalman Filter [239-241](#)  
results for Tikhonov Regularization [224-226](#)  
results for Truncated Singular Value Decomposition [234-237](#)  
Step Change in Surface Heat Flux [211](#), [246](#)  
Triangular Heat Flux [211](#), [212](#), [213](#), [246](#)  
with Units [213](#), [214](#), [215](#), [217](#)  
Tikhonov, A. N. [128](#)

Tikhonov Regularization [127](#)-133

    sequential [134](#)

Truncated Singular Value Decomposition see [TSVD](#)

TSVD [144](#), [145](#), [145](#), [202](#), [235](#), [236](#), [237](#), [237](#), [238](#)

    filter matrix [234](#)

two-dimensional IHCP [250](#), [252](#)

## ***u***

unsteady surface element method see [surface element method](#)

unsteady time [32](#), [41](#), [57](#), [59](#)

## ***v***

variance [187](#)

## ***w***

well-posed problem [2](#)

## ***x***

XI0BKT0 [28](#), [29](#), [31](#)

    X10B1T0 [33](#)

    X10B2T0 [35](#)

    X20B1T0 [37](#), [44](#), [46](#), [47](#)

    X20B2T0 [40](#)

X1C11B10T0 [174](#), [175](#)

X21B10T0 [166](#), [168](#), [184](#)

XI2BK0T0 [26](#)

X12B10T0 [28](#), [32](#), [33](#), [35](#), [36](#), [37](#), [39](#), [57](#), [64](#), [66-69](#), [72](#), [75](#), [310-313](#), [315](#)

X12B20T0 [28](#), [34](#), [35](#), [37](#), [57](#), [64](#), [67](#), [69](#), [70](#), [72](#), [75](#), [312](#), [313](#)

X22B10T0 [28](#), [29](#), [32](#), [37](#), [38](#), [39](#), [41](#), [57](#), [63](#), [64](#), [75](#), [76-80](#), [82](#), [84](#), [88](#), [95](#), [99](#), [211](#), [315-317](#)

X22B20T0 [28](#), [40](#), [42](#), [57](#), [75](#), [78](#), [80](#), [82](#), [84](#), [99](#), [316](#), [317](#)

XI2BK0T1

X12B10T1 [28](#), [33](#)

X12B20T1 [28](#), [35](#), [67](#)

X12B30T1 [73](#)

X22B10T1 [28](#), [38](#), [43](#), [59](#), [90](#)

X22B20T1 [28](#), [40](#), [78](#), [82](#)

X22B30T1 [83](#)

X32B00T1 [27](#)

XI2B-0T1

X12B-0T1 [27](#), [63](#), [64](#), [81](#), [309](#)

X22B-0T1 [27](#), [62](#), [63](#), [75](#), [110](#), [137](#), [153](#), [210](#), [214](#)

X22B--T1 [172](#)

X32B-0T1 [26](#), [27](#)

X20B(y1pt1)Y20B0T0 [302](#)

X20B(y1pt1)Y22B00T0 [302](#)

X22B(y1pt1)0Y20B0T0 [301](#), [319](#)

X22B(y1pt1)0Y22B00T1 [42](#)

X22B(y1pt1)0Y22B00T0 [43](#), [56](#), [57](#), [85](#), [86](#), [88](#), [90](#), [92](#), [95](#),  
[98](#), [115](#), [249](#), [301](#)

X22B(y-t1)0Y22B00T1 [85](#)

X22B(y-t-)0Y22B00T1 [90](#), [317](#)