Hsien-Chung Wu

Mathematical Foundation of Fuzzy Sets

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Preface

The concept of fuzzy set, introduced by L.A. Zadeh in 1965, tried to extend classical set theory. It is well known that a classical set corresponds to an indicator function whose values are only taken to be 0 and 1. With the aid of a membership function associated with a fuzzy set, each element in a set is allowed to take any values between 0 and 1, which can be regarded as the degree of membership. This kind of imprecision draws forth a bunch of applications.

This book is intended to present the mathematical foundations of fuzzy sets, which can rigorously be used as a basic tool to study engineering and economics problems in a fuzzy environment. It may also be used as a graduate level textbook. The main prerequisites for most of the material in this book are mathematical analysis including semi-continuities, supremum, convexity, and basic topological concepts of Euclidean space, $\mathbb{R}^n$. This book presents the current state of affairs in set operations of fuzzy sets, arithmetic operations of fuzzy intervals and fuzzification of crisp functions that are frequently adopted to model engineering and economics problems with fuzzy uncertainty. Especially, the concepts of gradual sets and gradual elements have been presented in order to cope with the difficulty for considering elements of fuzzy sets such as considering elements of crisp sets.

- Chapter 1 presents the mathematical tools that are used to study the essence of fuzzy sets. The concepts of supremum and semi-continuity and their properties are frequently invoked to establish the equivalences among the different settings of set operations and arithmetic operations of fuzzy sets.
- Chapter 2 introduces the basic concepts and properties of fuzzy sets such as membership functions and level sets. The fuzzy intervals are categorized as different types based on the different assumptions of membership functions in order to be used for the different purposes of applications.
- Chapter 3 deals with the intersection and union of fuzzy sets including the complement of fuzzy sets. The general settings by considering aggregation functions have been presented to study the intersection and union of fuzzy sets that cover the conventional ones such as using minimum and maximum functions (t-norm and s-norm) for intersection and union, respectively.
- Chapter 4 extends the conventional extension principle to the so-called generalized extension principle by using general aggregation functions instead of using minimum function or t-norm to fuzzify crisp functions. Fuzzifications of real-valued and vector-valued functions are frequently adopted in engineering and economics problems that involve fuzzy data, which means that the real-valued data cannot be exactly collected owing to the fluctuation of an uncertain situation.

- Chapter 5 presents the methodology for generating fuzzy sets from a nested family or non-nested family of subsets of Euclidean space $\mathbb{R}^n$. Especially, generating fuzzy intervals from a nested family or non-nested family of bounded closed intervals is useful for fuzzifying the real-valued data into fuzzy data. Based on a collection of real-valued data, we can generate a fuzzy set that can essentially represent this collection of real-valued data.
- Chapter 6 deals with the fuzzification of crisp functions. Using the extension principle presented in Chapter 4 can fuzzify crisp functions. This chapter studies another methodology to fuzzify crisp functions using the mathematical expression in the well-known decomposition theorem. Their equivalences are also established under some mild assumptions.
- Chapter 7 studies the arithmetic operations of fuzzy sets. The conventional arithmetic operations of fuzzy sets are based on the extension principle presented in Chapter 4. Many other arithmetic operations using the general aggregation functions haven also been studied. The equivalences among these different settings of arithmetic operations are also established in order to demonstrate the consistent usage in applications.
- Chapter 8 gives a comprehensive and accessible study regarding inner product of fuzzy vectors that can be treated as an application using the methodologies presented in Chapter 7. The potential applications of inner product of fuzzy vectors are fuzzy linear programming problems and the engineering problems that are formulated using the form of inner product involving fuzzy data.
- Chapter 9 introduces the concepts of gradual sets and gradual elements that can be used to propose the concept of elements of fuzzy sets such as the concept of elements of crisp sets. Roughly speaking, a fuzzy set can be treated as a collection of gradual elements. In other words, a fuzzy set consists of gradual elements. In this case, the set operations and arithmetic operations of fuzzy sets can be defined as the operations of gradual elements, like the operations of elements of crisp sets. The equivalences with the conventional set operations and arithmetic operations of fuzzy sets are also established under some mild assumptions.
- Chapter 10 deals with the concept of duality of fuzzy sets by considering the lower α-level sets. The conventional α-level sets are treated as upper α-level sets. This chapter considers the lower α-level sets that can be regarded as the dual of upper α-level sets. The well-known extension principle and decomposition theorem are also established based on the lower α-level sets, and are called the dual extension principle and dual decomposition theorem. The so-called dual arithmetics of fuzzy sets are also proposed based on the lower α-level sets, and a duality relation with the conventional arithmetics of fuzzy sets is also established.

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Mathematical Analysis

We present some materials from mathematical analysis, which will be used throughout this book. More detailed arguments can be found in any mathematical analysis monograph.

1.1 Infimum and Supremum

Let S be a subset of $\mathbb{R}$. The upper and lower bounds of S are defined below.

- We say that u is an **upper bound** of S when there exists a real number u satisfying $x \leq u$ for every $x \in S$. In this case, we also say that S is bounded above by u.
- We say that l is a **lower bound** of S when there exists a real number l satisfying $x \geq l$ for every $x \in S$. In this case, we also say that S is bounded below by l.

The set S is said to be unbounded above when the set S has no upper bound. The set S is said to be unbounded below when the set S has no lower bound. The maximal and minimal elements of S are defined below.

- We say that u^* is a **maximal element** of S when there exists a real number $u^* \in S$ satisfying $x \leq u^*$ for every $x \in S$. In this case, we write $u^* = \max\ S$.
- We say that l^* is a **minimal element** of S when there exists a real number $l^* \in S$ satisfying $x \geq l^*$ for every $x \in S$. In this case, we write $l^* = \min\ S$.

Example 1.1.1 We provide some concrete examples.

(i) The set $\mathbb{R}^+ = (0, +\infty)$ is unbounded above. It has no upper bounds and no maximal element. It is bounded below by 0, but it has no minimal element.
(ii) The closed interval $S = [0,1]$ is bounded above by 1 and is bounded below by 0. We also have $\max\ S = 1$ and $\min\ S = 0$.
(iii) The half-open interval $S = [0,1)$ is bounded above by 1, but it has no maximal element. However, we have $\min\ S = 0$.

Although the set $S = [0,1)$ is bounded above by 1, it has no maximal element. This motivates us to introduce the concepts of supremum and infinum.

Mathematical Foundations of Fuzzy Sets, First Edition. Hsien-Chung Wu.

Definition 1.1.2 Let S be a subset of $\mathbb{R}$.

(i) Suppose that S is bounded above. A real number $\bar{u} \in \mathbb{R}$ is called a **least upper bound** or **supremum** of S when the following conditions are satisfied.
- $\bar{u}$ is an upper bound of S.
- If u is any upper bound of S, then $u \geq \bar{u}$.

In this case, we write $\bar{u} = \sup S$. We say that the supremum $\sup S$ is *attained* when $\bar{u} \in S$.

(ii) Suppose that S is bounded below. A real number $\bar{l} \in \mathbb{R}$ is called a **greatest lower bound** or **infimum** of S when the following conditions are satisfied.
- $\bar{l}$ is a lower bound of S.
- If l is any lower bound of S, then $l \leq \bar{l}$.

In this case, we write $\bar{l} = \inf S$. We say that the infimum $\inf S$ is *attained* when $\bar{l} \in S$.

It is clear to see that if the supremum $\sup S$ is attained, then $\max S = \sup S$. Similarly, if the infimum $\inf S$ is attained, then $\min S = \inf S$.

Example 1.1.3 Let $S = [0,1]$. Then, we have

$$\max S = \sup S = 1 \text{ and } \inf S = \min S = 0.$$

If $S = [0,1)$, then $\max S$ does not exists. However, we have $\sup S = 1$.

Proposition 1.1.4 *Let S be a subset of $\mathbb{R}$ with $\bar{u} = \sup S$. Then, given any $s < \bar{u}$, there exists $t \in S$ satisfying $s < t \leq \bar{u}$.*

Proof. We are going to prove it by contradiction. Suppose that we have $t \leq s$ for all $t \in S$. Then s is an upper bound of S. According to the definition of supremum, we also have $s \geq \bar{u}$. This contradiction implies that $s < t$ for some $t \in S$, and the proof is complete. ∎

Proposition 1.1.5 *Given any two nonempty subsets A and B of $\mathbb{R}$, we define $C = A + B$ by*

$$C = \{x + y : x \in A \text{ and } y \in B\}.$$

Suppose that the supremum $\sup A$ and $\sup B$ are attained. Then, the supremum $\sup C$ is attained, and we have

$$\sup C = \sup A + \sup B.$$

Proof. We first have

$$\sup A = \max A \text{ and } \sup B = \max B.$$

We write $a = \sup A$ and $b = \sup B$. Given any $z \in C$, there exist $x \in A$ and $y \in B$ satisfying $z = x + y$. Since $x \leq a$ and $y \leq b$, we have $z = x + y \leq a + b$, which says that $a + b$ is an upper bound of C. Therefore, the definition of $c = \sup C$ says that $c \leq a + b$. Next, we want to show that $a + b \leq c$. Given any $\epsilon > 0$, Proposition 1.1.4 says that there exist $x \in A$ and $y \in B$ satisfying $a - \epsilon < x$ and $b - \epsilon < y$. We also see that $x + y \leq c$. Adding these inequalities, we obtain

$$a + b - 2\epsilon < x + y \leq c,$$

which says that $a + b < c + 2\epsilon$. Since ϵ can be any positive real number, we must have $a + b \leq c$. This completes the proof. ∎

Proposition 1.1.6 *Let A and B be any two nonempty subsets of $\mathbb{R}$ satisfying $a \leq b$ for any $a \in A$ and $b \in B$. Suppose that the supremum* $\sup B$ *is attained. Then, the supremum* $\sup A$ *is attained and* $\sup A \leq \sup B$.

Proof. It is left as an exercise. ∎

1.2 Limit Inferior and Limit Superior

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. The **limit superior** of $\{a_n\}_{n=1}^{\infty}$ is defined by

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{k\geq n} a_k,$$

and the **limit inferior** of $\{a_n\}_{n=1}^{\infty}$ is defined by

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} (-a_n).$$

Moreover, we can see that

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k.$$

Let

$$b_n = \sup_{k\geq n} a_k \text{ and } c_n = \inf_{k\geq n} a_k \tag{1.1}$$

It is clear to see that $\{b_n\}_{n=1}^{\infty}$ is a decreasing sequence and $\{c_n\}_{n=1}^{\infty}$ is an increasing sequence. In this case, we have

$$\inf_{n\geq 1} b_n = \lim_{n\to\infty} b_n \text{ and } \sup_{n\geq 1} c_n = \lim_{n\to\infty} c_n,$$

which also says that

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} b_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sup_{k\geq n} a_k \tag{1.2}$$

and

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} c_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \inf_{k\geq n} a_k. \tag{1.3}$$

Some useful properties are given below.

Proposition 1.2.1 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then, the following statements hold true.*

(i) *We have*

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

(ii) *We have*

$$\lim_{n\to\infty} a_n = a$$

if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \textit{ with } |a| < +\infty.$$

(iii) *The sequence diverges to $+\infty$ if and only if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = +\infty.$$

(iv) *The sequence diverges to* $-\infty$ *if and only if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = -\infty.$$

(v) *Let* $\{b_n\}_{n=1}^{\infty}$ *be another sequence satisfying* $a_n \le b_n$ *for all* n. *Then, we have*

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \text{ and } \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n.$$

Proof. To prove part (i), from (1.1), we see that $c_n \le b_n$ for all n. Using (1.2) and (1.3), we obtain

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n \le \lim_{n\to\infty} b_n = \limsup_{n\to\infty} a_n.$$

To prove part (ii), suppose that

$$\lim_{n\to\infty} a_n = a.$$

Then, given any $\epsilon > 0$, there exists an integer N satisfying

$$a - \frac{\epsilon}{2} < a_n < a + \frac{\epsilon}{2} \text{ for } n \ge N,$$

which implies

$$a - \frac{\epsilon}{2} \le \inf_{k\ge n} a_k = c_n \text{ and } b_n = \sup_{k\ge n} a_k \le a + \frac{\epsilon}{2} \text{ for } n \ge N.$$

In other words, we have

$$a - \frac{\epsilon}{2} \le c_n \le b_n \le a + \frac{\epsilon}{2} \text{ for } n \ge N,$$

which also implies

$$\left|c_n - a\right| \le \frac{\epsilon}{2} < \epsilon \text{ and } \left|b_n - a\right| \le \frac{\epsilon}{2} < \epsilon \text{ for } n \ge N.$$

Therefore, we obtain

$$\lim_{n\to\infty} c_n = a = \lim_{n\to\infty} b_n,$$

which implies, by using (1.2) and (1.3),

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a.$$

For the converse, from (1.1) again, we see that $c_n \le a_n \le b_n$ for all $n \ge 1$. Since

$$a = \lim_{n\to\infty} \inf_{k\ge n} a_k = \lim_{n\to\infty} c_n \text{ and } a = \lim_{n\to\infty} \sup_{k\ge n} a_k = \lim_{n\to\infty} b_n.$$

Using the pinching theorem, we obtain the desired limit. The remaining proofs are left as exercise, and the proof is complete. ∎

Proposition 1.2.2 *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be any two sequences in* $\mathbb{R}$. *Then, we have*

$$\limsup_{n\to\infty} \left(a_n + b_n\right) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty} \left(a_n + b_n\right) \ge \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

Proof. For $k \geq n$, we have

$$a_k + b_k \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k,$$

which says that

$$\sup_{k\geq n} \left(a_k + b_k\right) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k. \tag{1.4}$$

Therefore, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \left(a_n + b_n\right) &= \inf_{n\geq 1} \sup_{k\geq n} \left(a_k + b_k\right) = \lim_{n\to\infty} \sup_{k\geq n} \left(a_k + b_k\right)\\ &\leq \lim_{n\to\infty} \left[\sup_{k\geq n} a_k + \sup_{k\geq n} b_k\right] \text{ (using (1.4))}\\ &= \lim_{n\to\infty} \sup_{k\geq n} a_k + \lim_{n\to\infty} \sup_{k\geq n} b_k \text{ (since the limits exist)}\\ &= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \text{ (using (1.2) and (1.3)).}\end{aligned}$$

We similarly have

$$\inf_{k\geq n} \left(a_k + b_k\right) \geq \inf_{k\geq n} a_k + \inf_{k\geq n} b_k. \tag{1.5}$$

Therefore, we also obtain

$$\begin{aligned}\liminf_{n\to\infty} \left(a_n + b_n\right) &= \sup_{n\geq 1} \inf_{k\geq n} \left(a_k + b_k\right) = \lim_{n\to\infty} \inf_{k\geq n} \left(a_k + b_k\right)\\ &\geq \lim_{n\to\infty} \left[\inf_{k\geq n} a_k + \inf_{k\geq n} b_k\right] \text{ (using (1.5))}\\ &= \lim_{n\to\infty} \inf_{k\geq n} a_k + \lim_{n\to\infty} \inf_{k\geq n} b_k \text{ (since the limits exist)}\\ &= \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \text{ (using (1.2) and (1.3)).}\end{aligned}$$

This completes the proof. ∎

Proposition 1.2.3 *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^m$ satisfying $A_{n+1} \subseteq A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = A$, and let f be a real-valued function defined on $\mathbb{R}^m$. Then*

$$\lim_{n\to\infty} \sup_{a\in A_n} f(a) = \sup_{a\in A} f(a) \textit{ and } \sup_{a\in A_n} f(a) \geq \sup_{a\in A_{n+1}} f(a)$$

and

$$\lim_{n\to\infty} \inf_{a\in A_n} f(a) = \inf_{a\in A} f(a) \textit{ and } \inf_{a\in A_n} f(a) \leq \inf_{a\in A_{n+1}} f(a).$$

Proof. Since

$$\inf_{a\in A} f(a) = -\sup_{a\in A} [-f(a)].$$

It suffices to prove the case of the supremum. Let

$$y_n^* = \sup_{a\in A_n} f(a) \text{ and } y^* = \sup_{a\in A} f(a).$$

Since $A_{n+1} \subseteq A_n$ for all n, we have that $\{y_n^*\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers. We also have $y_n^* \geq y^*$ for all n, which implies

$$\liminf_{n\to\infty} y_n^* \geq y^*. \tag{1.6}$$

Given any $\epsilon > 0$, according to the concept of supremum, there exists $a_n \in A_n$ satisfying

$$y_n^* - \epsilon \leq f(a_n). \tag{1.7}$$

Let $b_n = \inf_{k\geq n} f(a_k)$. We consider the subsequence $\{\bar{a}_m\}_{m=1}^{\infty}$ defined by $\bar{a}_m = a_{m+n-1}$ in the sense of

$$\left\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m, \ldots\right\} = \left\{a_n, a_{n+1}, \ldots, a_{m+n-1}, \ldots\right\}.$$

Then $b_n = \inf_{m\geq n} f(\bar{a}_m)$ and $b_n \leq f(\bar{a}_m)$ for all m. Since $A_{n+1} \subseteq A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = A$, the "last term" of the sequence $\{\bar{a}_m\}_{m=1}^{\infty}$ must be in A, a claim that will be proved below. Since $\bar{a}_k \in A_k \subseteq A_n$ for all $k \geq n$, we have the subsequence $\{\bar{a}_k\}_{k=n}^{\infty} \subseteq A_n$, which also implies

$$\bar{A} \equiv \bigcap_{n=1}^{\infty} \left(\{\bar{a}_k\}_{k=n}^{\infty}\right) \subseteq \bigcap_{n=1}^{\infty} A_n = A,$$

where $\bar{A}$ can be regarded as the "last term" and $\bar{A} \subseteq \{\bar{a}_m\}_{m=1}^{\infty}$. Since y^* is the supremum of f on A, it follows that $f(\bar{a}) \leq y^*$ for each $\bar{a} \in \bar{A} \subseteq A$. Since $b_n \leq f(\bar{a}_m)$ for all m, we see that $b_n \leq y^*$ for all n. Therefore, we obtain

$$\liminf_{n\to\infty} f(a_n) = \sup_{n\geq 1} \inf_{k\geq n} f(a_k) = \sup_{n\geq 1} b_n \leq y^*,$$

which implies, by (1.7),

$$\liminf_{n\to\infty} y_n^* - \epsilon \leq \liminf_{n\to\infty} f(a_n) \leq y^*.$$

Since ϵ is any positive number, we obtain

$$\liminf_{n\to\infty} y_n^* \leq y^*. \tag{1.8}$$

Combining (1.6) and (1.8), we obtain

$$\sup_{n\geq 1} \inf_{k\geq n} y_k^* = \liminf_{n\to\infty} y_n^* = y^*.$$

Since $\{y_n^*\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers, we conclude that

$$\inf_{n\geq 1} y_n^* = \lim_{n\to\infty} y_n^* = \liminf_{n\to\infty} y_n^* = y^*,$$

and the proof is complete. ■

Proposition 1.2.4 *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^m$ satisfying $A_n \subseteq A_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} A_n = A$, and let f be a real-valued function defined on $\mathbb{R}^m$. Then*

$$\lim_{n\to\infty} \sup_{a\in A_n} f(a) = \sup_{a\in A} f(a) \text{ and } \sup_{a\in A_n} f(a) \leq \sup_{a\in A_{n+1}} f(a)$$

and

$$\lim_{n\to\infty} \inf_{a\in A_n} f(a) = \inf_{a\in A} f(a) \text{ and } \inf_{a\in A_n} f(a) \geq \inf_{a\in A_{n+1}} f(a).$$

Proof. It suffices to prove the case of the supremum. Let

$$y_n^* = \sup_{a\in A_n} f(a) \text{ and } y^* = \sup_{a\in A} f(a).$$

Since $A_n \subseteq A_{n+1}$ for all n, we have that $\{y_n^*\}_{n=1}^{\infty}$ is an increasing sequence of real numbers. We also have $y_n^* \leq y^*$ for all n, which implies

$$\limsup_{n\to\infty} y_n^* \leq y^*. \tag{1.9}$$

Given any $\epsilon > 0$, according to the concept of supremum, there exists $a^* \in A$ satisfying $y^* - \epsilon \leq f(a^*)$. Since

$$a^* \in A = \bigcup_{n=1}^{\infty} A_n,$$

we have that $a^* \in A_{n^*}$ for some integer n^*. We construct a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying $a_n \in A_n$ for all $n < n^*$ and $a_n = a^*$ for all $n \geq n^*$. Since $A_n \subseteq A_{n+1}$ for all n, it follows that $a_n \in A_n$ for all $n \geq n^*$. Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_n \in A_n$ for all n and

$$a^* \in \{a_k\}_{k=n}^{\infty} \text{ for all } n,$$

which means that a^* is the "last term" of the sequence $\{a_n\}_{n=1}^{\infty}$. We also have

$$y_n^* \geq f(a_n). \tag{1.10}$$

Let $b_n = \sup_{k\geq n} f(a_k)$. We consider the subsequence $\{\bar{a}_p\}_{p=1}^{\infty}$ defined by $\bar{a}_p = a_{p+n-1}$ in the sense of

$$\left\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_p, \ldots\right\} = \left\{a_n, a_{n+1}, \ldots, a_{p+n-1}, \ldots\right\}.$$

Then $b_n = \sup_{p\geq n} f(\bar{a}_p)$ and $b_n \geq f(\bar{a}_p)$ for all p. Since a^* is the "last term" of the sequence $\{a_n\}_{n=1}^{\infty}$, it follows that a^* is also the "last term" of the sequence $\{\bar{a}_m\}_{m=1}^{\infty}$. Therefore, we have $b_n \geq f(a^*) \geq y^* - \epsilon$ for all n, which implies

$$\limsup_{n\to\infty} f(a_n) = \inf_{n\geq 1} \sup_{k\geq n} f(a_k) = \inf_{n\geq 1} b_n \geq y^* - \epsilon.$$

Since ϵ is any positive number, it follows that

$$\limsup_{n\to\infty} f(a_n) \geq y^*.$$

Using (1.10), we obtain

$$\limsup_{n\to\infty} y_n^* \geq \limsup_{n\to\infty} f(a_n) \geq y^*. \tag{1.11}$$

Combining (1.9) and (1.11), we obtain

$$\inf_{n\geq 1} \sup_{k\geq n} y_k^* = \limsup_{n\to\infty} y_n^* = y^*.$$

Since $\{y_n^*\}_{n=1}^{\infty}$ is an increasing sequence of real numbers, we conclude that

$$\sup_{n\geq 1} y_n^* = \lim_{n\to\infty} y_n^* = \limsup_{n\to\infty} y_n^* = y^*,$$

and the proof is complete. ∎

Given any $x = (x^{(1)}, \ldots, x^{(m)})$ and $y = (y^{(1)}, \ldots, y^{(m)})$ in $\mathbb{R}^m$. The Euclidean distance between x and y is defined by

$$\| x - y \| = \sqrt{(x^{(1)} - y^{(1)})^2 + \cdots + (x^{(m)} - y^{(m)})^2}.$$

Given a point $x \in \mathbb{R}^m$, we consider the open ϵ-ball

$$B(x;\epsilon) = \left\{ y \in \mathbb{R}^m : \| x - y \| < \epsilon \right\}. \tag{1.12}$$

The concept of closure based on open balls will be frequently used throughout this book. For the general concept refer to Kelley and Namioka [55]. In this book, we are going to consider the closure of a subset of $\mathbb{R}^m$, which is given below.

Definition 1.2.5 Let A be a subset of $\mathbb{R}^m$. The **closure** of A is denoted and defined by

$$\mathrm{cl}(A) = \left\{ x \in \mathbb{R}^m : A \cap B(x;\epsilon) \neq \emptyset \text{ for any } \epsilon > 0 \right\}.$$

We say that A is a closed subset of $\mathbb{R}^m$ when $A = \mathrm{cl}(A)$.

Remark 1.2.6 Given any $x \in \mathrm{cl}(A)$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A satisfying $\| x_n - x \| \to 0$ as $n \to \infty$. In particular, for $m = 1$, we see that $x_n \to x$ as $n \to \infty$.

Proposition 1.2.7 *Let A be a subset of $\mathbb{R}$, and let f be a continuous function defined on cl(A). Then*

$$\sup_{a\in A} f(a) = \sup_{a\in cl(A)} f(a) \textit{ and } \inf_{a\in A} f(a) = \inf_{a\in cl(A)} f(a).$$

Proof. It suffices to prove the case of the supremum, since

$$\inf_{a\in A} f(a) = -\sup_{a\in A}\,[-f(a)].$$

It is obvious that

$$\sup_{a\in A} f(a) \leq \sup_{a\in \mathrm{cl}(A)} f(a).$$

Given any $\epsilon > 0$, according to the concept of supremum, there exists $a^* \in \mathrm{cl}(A)$ satisfying

$$\sup_{a\in \mathrm{cl}(A)} f(a) - \epsilon \leq f\left(a^*\right).$$

We also see that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in A satisfying $a_n \to a^*$. Since f is continuous on cl(A), we also have $f(a_n) \to f(a^*)$ as $n \to \infty$. Therefore, we obtain

$$\sup_{a\in \mathrm{cl}(A)} f(a) - \epsilon \leq f\left(a^*\right) = \lim_{n\to\infty} f\left(a_n\right) \leq \lim_{n\to\infty}\left[\sup_{a\in A} f(a)\right] = \sup_{a\in A} f(a).$$

Since ϵ can be any positive number, it follows that

$$\sup_{a\in \mathrm{cl}(A)} f(a) \leq \sup_{a\in A} f(a).$$

This completes the proof. ■

Let S be a subset of $\mathbb{R}$. For $a \in S$ and a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, we write $a_n \uparrow a$ to mean that the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and converges to a. We also write $a_n \downarrow a$ to mean that the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing and converges to a.

Proposition 1.2.8 *Let A be a subset of $\mathbb{R}$. The following statements hold true.*

(i) *Let f be a right-continuous function defined on cl(A). Given any fixed $r \in \mathbb{R}$, suppose that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in A satisfying $a_n \downarrow r$ as $n \to \infty$ and $a_n > r$ for all n. Then, we have*

$$\sup_{\{a\in A : a>r\}} f(a) = \sup_{\{a\in A : a\geq r\}} f(a) \textit{ and } \inf_{\{a\in A : a>r\}} f(a) = \inf_{\{a\in A : a\geq r\}} f(a).$$

(ii) *Let f be a continuous function defined on cl(A). Given any fixed $r \in \mathbb{R}$, suppose that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in A satisfying $a_n \to r$ as $n \to \infty$ and $a_n > r$ for all n. Then, we have*

$$\sup_{\{a\in A:a>r\}} f(a) = \sup_{\{a\in A:a\geq r\}} f(a) \text{ and } \inf_{\{a\in A:a>r\}} f(a) = \inf_{\{a\in A:a\geq r\}} f(a).$$

In particular, we can assume $r \in cl(\{a \in A : a > r\})$.

Proof. It suffices to prove the case of the supremum. It is obvious that

$$\sup_{\{a\in A:a>r\}} f(a) \leq \sup_{\{a\in A:a\geq r\}} f(a).$$

To prove part (i), given any $\epsilon > 0$, according to the concept of supremum $\sup_{\{a\in A:a\geq r\}} f(a)$, there exists $a^* \in A$ with $a^* \geq r$ satisfying

$$\sup_{\{a\in A:a\geq r\}} f(a) - \epsilon \leq f(a^*).$$

We consider the following two cases.

- Suppose that $a^* > r$. Then, we have

$$\sup_{\{a\in A:a\geq r\}} f(a) - \epsilon \leq f(a^*) \leq \sup_{\{a\in A:a>r\}} f(a).$$

- Suppose that $a^* = r$. The assumption says that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in A satisfying $a_n \downarrow a^*$ as $n \to \infty$ and $a_n > r$ for all n. Since f is right-continuous and $a^* \in$ cl(A), we also have $f(a_n) \to f(a^*)$ as $n \to \infty$. Therefore, we obtain

$$\sup_{\{a\in A:a\geq r\}} f(a) - \epsilon \leq f(a^*) = \lim_{n\to\infty} f\left(a_n\right) \leq \lim_{n\to\infty}\left[\sup_{\{a\in A:a>r\}} f(a)\right] = \sup_{\{a\in A:a>r\}} f(a).$$

  Since ϵ can be any positive number, it follows that

$$\sup_{\{a\in A:a\geq r\}} f(a) \leq \sup_{\{a\in A:a>r\}} f(a).$$

Part (ii) can be similarly obtained, and the proof is complete. ∎

Proposition 1.2.9 *Let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two sequences of subsets of $\mathbb{R}$ satisfying*

$$A_{n+1} \subseteq A_n \text{ and } B_{n+1} \subseteq B_n \text{ for all } n$$

and

$$\bigcap_{n=1}^{\infty} A_n = A \text{ and } \bigcap_{n=1}^{\infty} B_n = B.$$

Then, we have

$$\lim_{n\to\infty} \sup_{a\in A_n} \inf_{b\in B_n} (a-b) = \lim_{n\to\infty}\left(\sup_{a\in A_n} a - \sup_{b\in B_n} b\right) = \sup_{a\in A} a - \sup_{b\in B} b = \sup_{a\in A} \inf_{b\in B} (a-b)$$

and

$$\lim_{n\to\infty} \sup_{b\in B_n} \inf_{a\in A_n} (a-b) = \lim_{n\to\infty}\left(\inf_{a\in A_n} a - \inf_{b\in B_n} b\right) = \inf_{a\in A} a - \inf_{b\in B} b = \sup_{b\in B} \inf_{a\in A} (a-b).$$

Proof. It is obvious that

$$\sup_{a\in A_n} \inf_{b\in B_n} (a-b) = \sup_{a\in A_n} a - \sup_{b\in B_n} b \text{ and } \sup_{b\in B_n} \inf_{a\in A_n} (a-b) = \inf_{a\in A_n} a - \inf_{b\in B_n} b.$$

The results follow immediately from Proposition 1.2.3. ∎

Proposition 1.2.10 *Let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two sequences of sets in $\mathbb{R}$ satisfying*

$$A_n \subseteq A_{n+1} \text{ and } B_n \subseteq B_{n+1} \text{ for all } n$$

and

$$\bigcup_{n=1}^{\infty} A_n = A \text{ and } \bigcup_{n=1}^{\infty} B_n = B.$$

Then, we have

$$\lim_{n\to\infty} \sup_{a\in A_n} \inf_{b\in B_n} (a-b) = \lim_{n\to\infty} \left(\sup_{a\in A_n} a - \sup_{b\in B_n} b \right) = \sup_{a\in A} a - \sup_{b\in B} b = \sup_{a\in A} \inf_{b\in B} (a-b)$$

and

$$\lim_{n\to\infty} \sup_{b\in B_n} \inf_{a\in A_n} (a-b) = \lim_{n\to\infty} \left(\inf_{a\in A_n} a - \inf_{b\in B_n} b \right) = \inf_{a\in A} a - \inf_{b\in B} b = \sup_{b\in B} \inf_{a\in A} (a-b).$$

Proof. The results follow immediately from Proposition 1.2.4. ■

Proposition 1.2.11 *Let f be a real-valued function defined on a subset A of $\mathbb{R}$, and let k be a constant. Then, we have*

$$\sup_{x\in A} \min\, \{f(x), k\} = \min \left\{ \sup_{x\in A} f(x), k \right\}$$

and

$$\inf_{x\in A} \max\, \{f(x), k\} = \max \left\{ \inf_{x\in A} f(x), k \right\}.$$

Proof. We have

$$\min \left\{ \sup_{x\in A} f(x), k \right\} = \begin{cases} k, & \text{if there exists } x \in A \text{satisfying } f(x) > k \\ \sup_{x\in A} f(x), & \text{if } f(x) \le k \text{ for all } x \in A. \end{cases}$$

and

$$\begin{aligned}
\sup_{x\in A} \min\, \{f(x), k\} &= \begin{cases} \max \left\{ \sup_{\{x\in A : f(x)>k\}} \min\{f(x),k\},\ \sup_{\{x\in A : f(x)\le k\}} \min\{f(x),k\} \right\}, \\ \qquad \text{if there exists } x \in A \text{ satisfying } f(x) > k \\ \sup_{x\in A} f(x), \qquad \text{if } f(x) \le k \text{ for all } x \in A \end{cases}. \\
&= \begin{cases} \max \left\{ k,\ \sup_{\{x\in A : f(x)\le k\}} f(x) \right\}, \\ \qquad \text{if there exists } x \in A \text{ satisfying } f(x) > k \\ \sup_{x\in A} f(x), \qquad \text{if } f(x) \le k \text{ for all } x \in A \end{cases}. \\
&= \begin{cases} k, & \text{if there exists } x \in A \text{ satisfying } f(x) > k \\ \sup_{x\in A} f(x), & \text{if } f(x) \le k \text{ for all } x \in A. \end{cases}
\end{aligned}$$

Another equality can be similarly obtained. This completes the proof. ■

1.3 Semi-Continuity

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a real-valued function defined on $\mathbb{R}^m$. We say that the supremum $\sup_{x\in S} f(x)$ is *attained* when there exists $x^* \in S$ satisfying $f(x) \leq f(x^*)$ for all $x \in S$ with $x \neq x^*$. Equivalently, the supremum $\sup_{x\in S} f(x)$ is attained if and only if

$$\sup_{x\in S} f(x) = \max_{x\in S} f(x).$$

Similarly, the infimum $\inf_{x\in S} f(x)$ is *attained* when there exists $x^* \in S$ satisfying $f(x) \geq f(x^*)$ for all $x \in S$ with $x \neq x^*$. Equivalently, the infimum $\inf_{x\in S} f(x)$ is attained if and only if

$$\inf_{x\in S} f(x) = \min_{x\in S} f(x).$$

Let $\mathbf{x} = (x_1, \ldots, x_m)$ be an element in $\mathbb{R}^m$. Recall that the Euclidean norm of $\mathbf{x}$ is given by

$$\| \mathbf{x} \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}.$$

Definition 1.3.1 Let S be a nonempty set in $\mathbb{R}^m$.

- A real-valued function $f : S \to \mathbb{R}$ defined on S is said to be **upper semi-continuous** at $\bar{\mathbf{x}}$ when the following condition is satisfied: for each $\epsilon > 0$, there exists $\delta > 0$ such that $\| \mathbf{x} - \bar{\mathbf{x}} \| < \delta$ implies $f(\mathbf{x}) < f(\bar{\mathbf{x}}) + \epsilon$ for any $\mathbf{x} \in S$.
- A real-valued function f defined on S is said to be **lower semi-continuous** at $\bar{\mathbf{x}}$ when the following condition is satisfied: for each $\epsilon > 0$, there exists $\delta > 0$ such that $\| \mathbf{x} - \bar{\mathbf{x}} \| < \delta$ implies $f(\bar{\mathbf{x}}) < f(\mathbf{x}) + \epsilon$ for any $\mathbf{x} \in S$.

Remark 1.3.2 We have the following interesting observations.

- If f is upper semi-continuou on S, then $-f$ is lower semi-continuous on S.
- If f is lower semi-continuou on S, then $-f$ is upper semi-continuous on S.
- The real-valued function f is continuous on S if and only if it is both lower and upper semi-continuous in S.
- If f is upper semi-continuous on $\mathbb{R}$, then $\{\mathbf{x} : f(\mathbf{x}) \geq \alpha\}$ is a closed subset of $\mathbb{R}^m$ for all α.
- If f is lower semi-continuous on $\mathbb{R}$, then $\{\mathbf{x} : f(\mathbf{x}) \leq \alpha\}$ is a closed subset of $\mathbb{R}^m$ for all α.

Proposition 1.3.3 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a multi-variable real-valued function, and let each real-valued function $g_i : \mathbb{R} \to \mathbb{R}$ be continuous at $x_0 \in \mathbb{R}$ for $i = 1, \ldots, n$. Then, the following statements hold true.*

(i) *Suppose that f is lower semi-continuous at $\mathbf{x}_0 \equiv (g_1(x_0), \ldots, g_m(x_0))$. Then, the composition function $h(x) = f\left(g_1(x), \ldots, g_m(x)\right)$ is lower semi-continuous at x_0.*

(ii) *Suppose that f is upper semi-continuous at $\mathbf{x}_0 \equiv (g_1(x_0), \ldots, g_m(x_0))$. Then, the composition function $h(x) = f\left(g_1(x), \ldots, g_m(x)\right)$ is upper semi-continuous at x_0.*

Proof. To prove part (i), since f is lower semi-continuous at $\mathbf{x}_0$, given any $\epsilon > 0$, there exists $\delta^* > 0$ such that

$$\| \mathbf{x} - \mathbf{x}_0 \| < \delta^* \text{ implies } f(\mathbf{x}_0) < f(\mathbf{x}) + \epsilon.$$

Since each g_i is continuous at x_0 for $i = 1, \dots, n$, given $\delta^*/\sqrt{n}$, there exists $\delta_i > 0$ such that

$$|x - x_0| < \delta_i \text{ implies } \left|g_i(x) - g_i(x_0)\right| < \frac{\delta^*}{\sqrt{n}} \text{ for } i = 1, \dots, n. \tag{1.13}$$

Let $\delta = \min\{\delta_1, \dots, \delta_m\}$. Then $|x - x_0| < \delta$ implies that the inequality (1.13) is satisfied for all $i = 1, \dots, n$. Let $\mathbf{x} \equiv (g_1(x), \dots, g_m(x))$. Then

$$\| \mathbf{x} - \mathbf{x}_0 \| = \sqrt{(g_1(x) - g_1(x_0))^2 + \cdots + (g_m(x) - g_m(x_0))^2} < \delta^*,$$

which implies

$$h(x_0) = f(g_1(x_0), \dots, g_m(x_0)) = f(\mathbf{x}_0) < f(\mathbf{x}) + \epsilon = f(g_1(x), \dots, g_m(x)) + \epsilon = h(x) + \epsilon,$$

which says that h is lower semi-continuous at x_0. Part (ii) can be similarly obtained. This completes the proof. ∎

Proposition 1.3.4 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a multi-variable real-valued function, and let each real-valued function $g_i : \mathbb{R} \to \mathbb{R}$ be left-continuous at $x_0 \in \mathbb{R}$ for $i = 1, \dots, n$. Then, the following statements hold true.*

(i) *Assume that the composition function $h(x) = f(g_1(x), \dots, g_m(x))$ is increasing. If f is lower semi-continuous at $\mathbf{x}_0 \equiv (g_1(x_0), \dots, g_m(x_0))$, then h is lower semi-continuous at x_0.*
(ii) *Assume that the composition function $h(x) = f(g_1(x), \dots, g_m(x))$ is decreasing. If f is upper semi-continuous at $\mathbf{x}_0 \equiv (g_1(x_0), \dots, g_m(x_0))$, then h is upper semi-continuous at x_0.*

Proof. To prove part (i), since f is lower semi-continuous at $\mathbf{x}_0$, given any $\epsilon > 0$, there exists $\delta^* > 0$ such that

$$\| \mathbf{x} - \mathbf{x}_0 \| < \delta^* \text{ implies } f(\mathbf{x}_0) < f(\mathbf{x}) + \epsilon.$$

Since each g_i is left-continuous at x_0 for $i = 1, \dots, n$, given $\delta^*/\sqrt{n}$, there exists $\delta_i > 0$ such that

$$0 < x_0 - x < \delta_i \text{ implies } \left|g_i(x) - g_i(x_0)\right| < \frac{\delta^*}{\sqrt{n}} \text{ for } i = 1, \dots, n.$$

The argument in the proof of Proposition 1.3.3 is still valid to show that there exists $\delta > 0$ such that

$$0 < x_0 - x < \delta \text{ implies } h(x_0) < h(x) + \epsilon.$$

For $0 < x - x_0 < \delta$, since h is increasing, it follows that

$$h(x_0) \leq h(x) < h(x) + \epsilon.$$

Therefore, we conclude that

$$|x_0 - x| < \delta \text{ implies } h(x_0) < h(x) + \epsilon,$$

which says that h is lower semi-continuous at x_0.

To prove part (ii), we can similarly show that there exists $\delta > 0$ such that

$$0 < x_0 - x < \delta \text{ implies } h(x) < h(x_0) + \epsilon.$$

For $0 < x - x_0 < \delta$, since h is decreasing, it follows that

$$h(x) \leq h(x_0) < h(x_0) + \epsilon,$$

which says that h is upper semi-continuous at x_0. This completes the proof. ∎

Proposition 1.3.5 *We have the following properties.*

(i) *Suppose that the real-valued functions f_1 and f_2 are lower semi-continuous on the closed interval $[a, b]$. Then, the addition $f_1 + f_2$ is also lower semi-continuous on the closed interval $[a, b]$.*
(ii) *Suppose that the real-valued functions g_1 and g_2 are upper semi-continuous on on the closed interval $[a, b]$. Then, the addition $g_1 + g_2$ is also upper semi-continuous on the closed interval $[a, b]$.*

Proof. To prove part (i), given $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \text{ implies } f_1(x_0) < f_1(x) + \frac{\epsilon}{2},$$

and that

$$|x - x_0| < \delta_2 \text{ implies } f_2(x_0) < f_2(x) + \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|x - x_0| < \delta$, we have

$$f_1(x_0) + f_2(x_0) < f_1(x) + \frac{\epsilon}{2} + f_2(x) + \frac{\epsilon}{2} = f_1(x) + f_2(x) + \epsilon$$

which shows that $f_1 + f_2$ is lower semi-continuous at x_0.

To prove part (ii), given $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \text{ implies } g_1(x_0) + \frac{\epsilon}{2} > g_1(x),$$

and that

$$|x - x_0| < \delta_2 \text{ implies } g_2(x_0) + \frac{\epsilon}{2} > g_2(x).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|x - x_0| < \delta$, we have

$$g_1(x_0) + g_2(x_0) + \epsilon = g_1(x_0) + \frac{\epsilon}{2} + g_2(x_0) + \frac{\epsilon}{2} > g_1(x) + g_2(x),$$

which shows that $g_1 + g_2$ is upper semi-continuous at x_0. This completes the proof. ∎

Proposition 1.3.6 *We have the following properties.*

(i) *Suppose that the real-valued functions f_1 and f_2 are lower semi-continuous on the closed interval $[a, b]$. Then, the real-valued functions* $\min\{f_1, f_2\}$ *and* $\max\{f_1, f_2\}$ *are also lower semi-continuous on the closed interval $[a, b]$.*
(ii) *Suppose that the real-valued functions g_1 and g_2 are upper semi-continuous on on the closed interval $[a, b]$. Then, the real-valued functions* $\min\{g_1, g_2\}$ *and* $\max\{g_1, g_2\}$ *are also upper semi-continuous on the closed interval $[a, b]$.*

Proof. To prove part (i), given $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \text{ implies } f_1(x_0) < f_1(x) + \epsilon,$$

and that

$$|x - x_0| < \delta_2 \text{ implies } f_2(x_0) < f_2(x) + \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|x - x_0| < \delta$, we have

$$\min\left\{f_1(x_0), f_2(x_0)\right\} < \min\left\{f_1(x) + \epsilon, f_2(x) + \epsilon\right\} = \min\left\{f_1(x), f_2(x)\right\} + \epsilon$$

and

$$\max\left\{f_1(x_0), f_2(x_0)\right\} < \max\left\{f_1(x) + \epsilon, f_2(x) + \epsilon\right\} = \max\left\{f_1(x), f_2(x)\right\} + \epsilon,$$

which show that $\min\{f_1, f_2\}$ and $\max\{f_1, f_2\}$ are lower semi-continuous at x_0.

To prove part (ii), given $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \text{ implies } g_1(x_0) + \epsilon > g_1(x),$$

and that

$$|x - x_0| < \delta_2 \text{ implies } g_2(x_0) + \epsilon > g_2(x).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|x - x_0| < \delta$, we have

$$\min\left\{g_1(x_0), g_2(x_0)\right\} + \epsilon = \min\left\{g_1(x_0) + \epsilon, g_2(x_0) + \epsilon\right\} > \min\left\{g_1(x), g_2(x)\right\}$$

and

$$\max\left\{g_1(x_0), g_2(x_0)\right\} + \epsilon = \max\left\{g_1(x_0) + \epsilon, g_2(x_0) + \epsilon\right\} > \max\left\{g_1(x), g_2(x)\right\},$$

which show that $\min\{g_1, g_2\}$ and $\max\{g_1, g_2\}$ are upper semi-continuous at x_0. This completes the proof. ∎

Proposition 1.3.7 *We have the following properties.*

(i) *Suppose that f is increasing on a subset D of $\mathbb{R}$. Then f is left-continuous on D if and only if f is lower semi-continuous on D.*
(ii) *Suppose that g is decreasing on a subset D of $\mathbb{R}$. Then g is left-continuous on D if and only if g is upper semi-continuous on D.*

Proof. To prove part (i), we first assume that f is left-continuous at $x_0 \in D$. Then, given any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < x_0 - x < \delta$ implies $|f(x_0) - f(x)| < \epsilon$, i.e. $f(x_0) < f(x) + \epsilon$. For $x_0 \in D$ with $0 < x - x_0 < \delta$, since f is increasing, we have

$$f(x_0) \leq f(x) < f(x) + \epsilon.$$

Therefore, we conclude that $|x_0 - x| < \delta$ implies $f(x_0) < f(x) + \epsilon$, which shows that f is lower semi-continuous at $x_0 \in D$.

Conversely, we assume that f is lower semi-continuous at $x_0 \in D$. Then, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|x_0 - x| < \delta$ implies $f(x_0) < f(x) + \epsilon$. If $0 < x_0 - x < \delta$ then we immediately have $f(x_0) - f(x) < \epsilon$ by the lower semi-continuity at x_0. Since f is increasing, we also have

$$f(x) \leq f(x_0) < f(x_0) + \epsilon.$$

Therefore, we conclude that $0 < x_0 - x < \delta$ implies $|f(x_0) - f(x)| < \epsilon$, which shows that f is left-continuous at $x_0 \in D$.

To prove part (ii), we first assume that g is left-continuous at $x_0 \in D$. Then, given any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < x_0 - x < \delta$ implies $|g(x_0) - g(x)| < \epsilon$, i.e. $g(x) < g(x_0) + \epsilon$. For $x_0 \in D$ with $0 < x - x_0 < \delta$, since g is decreasing, we have

$$g(x_0) + \epsilon \geq g(x) + \epsilon > g(x).$$

Therefore, we conclude that $|x_0 - x| < \delta$ implies $g(x) < g(x_0) + \epsilon$, which shows that g is upper semi-continuous at $x_0 \in D$.

Conversely, we assume that f is upper semi-continuous at $x_0 \in D$. Then, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|x_0 - x| < \delta$ implies $g(x) < g(x_0) + \epsilon$. If $0 < x_0 - x < \delta$, then we immediately have $g(x) - g(x_0) < \epsilon$ by the upper semi-continuity at x_0. Since g is decreasing, we also have

$$g(x_0) \leq g(x) < g(x) + \epsilon.$$

Therefore, we conclude that $0 < x_0 - x < \delta$ implies $|g(x_0) - g(x)| < \epsilon$, which shows that g is left-continuous at $x_0 \in D$. This completes the proof. ∎

Let A be a subset of $\mathbb{R}^m$. The **characteristic function** or **indicator function** of A is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A. \end{cases} \tag{1.14}$$

Proposition 1.3.8 *Let S be a subset of $\mathbb{R}$, and let $\zeta^L : S \to \mathbb{R}$ and $\zeta^U : S \to \mathbb{R}$ be two bounded real-valued functions defined on S satisfying $\zeta^L(\alpha) \leq \zeta^U(\alpha)$ for each $\alpha \in S$. Suppose that ζ^L is lower semi-continuous on S, and that ζ^U is upper semi-continuous on S. Let $M_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$ for $\alpha \in S$ be closed intervals. Then, for any fixed $x \in \mathbb{R}$, the function $\zeta(\alpha) = \alpha \cdot \chi_{M_\alpha}(x)$ is upper semi-continuous on S.*

Proof. For any fixed $\alpha_0 \in S$, we are going to show that, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\alpha - \alpha_0| < \delta \text{ implies } \zeta(\alpha_0) + \epsilon > \zeta(\alpha).$$

We consider the cases of $x \in M_{\alpha_0}$ and $x \notin M_{\alpha_0}$. For $x \in M_{\alpha_0}$, we have $\zeta(\alpha_0) = \alpha_0$. If $|\alpha - \alpha_0| < \delta = \epsilon$, we have $\alpha_0 + \epsilon > \alpha$. We consider the following cases.

- Suppose that $x \notin M_\alpha$. Then, we have $\zeta(\alpha) = 0$. Therefore, we obtain

  $$\zeta(\alpha_0) + \epsilon = \alpha_0 + \epsilon > 0 = \zeta(\alpha).$$

- Suppose that $x \in M_\alpha$. Then, we have $\zeta(\alpha) = \alpha$. Therefore, we obtain

  $$\zeta(\alpha_0) + \epsilon = \alpha_0 + \epsilon > \alpha = \zeta(\alpha).$$

Now, we consider the case of $x \notin M_{\alpha_0}$, i.e. $x < \zeta^L(\alpha_0)$ or $x > \zeta^U(\alpha_0)$. In this case, we have $\zeta(\alpha_0) = 0$.

- For $x < \zeta^L(\alpha_0)$, let $\epsilon = \zeta^L(\alpha_0) - x$. Since ζ^L is lower semi-continuous at α_0, there exists $\delta > 0$ such that $|\alpha - \alpha_0| < \delta$ implies $\zeta^L(\alpha_0) < \zeta^L(\alpha) + \epsilon$. Therefore, we obtain

  $$\zeta^L(\alpha) > \zeta^L(\alpha_0) - \epsilon = \zeta^L(\alpha_0) + x - \zeta^L(\alpha_0) = x.$$

  This also says that $x \notin M_\alpha$, i.e. $\zeta(\alpha) = 0$ for $|\alpha - \alpha_0| < \delta$.

- For $x > \zeta^U(\alpha_0)$, let $\epsilon = x - \zeta^U(\alpha_0)$. Since ζ^U is upper semi-continuous at α_0, there exists $\delta > 0$ such that $|\alpha - \alpha_0| < \delta$ implies $\zeta^U(\alpha) < \zeta^U(\alpha_0) + \epsilon$. Therefore, we obtain

$$\zeta^U(\alpha) < \zeta^U(\alpha_0) + \epsilon = \zeta^U(\alpha_0) + x - \zeta^U(\alpha_0) = x.$$

This also says that $x \notin M_\alpha$, i.e. $\zeta(\alpha) = 0$ for $|\alpha - \alpha_0| < \delta$.

The above two cases conclude that

$$\zeta(\alpha_0) + \epsilon = \epsilon > 0 = \zeta(\alpha)$$

for $|\alpha - \alpha_0| < \delta$. This completes the proof. ∎

Proposition 1.3.9 *Let S be a subset of $\mathbb{R}$, and let $\zeta^L : S \to \mathbb{R}$ and $\zeta^U : S \to \mathbb{R}$ be two bounded real-valued functions defined on S satisfying $\zeta^L(\alpha) \leq \zeta^U(\alpha)$ for each $\alpha \in S$. Suppose that the following conditions are satisfied.*

- *ζ^L is an increasing function and ζ^U is a decreasing function on S.*
- *ζ^L and ζ^U are left-continuous on S.*

Let $M_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$ for $\alpha \in S$ be closed intervals. Then, for any fixed $x \in \mathbb{R}$, the function $\zeta(\alpha) = \alpha \cdot \chi_{M_\alpha}(x)$ is upper semi-continuous on S.

Proof. The result follows immediately from Propositions 1.3.8 and 1.3.7. ∎

Proposition 1.3.10 *Let S be a subset of $\mathbb{R}$. For each $i = 1, \ldots, n$, let $\zeta_i^L : S \to \mathbb{R}$ and $\zeta_i^U : S \to \mathbb{R}$ be bounded real-valued functions defined on S satisfying $\zeta_i^L(\alpha) \leq \zeta_i^U(\alpha)$ for each $\alpha \in S$. Suppose that the following conditions are satisfied.*

- *ζ_i^L are increasing function and ζ_i^U are decreasing function on S for $i = 1, \ldots, n$.*
- *ζ_i^L and ζ_i^U are left-continuous on S for $i = 1, \ldots, n$.*

Let $M_\alpha^{(i)} = [\zeta_i^L(\alpha), \zeta_i^U(\alpha)]$ for $\alpha \in S$ and for $i = 1, \ldots, n$ be closed intervals, and let

$$M_\alpha = M_\alpha^{(1)} \times \cdots \times M_\alpha^{(n)} \subset \mathbb{R}^n.$$

Given any fixed $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the function $\zeta(\alpha) = \alpha \cdot \chi_{M_\alpha}(\mathbf{x})$ is upper semi-continuous on S.

Proof. Proposition 1.3.9 says that the functions $\zeta_i(\alpha) = \alpha \cdot \chi_{M_\alpha^{(i)}}(x_i)$ are upper semi-continuous on S for $i = 1, \ldots, n$. For $r \in S$, we define the sets

$$F_r = \{\alpha \in S : \zeta(\alpha) \geq r\} \text{ and } F_r^{(i)} = \left\{\alpha \in S : \zeta_i(\alpha) \geq r\right\} \text{ for } i = 1, \ldots, n.$$

The upper semi-continuity of ζ_i says that $F_r^{(i)}$ is a closed set for $i = 1, \ldots, n$. We want to claim $F_r = \bigcap_{i=1}^n F_r^{(i)}$. Given any $\alpha \in F_r$, it follows that $\mathbf{x} \in M_\alpha$ and $\alpha \geq r$, i.e. $x_i \in M_\alpha^{(i)}$ and $\alpha \geq r$ for $i = 1, \ldots, n$, which also implies $\zeta_i(\alpha) \geq r$ for $i = 1, \ldots, n$. Therefore, we obtain the inclusion $F_r \subseteq \bigcap_{i=1}^n F_r^{(i)}$. On the other hand, suppose that $\alpha \in F_r^{(i)}$ for $i = 1, \ldots, n$. It follows that $x_i \in M_\alpha^{(i)}$ and $\alpha \geq r$ for $i = 1, \ldots, n$, i.e. $\mathbf{x} \in M_\alpha$ and $\alpha \geq r$. Therefore, we obtain the equality $F_r = \bigcap_{i=1}^n F_r^{(i)}$, which also says that F_r is a closed set, since each $F_r^{(i)}$ is a closed set for $i = 1, \ldots, n$. Therefore, we conclude that ζ is indeed upper semi-continuous on S. This completes the proof. ∎

We say that S is a disjoint union of intervals in $\mathbb{R}$ when S can be expressed as

$$S = \bigcup_{i=1}^{\infty} I_i$$

satisfying $I_i \cap I_j = \emptyset$ for $i \neq j$, where each I_i is an interval in $\mathbb{R}$.

Proposition 1.3.11 *Let S be a disjoint union of intervals in $\mathbb{R}$, and let $\zeta^L : S \to \mathbb{R}$ and $\zeta^U : S \to \mathbb{R}$ be two bounded real-valued functions defined on S satisfying $\zeta^L(\alpha) \leq \zeta^U(\alpha)$ for each $\alpha \in S$. For $\alpha \in S$, we define the functions*

$$l(\alpha) = \inf_{\{x \in S : x \geq \alpha\}} \zeta^L(x) \text{ and } u(\alpha) = \sup_{\{x \in S : x \geq \alpha\}} \zeta^U(x).$$

Then l and u are left-continuous on S. Moreover, l is lower semi-continuous on S and u is upper semi-continuous on S.

Proof. Given $\alpha \in S$, since S is a disjoint union of intervals, there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in S satisfying $\alpha_n \uparrow \alpha$ as $n \to \infty$, where we allow $\alpha_n = \alpha$ for some n. Let

$$A_n = \{x \in S : x \geq \alpha_n\} \text{ and } A = \{x \in S : x \geq \alpha\}.$$

Then it is obvious that $A_{n+1} \subseteq A_n$ for all n and $A \subseteq \bigcap_{n=1}^{\infty} A_n$. For $x \in \bigcap_{n=1}^{\infty} A_n$, it means $x \in S$ and $x \geq \alpha_n$ for all n. By taking limit, we obtain $x \geq \alpha$, i.e. $x \in A$. This shows that $A = \bigcap_{n=1}^{\infty} A_n$. Using Proposition 1.2.3, we obtain

$$l(\alpha_n) = \inf_{t \in A_n} \zeta^L(x) \to \inf_{t \in A} \zeta^L(x) = l(\alpha) \text{ for } \alpha_n \uparrow \alpha.$$

This says that l is left-continuous on S. We can similarly show that u is left-continuous on S. Since l is decreasing and u is increasing on S, Proposition 1.3.7 says that l is lower semi-continuous on S and u is upper semi-continuous on S. This completes the proof. ■

Let S be a disjoint union of intervals in $\mathbb{R}$. We write $\partial^L(S)$ to denote the set of all left endpoints of subintervals in S, and write $\partial^R(S)$ to denote the set of all right endpoints of subintervals in S. For any $\alpha \in S \backslash \partial^R(S)$, i.e. $\alpha \in S$ and $\alpha \notin \partial^R(S)$, it is clear to see that there exists a sequence in S satisfying $\alpha_n \downarrow \alpha$ as $n \to \infty$ with $\alpha_n > \alpha$ for all n.

Proposition 1.3.12 *Let S be a disjoint union of closed intervals in $\mathbb{R}$, and let $\zeta^L : S \to \mathbb{R}$ and $\zeta^U : S \to \mathbb{R}$ be two bounded and right-continuous real-valued functions defined on S satisfying $\zeta^L(\alpha) \leq \zeta^U(\alpha)$ for each $\alpha \in S$. Let $M_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$ for $\alpha \in S$ be closed intervals. Then, the functions*

$$l(\alpha) = \inf_{\{x \in S : x \geq \alpha\}} \zeta^L(x) \text{ and } u(\alpha) = \sup_{\{x \in S : x \geq \alpha\}} \zeta^U(x)$$

are continuous on $S \backslash \partial^R(S)$. Moreover, for $\alpha \in S \backslash \partial^R(S)$ and $\alpha_n \downarrow \alpha$ as $n \to \infty$ with $\alpha_n > \alpha$ for all n, we have $l(\alpha_n) \downarrow l(\alpha)$ and $u(\alpha_n) \uparrow u(\alpha)$ as $n \to \infty$.

Proof. According to Proposition 1.3.11, we remain to show that l and u are right-continuous on $S \backslash \partial^R(S)$. We first note that S is a closed set, i.e. $\mathrm{cl}(S) = S$. We are going to use part (i) of Proposition 1.2.8. Given $\alpha \in S \backslash \partial^R(S)$, there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in S satisfying $\alpha_n \downarrow \alpha$ as $n \to \infty$ with $\alpha_n > \alpha$ for all n. Let

$$A_n = \{x \in S : x \geq \alpha_n\} \text{ and } A^* = \{x \in S : x > \alpha\}.$$

It is clear to see that $A_n \subseteq A_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} A_n \subseteq A^*$. For $x \in A^*$, i.e. $x \in S$ and $x > \alpha$, since $\alpha_n \downarrow \alpha$, there exists α_{n^*} satisfying $\alpha \le \alpha_{n^*} < x$, which says that $x \in \bigcup_{n=1}^{\infty} A_n$. Therefore, we obtain $\bigcup_{n=1}^{\infty} A_n = A^*$. Using Proposition 1.2.4 and part (i) of Proposition 1.2.8, for $\alpha_n \downarrow \alpha$ with $\alpha_n > \alpha$, we have

$$l(\alpha_n) = \inf_{x \in A_n} \zeta^L(x) \to \inf_{x \in A^*} \zeta^L(x) = \inf_{\{x \in S : x \ge \alpha\}} \zeta^L(x) = l(\alpha).$$

Therefore, we conclude that l is continuous on S. We can similarly show that u is continuous on S. Since l is increasing and u is decreasing, we also have $l(\alpha_n) \downarrow l(\alpha)$ and $u(\alpha_n) \uparrow u(\alpha)$ as $n \to \infty$ for $\alpha_n \downarrow \alpha$ as $n \to \infty$ with $\alpha_n > \alpha$ for all n, and the proof is complete. ■

Proposition 1.3.13 *Let S be a closed subset of $\mathbb{R}$, and let $\zeta^L : S \to \mathbb{R}$ and $\zeta^U : S \to \mathbb{R}$ be two bounded real-valued functions defined on S satisfying $\zeta^L(\alpha) \le \zeta^U(\alpha)$ for each $\alpha \in S$. Suppose that ζ^L is lower semi-continuous on S, and that ζ^U is upper semi-continuous on S. Let $M_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$ for $\alpha \in S$ be closed intervals. Then, we have*

$$\bigcup_{\{\beta \in S : \beta \ge \alpha\}} M_\beta = \left[\inf_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta), \sup_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta) \right]$$
$$= \left[\min_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta), \max_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta) \right] \tag{1.15}$$

for any $\alpha \in S$.

Proof. Since S is a closed set, by Proposition 1.4.4 (which will be given below), the semi-continuities say that the imfimum and supremum are attained given by

$$\inf_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta) = \min_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta) \text{ and } \sup_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta) = \max_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta).$$

For $x \in \bigcup_{\{\beta \in S : \beta \ge \alpha\}} M_\beta$, there exists $\beta_0 \ge \alpha$ satisfying $x \in M_{\beta_0}$, i.e. $\zeta^L(\beta_0) \le x \le \zeta^U(\beta_0)$. Then, we have

$$x \ge \zeta^L(\beta_0) \ge \min_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta) \text{ and } x \le \zeta^U(\beta_0) \le \max_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta);$$

that is,

$$x \in \left[\min_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta), \max_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta) \right].$$

To prove the other direction of inclusion, given any x satisfying

$$\min_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta) \le x \le \max_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta), \tag{1.16}$$

we want to lead to a contradiction by assuming $x \notin M_\beta$ for each $\beta \in S$ with $\beta \ge \alpha$. Under this assumption, since each M_β is a bounded closed interval, it follows that $x < \zeta^L(\beta)$ for each $\beta \in S$ with $\beta \ge \alpha$ or $x > \zeta^U(\beta)$ for each $\beta \in S$ with $\beta \ge \alpha$. Since the infimum and supremum are attained, we obtain

$$x < \min_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta) = \inf_{\{\beta \in S : \beta \ge \alpha\}} \zeta^L(\beta) \text{ or } x > \max_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta) = \sup_{\{\beta \in S : \beta \ge \alpha\}} \zeta^U(\beta),$$

which contradicts (1.16). Therefore, there exists $\beta_0 \in S$ with $\beta_0 \ge \alpha$ satisfying $x \in M_{\beta_0}$. This completes the proof. ■